Research Statement

Preliminaries

My primary research interests are in geometric inverse semigroup theory and its connections with other fields of mathematics.

A semigroup $M$ is called an inverse semigroup if for every element $a \in M$, there exists a unique element $b \in M$ such that $a = aba$ and $b = bab$. Let $A$ and $B$ be subsets of a nonempty set $X$ with same cardinality. Then a bijective map $\phi : A \rightarrow B$ is called a partial one to one map of $X$. If we consider the set $I_X$ of all partial one to one maps of a nonempty set $X$, then $I_X$ forms an inverse semigroup under the binary operation of composition of partial functions. In the literature, this inverse semigroup is known as the Symmetric Inverse Monoid on the set $X$. One of the earliest result proved about inverse semigroups is the Wagner-Preston theorem. This theorem is analogous to the Cayley’s theorem for groups. The Wagner-Preston theorem says that every inverse semigroup embeds in a suitable symmetric inverse monoid. The basic details about the structure of inverse semigroups and their connection with other fields can be found in [8].

A congruence on a semigroup $S$ is an equivalence relation $\rho$ which is compatible with the semigroup operation: i.e., if $apb$ then $(ac)\rho(bc)$ and $(ca)\rho(cb)$ for all $a, b, c \in S$. The congruence class of $a$ is denoted by $a\rho$. The product of any two congruence class $ap$ and $bp$ is given by $(ap)(bp) = (ab)\rho$ and forms the quotient semigroup $S/\rho$. If $X$ is an alphabet and $X^{-1}$ denotes a disjoint set of inverses of elements of $X$. Then the Vagner congruence $\rho$ is the congruence generated by

$$\{(u, uu^{-1}u), (vu^{-1}vv, vv^{-1}uu^{-1})|u, v \in (X \cup X^{-1})^*\}.$$ 

The semigroup $(X \cup X^{-1})^*/\rho$ is called the free inverse monoid on $X$ and denoted by $FIM(X)$. If $R \subset (X \cup X^{-1}) \times (X \cup X^{-1})$ and $\tau$ is the congruence generated by $\rho \cup R$, where $\rho$ is the Vagner congruence, then $M = Inv\langle X|R \rangle := (X \cup X^{-1})^*/\tau$ is the inverse semigroup presented by the set $X$ of generators and the $R$ of relations.

If $R \subset X^+ \times X^+$, then the inverse semigroup given $M = Inv\langle X|R \rangle$ is called a positively presented inverse semigroup. My research focuses on the positively presented inverse semigroups.

The word problem for the inverse semigroup $M = Inv\langle X|R \rangle$ is the question of whether there is an algorithm which, given any two words $w_1, w_2 \in (X \cup X^{-1})^*$, will determine whether $w_1 = w_2$ in $M$.

In order to study the word problem for inverse semigroups, J. B. Stephen introduce the notion of Schützenberger graph in [12]. If $M = Inv\langle X|R \rangle$, then the Cayley graph of $M$ is denoted by $\Gamma(M, X)$. The vertices of this graph are the elements of $M$ and there is an edge labeled by $x \in X \cup X^{-1}$ starting from $m$ and ending at $mx$ for each $m \in M$. The Cayley graph $\Gamma(M, X)$ is not strongly connected in general(unless $M$ happens to be a group), because there may not be an edge labeled by $x^{-1}$ starting from $mx$ and ending at $m$. For example, if $M$ is an inverse semigroup that contains zero. Then there will be an edge starting from every nonzero element of $M$ and ending at zero, but there will be no edge starting from zero and ending at a vertex that is labeled by a nonzero element of $M$. So unlike the group case, the Cayley graph of an inverse semigroup is not a geodesic metric space relative to the a word metric.

To overcome this difficulty, we only consider the strongly connected components of $\Gamma(M, X)$. A strongly connected component of the Cayley graph $\Gamma(M, X)$ corresponds to an $R$–class of $M$. These strongly connected components are called the Schützenberger graphs of $M$. In more detail, for each word $w \in (X \cup X^{-1})^*$, we denote by $\Sigma(M, X, w)$ the graph with set $R_w = \{m \in M : mnw^{-1} = ww^{-1}inM\}$ of vertices, and with an edge labeled by $x \in (X \cup X^{-1}$ from $m$ to $mx$ if $m, mx \in R_w$. It is easy to see that if $x$ labels an edge from $m$ to $mx$ in $\Sigma(M, X, w)$ then $x^{-1}$
labels an edge from $mx$ to $m$ in this graph. Thus, the Schützenberger graphs $ST(M,X,w)$ are geodesic metric spaces with respect to the word metric.

It is useful to consider the **Schützenberger automaton** $A(M,X,w) = (ww^{-1}, ST(M,X,w), w)$ with initial state (vertex) $ww^{-1} \in M$, terminal state $w \in M$ and the set $ST(M,X,w)$ of states. The language accepted by this automaton is the set

$$L(w) = \{ u \in M : u \text{ labels a path in } A(M,X,w) \text{ from } ww^{-1} \text{ to } w \text{ in } ST(M,X,w) \}.$$  

(Here $w$ and $u$ are regarded as words in $(X \cup X^{-1})^*$ and as elements of $M$, and we regard the language $L(w)$ as a subset of $M$.) Further details about Schützenberger graphs can be found in [12] and [9].

The following theorem of Stephen [12] is very useful in deciding the word problem for an inverse semigroup.

**Theorem 0.1.** Let $M = \text{Inv}(X|R)$ and let $u,v \in (X \cup X^{-1})^*$ (also interpreted as elements of $M$ as above). Then

1. $L(u) = \{ v \in M : v \geq u \text{ in the natural partial order on } M \text{ (defined below) } \}$.

2. $u = v$ in $M$ iff $L(u) = L(v)$ iff $v \in L(u)$ and $u \in L(v)$ iff $A(u)$ and $A(v)$ are isomorphic as birooted edge-labeled graphs.

The natural partial order on the elements of $M$ is defined by $a \leq b$ iff $a = aa^{-1}b$. The equivalence relation $\sigma$ on $M$ is defined by $a\sigma b$ iff there exists an element $u \in M$ such that $u \leq a,b$. $\sigma$ is a congruence relation with the following properties:

1. $M/\sigma$ is a group such that $M/\sigma$ is isomorphic to the group $G = Gp(X|R)$.

2. If $\mu$ is a congruence on $M$ such that $M/\mu$ is a group then then $\sigma \subseteq \mu$.

The proofs of the above facts can be found in [8]. There exists a natural homomorphism $\sigma : M \rightarrow M/\sigma$. The group $G$ is the **maximal group homomorphic image** of $M$.

The inverse semigroup $M$ is called $E$-unitary if $\sigma^{-1}(1) = \{ e \in M : e^2 = e \}$.

**Adian Presentations**

Let $X$ be a nonempty set and $R \subset (X \cup X^{-1})^+ \times (X \cup X^{-1})^+$ then the pair $\langle X|R \rangle$ is called a presentation. For each relation $(r,s) \in R$ the words $r$ and $s$ are called $R$-words. If each $R$-word $r$ is an element of $X^+$, then the presentation $\langle X|R \rangle$ is called a positive presentation.

Let $\langle X|R \rangle$ be a positive presentation. Then the **left graph** of $\langle X|R \rangle$ is denoted by $LG(\langle X|R \rangle)$. It is an undirected graph whose set of vertices is $X$. For each relation $(r,s) \in R$ there is an undirected edge joining the first letter of $r$ to the first letter of $s$. Similarly, the **right graph** of the presentation $\langle X|R \rangle$ is denoted by $RG(\langle X|R \rangle)$ and can be constructed dually to the left graph. A closed path in a graph is called a **cycle**. If both the left and the right graphs of the presentation $\langle X|R \rangle$ are cycle free, then $\langle X|R \rangle$ is called a **cycle free presentation** or an Adian presentation.

If $\langle X|R \rangle$ is an Adian presentation, then the semigroup $Sg(\langle X|R \rangle)$ is called an Adian semigroup, the inverse semigroup $Inv(\langle X|R \rangle)$ is called an Adian inverse semigroup and the group $Gp(\langle X|R \rangle)$ is called an Adian group. In [1], Adian proved that a finitely presented Adian semigroup $Sg(\langle X|R \rangle)$ embeds into the Adian group $Gp(\langle X|R \rangle)$. Subsequently in [10] also provided the proof of the same fact by using a geometrical approach. Remmers’ proof was stronger than the Adian’s proof in the sense that it was also valid for infinite presentations.
In [2], Adian studies the word problem for some special classes of semigroups and groups and conjectured that the word problem is decidable for Adian semigroups.

Sarkisian proposed a solution to Adian’s conjecture in [11], but subsequently it was recognized that the proof is only valid for a particular class of Adian semigroups. So the problem is still unsolved.

We observed the following facts:

Proposition 0.2. (Muhammad Inam and Robert Ruyle) An Adian semigroup $Sg\langle X | R \rangle$ embeds into the corresponding Adian inverse semigroup $Inv\langle X | R \rangle$.

Proposition 0.3. (Muhammad Inam, John Meakin and Robert Ruyle) Let $\langle X | R \rangle$ be an Adian presentation, then the word problem for $Sg\langle X | R \rangle$ and $Gp\langle X | R \rangle$ is decidable if

1. the inverse semigroup $Inv\langle X | R \rangle$ is $E$–unitary.

2. the word problem for $Inv\langle X | R \rangle$ is decidable.

We proved the following theorem:

Theorem 0.4. (Muhammad Inam, John Meakin and Robert Ruyle) Adian inverse semigroups are $E$-unitary.

A Structural Property Of Adian Inverse Semigroups

In this section we briefly outline the proof of the fact that an Adian inverse semigroup is $E$–unitary. The proof makes use of the notion of a Van-Kampen diagram.

Informally, for a group $G = Gp\langle X | R \rangle$, a Van Kampen diagram is a planer diagram whose boundary is label is an element of $(X \cup X^{-1})^*$ and is equal to the identity of $G$. A formal definition can be found in [6].

Remmers proved the following results about a Van-Kampen diagram over an Adian presentation in [10]

Lemma 0.5. Let $\Delta$ be a Van-Kampen diagram for a word $w$ over an Adian presentation $\langle X | R \rangle$. Then :

1. $\Delta$ has no interior sources and no interior sinks.

2. $\Delta$ contains no directed (i.e. positively labeled) cycles.

3. Every positively labeled interior edge of $\Delta$ can be extended to a directed transversal of $\Delta$.

For our purpose we introduce the following definitions.

Definition 0.1. A subdiagram $\Delta'$ of a Van-Kampen diagram $\Delta$ is called a simple component of $\Delta$ if it is a maximal subdiagram whose boundary is a simple closed curve.

Definition 0.2. For a Van-Kampen diagram $\Delta$ over an Adian presentation $\langle X | R \rangle$ a transversal subdiagram $\Delta'$ is a subdiagram of a simple component of $\Delta$ such that $\Delta'$ has a boundary cycle of the form $pq$, where $p$ is a directed transversal and $q$ is a subpath of a boundary cycle of the simple component in which $\Delta'$ is contained.

Definition 0.3. For a Van-Kampen diagram $\Delta$ over an Adian presentation $\langle X | R \rangle$, a special 2-cell is a 2-cell one of whose two sides lies entirely on the boundary of $\Delta$.

Then we obtained the following lemmas, which enabled us to prove our main theorem.
Lemma 0.6. If $\Delta$ is a Van Kampen diagram with exactly one simple component and no extremal vertex then $\Delta$ has a directed transversal if and only if it has more than one 2-cell. Furthermore, any directed transversal of $\Delta$ divides $\Delta$ into two transversal subdiagrams each of which may be viewed as a Van-Kampen diagram with exactly one simple component and no extremal vertex.

Lemma 0.7. Let $\Delta$ be a Van-Kampen diagram over an Adian presentation $\langle X | R \rangle$ that has more than one 2-cell and suppose that $\Delta$ has just one simple component and no extremal vertex. Then $\Delta$ contains at least two special 2-cells.

Lemma 0.8. Let $\Delta$ be a Van-Kampen diagram over an Adian presentation $\langle X | R \rangle$ and suppose that $\Delta$ has just one simple component and no extremal vertex. Then any word labeling a boundary cycle of $\Delta$, starting and ending at any vertex $0$ on the boundary of $\Delta$, is an idempotent in the inverse monoid $S = \text{Inv}(X | R)$.

Theorem 0.9. (Muhammad Inam, John Meakin and Robert Ruyle) Adian inverse semigroups are E-unitary.

The proof of the theorem 0.9 follows from the above lemmas and by applying induction on the number of simple components of a Van-Kampen diagram.

The Word Problem for One Relator Adian Inverse Semigroups

In this section I briefly describe some subclasses of one relator Adian inverse semigroup for which the word problem is decidable.

In [12] Stephen proved that if $\text{Inv}(X | R)$ is a positively presented inverse semigroup, where $X$ is finite and $|r| = |s|$ for all $(r, s) \in R$, then $\text{Inv}(X | R)$ has decidable word problem. So, the word problem for $\text{Inv}(X | u = v)$ is decide if $|u| = |v|$.

Let $M = \text{Inv}(X | u = v)$ be an Adian inverse semigroup, where $X$ is finite and $|u| > |v|$.

Proposition 0.10. The word problem for $M$ is decidable if either no suffix of $u$ is a prefix of $v$, or no prefix of $u$ is a suffix of $v$.

The case, when a prefix of $u$ is a suffix of $v$ and a suffix of $u$ is a prefix of $v$ is more complicated and is still open. However, I have proved the following result:

Proposition 0.11. The word problem for the Baumslag-Solitar inverse semigroup $\text{Inv}\langle a, b | ab^n = b^n a \rangle$, where $m, n \in \mathbb{N}$, is decidable.

The Word Problem for a Class of Positively Presented Semigroups and Inverse Semigroups

My work is also concerned with other classes of positively presented inverse semigroups that satisfy condition (*) (defined below).

Definition 0.4. We say that a positive presentation $\langle X | R \rangle$ satisfies condition (*), if no prefix of an $R$–word is a suffix of itself or any other $R$–word,

I observed that if I impose an extra condition on those presentations which satisfy condition (*), then we we obtain the following result:
Proposition 0.12. Let \( M = \text{Inv}(X|R) \) be an inverse semigroup, where \( (X|R) \) is a positive presentation that satisfies the condition (*) and no \( R \)-word is a subword of another \( R \)-word. Then the Schützenberger graph of every word of \( (X \cup X^{-1})^* \) is finite. Hence the word problem for \( M \) is decidable.

We define a directed graph for a positive presentation, called the bi-sided graph of the presentation:

Definition 0.5. The bi-sided graph of a positive presentation \( (X|R) \) is a directed graph whose vertex set is the set of all \( R \)-words. For each relation \( (u_i, v_1u_jv_2) \in R \), where \( v_1, v_2 \in X^+ \) and \( u_i \) and \( u_j \), are not necessarily distinct \( R \)-words, there exists a directed edge from the vertex labeled by \( u_i \) to the vertex labeled by \( u_j \).

A closed directed path in a bi-sided graph is called a cycle.

Definition 0.6. A presentation \( (x|R) \) is called acyclic presentation if the corresponding bi-sided graph is cycle free.

We have proved the following proposition:

Proposition 0.13. Let \( M = \text{Inv}(X|R) \) be an inverse semigroup, where \( (X|R) \) is a positive, acyclic presentation that satisfies condition (*), then the Schützenberger graph of every word of \( (X \cup X^{-1})^* \) is finite. Hence the word problem for \( M \) is decidable.

We have also proved the following proposition:

Theorem 0.14. Let \( (X|R) \) be a positive presentation that satisfies condition (*) and assumes that \( R \) contains no relation of the form \( pq = psq \) for some \( p, s, q \in X^+ \). Then the semigroup \( S = Sg(X|R) \) embeds into the inverse semigroup \( M = \text{Inv}(X|R) \).

So, if the word problem for \( M \) is decidable then the word problem for \( S \) is also decidable. Based on all the results we have stated above, we conjecture the following statement:

Conjecture 0.15. Let \( M = \text{Inv}(X|R) \) be an inverse semigroup, where \( (X|R) \) is a positive presentation that satisfies the condition (*), then the word problem for \( M \) is decidable.

References