A NOTE ON A CONJECTURE OF WATANABE AND YOSHIDA

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Abstract. We consider a conjecture of Watanabe and Yoshida in [12] concerning the Hilbert - Kunz multiplicity of an ideal in a Cohen-Macaulay ring and provide a proof of the conjecture in the case the ring is graded.

1. Introduction

The purpose of this note is to examine a conjecture of Watanabe and Yoshida and to provide a proof of this conjecture in the case we have a graded ring R. We begin with some notation. Throughout this note all rings are assumed to be commutative Noetherian of prime characteristic $p > 0$ unless otherwise noted. Let (R, m) be a local ring of dimension d with maximal ideal $\mathfrak{m}$. For an ideal I of R and $q = p^e$ (for some e), we let $I^{[q]}$ denote the ideal of R generated by the set $\{i^q : i \in I\}$. Given an m-primary ideal I of R, let $\lambda_R(R/I)$ denote the length of R/I as an R-module. One then defines the Hilbert - Kunz multiplicity of I by

$$e_{HK}(I, R) := \lim_{q\to\infty} \frac{\lambda_R(R/I^{[q]})}{q^d}.$$

It was shown by Monsky [8] that this limit exists and is positive for all such ideals.

In [12], Watanabe and Yoshida make the following conjecture:

Conjecture 1.1. *Let (R, m) be a Cohen-Macaulay local ring of characteristic $p > 0$. Then*

(1) For any m-primary ideal I, one has $e_{HK}(I, R) \geq \lambda_R(R/I)$.

(2) For any m-primary ideal I with $\mathrm{pd}_R(R/I) < \infty$, one has $e_{HK}(I, R) = \lambda_R(R/I)$.

Dutta [2] has shown Conjecture 1.1 holds when R is a complete intersection ring (but not necessarily graded). However, neither part of the conjecture holds for all Cohen-Macaulay

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rings. Miller and Singh [7] have constructed a module M of finite projective dimension over a Gorenstein ring R of dimension 5 with

$$220 = \lim_{n \to \infty} \frac{\lambda_R(F_R^n(M))}{p^{5n}} < \lambda_R(M) = 222.$$

From the proof of Theorem 6.4 in [5], there exist $\mathfrak{m}$-primary ideals $J, I_1, \ldots, I_t$ of finite projective dimension such that $I_1, \ldots, I_t$ are parameter ideals and

(1) $\lambda_R(M) = \lambda_R(R/J) - \sum_{i=1}^t \lambda_R(R/I_i)$, and
(2) $\lim_{n\to\infty} \frac{\lambda_R(F^n(M))}{p^{5n}} = e_{HK}(J,R) - \sum_{i=1}^t \lambda_R(R/I_i)$.

Thus, for the ideal $J \subset R$, we have $e_{HK}(J,R) < \lambda_R(R/J)$, contradicting both parts of the conjecture above.

In this note all graded rings $R = \oplus_{i\geq 0} R_i$ are finitely generated over the Artinian local ring R_0. What we are able to prove, using basic properties of Poincaré series and a result of Avramov and Buchweitz [1], is the following:

Theorem 1.2. *Let R be a graded ring of characteristic $p > 0$ and dimension d, and I a homogeneous ideal with $\lambda(R/I) < \infty$ and $\operatorname{pd}(R/I) < \infty$. Then for every $q = p^e$, one has $\lambda(R/I^{[q]}) = q^d \lambda(R/I)$. In particular, $e_{HK}(I,R) = \lambda(R/I)$.*

Theorem 1.2 also follows from a conjecture of Szpiro [11, Conjecture C2]. Szpiro sketches a proof of this conjecture in the graded case, using different methods than what we employ here.

2. Proof of Theorem 1.2

Let R be a graded ring and M a nonzero finitely generated graded R-module. Let $P_M(t) = \sum_{i\in\mathbb{Z}} \lambda_{R_0}(M_i) t^i$ denote the Hilbert series for M. Note that if $\lambda_R(M) < \infty$, we have $\lambda_R(M) = P_M(1)$. Further note $P_{M(-k)}(t) = t^k P_M(t)$ for any $k \in \mathbb{Z}$, where if $M = \oplus_{i\in\mathbb{Z}} M_i$, $M(-k)$ denotes the graded R-module with $M(-k)_i = M_{i-k}$. We recall the following proposition concerning the Hilbert series for M.

Proposition 2.1. (cf. [10, Theorem 5.5]) *Let R be a graded Noetherian ring and M a nonzero finitely generated graded R-module of dimension l. Then there exist positive integers*

$s_1, \ldots, s_l$ and a polynomial $p(t) \in \mathbb{Z}\left[t, t^{-1}\right]$ with $p(1) \neq 0$ such that

$$P_M(t) = \frac{p(t)}{\prod_{i=1}^{l}(1 - t^{s_i})}.$$

The following result is a special case of a result in [1]. Since the proof is not difficult, we include it here for completeness.

Proposition 2.2. [1, Lemma 7] *Let M be a finitely generated graded R-module with finite length and finite projective dimension. Let $\mathfrak{m}$ be the maximal ideal of R_0 and let $K = R/(\mathfrak{m} + R_+) = R_0/\mathfrak{m}$. Set $\chi_M^R(t) := \sum_i (-1)^i P_{\operatorname{Tor}_i^R(M,K)}(t)$. Then one has $P_M(t) = \chi_M^R(t)\, P_R(t)$.*

Proof. Let $F_{\cdot} : 0 \to F_s \to \cdots \to F_1 \to F_0 \to 0$ be a minimal free resolution for M where $F_i \cong \oplus_{j=0}^{r_i} R(-j)^{b_{ij}}$ with $b_{ij} \in \mathbb{N}$. Then evaluating the Hilbert series for M using the resolution, one gets

$$P_M(t) = \sum_{i,j \in \mathbb{Z}} (-1)^i b_{ij} P_R(t)\, t^j. \tag{2.1}$$

Note this sum is well-defined as there are only finitely many nonzero b_{ij}. Moreover, $\chi_M^R(t)$ is also well-defined as $\operatorname{Tor}_i^R(M,K)$ is finitely generated graded and $\operatorname{Tor}_i^R(M,K) = 0$ for $i > \operatorname{pd}_R M$. Now, tensoring $F_{\cdot}$ with K, we obtain the complex

$$\cdots \to \oplus K(-j)^{b_{sj}} \to \cdots \to \oplus K(-j)^{b_{0j}} \to 0$$

where each map is the zero map. Note that for $n \in \mathbb{Z}$, $\operatorname{Tor}_i^R(M,K)_n = H_i(F_{\cdot} \otimes K)_n$, so $\dim_K \operatorname{Tor}_i^R(M,K)_n = \dim_K\left(\left(\oplus_{j\in\mathbb{Z}} K(-j)^{b_{ij}}\right)_n\right) = \sum_{j\in\mathbb{Z}} b_{ij} \dim_K K_{n-j} = b_{in} < \infty$ (as $K_i = 0$ for $i \neq 0$). Thus, $P_{\operatorname{Tor}_i^R(M,K)}(t) = \sum_{j\in\mathbb{Z}} b_{ij} t^j$.

Now, by the invariance of Euler-Poincare characteristics, we have

$$\begin{aligned}
\chi_M^R(t) &= \sum_{i\in\mathbb{Z}} (-1)^i P_{\operatorname{Tor}_i^R(M,K)}(t) \\
&= \sum_{i,j\in\mathbb{Z}} (-1)^i b_{ij} t^j \\
&= \frac{P_M(t)}{P_R(t)}. \qquad \text{(by equation (2.1))}
\end{aligned}$$

This gives $P_R(t)\,\chi_M^R(t) = P_M(t)$. Note also that we have $\chi_M^R(t) \in \mathbb{Z}[t, t^{-1}]$. □

Note that if we let F denote the Frobenius functor, then the Hilbert-Kunz multiplicity of an ideal is found by taking a limit of increasing iterations of the Frobenius of the R-module R/I, i.e. $e_{HK}(I,R) = \lim_{e\to\infty} F^e(R/I)/p^{ed}$. Now, if we can show $\lambda_R(F^e(R/I)) = p^{ed}\lambda_R(R/I)$ for all e sufficiently large, then $e_{HK}(I,R) = \lambda_R(R/I)$. We now prove the main theorem.

Theorem 2.3. *Let R be a graded ring of characteristic p and let M be a finitely generated graded R-module with $\lambda_R(M) < \infty$ and $\operatorname{pd}_R(M) < \infty$. Then $\lambda_R(F^e(M)) = q^d\lambda_R(M)$ for all $q = p^e$. In particular, one has $e_{HK}(I,R) = \lambda_R(R/I)$ for all zero-dimensional homogeneous ideals I of finite projective dimension.*

Proof. Let $G.$ be a minimal graded free resolution for M, with $G_i = \overset{r_i}{\underset{j=0}{\oplus}} R(-j)^{b_{ij}}$ and $b_{ij} \in \mathbb{N}$. Let $F^e(-)$ denote the Frobenius functor and $q = p^e$. Note that $L. = F^e(G.)$ is a minimal graded free resolution for $F^e(M)$ by [9, Theorem 1.7] where each twist by j in $G.$ is multiplied by a factor of q and the b_{ij} remain the same. That is, $L_i = \overset{r_i}{\underset{j=1}{\oplus}} R(-jq)^{b_{ij}}$. Now, by the lemma above, we have $P_M(t) = \chi_M^R(t)\,P_R(t)$ and $P_{F^e(M)}(t) = \chi_{F^e(M)}^R(t)\,P_R(t)$.

By Proposition 2.1, we can write $P_R(t) = \dfrac{p(t)}{\prod_{i=1}^d (1-t^{s_i})}$ where $d = \dim R$ and the $s_i \in \mathbb{N}$ are nonzero. Each term in the denominator can be factored as $1 - t^{s_i} = (1-t)g_i(t)$ with $g_i(t) \in \mathbb{Z}[t]$ and $g_i(1) \neq 0$. Letting $g(t) = \prod_i g_i(t)$, we can rewrite $P_R(t) = \dfrac{p(t)}{(1-t)^d g(t)}$ where $p(1)/g(1)$ is a nonzero rational number.

Since $\lambda_R(M) < \infty$, $P_M(t) \in \mathbb{Z}\left[t, t^{-1}\right]$. So, we have

$$P_M(t) = \chi_M^R(t)\,P_R(t) = \chi_M^R(t)\,\frac{p(t)}{(1-t)^d g(t)} \in \mathbb{Z}[t, t^{-1}].$$

Since $p(1) \neq 0$, we must have $(1-t)^d$ divides $\chi_M^R(t)$ in $\mathbb{Z}[t,t^{-1}]$; say $\chi_M^R(t) = \tilde{\chi}_M^R(t) \cdot (1-t)^d$, for some $\tilde{\chi}_M^R(t) \in \mathbb{Z}[t,t^{-1}]$. So,

$$P_M(t) = \tilde{\chi}_M^R(t)\,\frac{p(t)}{g(t)}. \tag{2.2}$$

In the proof of Proposition 2.2, we see that $\chi_M^R(t) = \sum_{i,j\in\mathbb{Z}} (-1)^i b_{ij} t^j$. Applying this to the resolution $L.$ of $F^e(M)$, we also have $\chi_{F^e(M)}^R(t) = \sum_{i,j\in\mathbb{Z}} (-1)^i b_{ij} t^{qj} = \chi_M^R(t^q) =$

$\tilde{\chi}_M^R(t^q) \cdot (1-t^q)^d$. Thus,

$$\begin{aligned} P_{F^e(M)}(t) &= \chi^R_{F^e(M)}(t)\,\frac{p(t)}{(1-t)^d\,g(t)} \\ &= \tilde{\chi}_M^R(t^q)\,(1-t^q)^d\,\frac{p(t)}{(1-t)^d\,g(t)} \\ &= \tilde{\chi}_M^R(t^q)\left(1+t+\cdots+t^{q-1}\right)^d\frac{p(t)}{g(t)}. \end{aligned}$$

Letting $t=1$, we get

$$\begin{aligned} \lambda_R(F^e(M)) &= P_{F^e(M)}(1) = \tilde{\chi}_M^R(1^q)\left(1+1+\cdots+1^{q-1}\right)^d\frac{p(1)}{g(1)} \\ &= \tilde{\chi}_M^R(1)\,(q)^d\,\frac{p(1)}{g(1)} \\ &= q^d P_M(1) \qquad \text{(by (2.2))} \\ &= q^d\lambda_R(M). \end{aligned}$$

Finally, if I is a zero-dimensional homogeneous ideal and $q=p^e$, we have $e_{HK}(I,R) = \lim_{q\to\infty}\lambda_R(F^e(R/I))/q^d = \lambda_R(R/I)$. □

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