

## 1. REGULAR LOCAL RINGS

Let  $(R, m)$  a Noetherian local ring and recall  $\text{embdim } R := \mu_R(m)$ . By Krull's Principal Ideal Theorem,  $\mu_R(m) \geq \dim R$ .

**Definition.**  $R$  is a **regular local ring (RLR)** if  $\mu_R(m) = \dim R$ .

**Proposition 1.1.** *RLRs are domains.*

*Proof.* Induct on  $\dim R$ . If  $\dim R = 0$ , then  $\mu_R(m) = 0$  implies  $m = (0)$  and so  $R$  is a field. So suppose  $\dim R > 0$  and let  $\text{Min}_R R = \{p_1, \dots, p_s\}$ . By Nakayama's Lemma,  $m \neq m^2$ . As  $\text{ht } m > 0$ , we know  $m \not\subseteq p_i$  for  $i = 1, \dots, s$ . Thus there exists  $x \in m \setminus (m^2 \cup p_1 \cup \dots \cup p_s)$  by prime avoidance. As  $x \in m \setminus m^2$ ,  $\text{embdim } R/(x) = \dim_{R/m} m/(m^2, x) = \dim_{R/m} m/m^2 - 1 = \mu_R(m) - 1$ . As  $R$  is local and  $x \notin p$  for all  $p \in \text{Min}_R R$ ,  $\dim R/(x) = \dim R - 1$ . So  $R/(x)$  is a RLR. By induction,  $R/(x)$  is a domain and so  $(x)$  is a prime ideal. As  $x \notin p$  for all  $p \in \text{Min}_R R$ ,  $\text{ht}(x) = 1$ . Let  $q \in \text{Min}_R R$  such that  $q \subseteq (x)$ . Note  $q = x(q : x)$  but  $x \in q$ , so  $(q : x) = q$  as  $q$  is prime. Therefore  $q = xq$ , that is,  $q = mq$ . By Nakayama's Lemma,  $q = 0$ , i.e.,  $0$  is a prime ideal. Therefore,  $R$  is a domain.  $\square$

The proof combined with the conclusion shows that for a RLR  $R$  and an element  $x \in m \setminus m^2$ ,  $R/(x)$  is a RLR. The converse is also true, for  $x \neq 0$ .

**Proposition 1.2.** *RLRs are Cohen Macaulay.*

*Proof.* Induct on  $\dim R$  to show  $\dim R = \text{depth } R$ . Since  $\text{depth } R \leq \dim R$ , the  $\dim R = 0$  case is trivial. So suppose  $\dim R > 0$ . Choose  $x \in m \setminus m^2$ . As  $R$  is a domain,  $x$  is a non-zero-divisor. Thus, by induction,  $\dim R - 1 = \dim R/(x) = \text{depth } R/(x) = \text{depth } R - 1$ .  $\square$

**Corollary 1.3.** *A local ring  $(R, m)$  is a RLR if and only if  $m$  is generated by a regular sequence.*

*Proof.* For the forward direction, say  $m = (x_1, \dots, x_d)$  where  $d = \dim R = \text{depth } R = \text{grade } m$ . By the last exercise,  $x_1, \dots, x_d$  is a regular sequence. For the backward direction, let  $m = (x_1, \dots, x_n)$  where  $x_1, \dots, x_n$  is a regular sequence. Then  $\dim R = \text{ht } m \leq n = \text{grade } m = \text{depth } R \leq \dim R$ . Thus  $\dim R = n \geq \mu_R(m) \geq \dim R$ .  $\square$

**Proposition 1.4.** *Let  $(R, m, k)$  be a RLR. Then*

- (1)  $\text{pd}_R k = \dim R$ .
- (2)  $\text{pd}_R M \leq \dim R$  for all finitely generated  $R$ -modules  $M$ .

Thus  $\dim R = \sup\{\text{pd}_R M \mid M \text{ is a finitely generated } R\text{-module}\}$ .

*Proof.* (1) Let  $m = (x_1, \dots, x_d)$  with  $d = \dim R$ . Then  $x_1, \dots, x_d$  is a regular sequence so  $K.(x_1, \dots, x_d; R)$  is exact except at degree 0 where  $H_0(K.(x_1, \dots, x_d; R)) = R/(x_1, \dots, x_d) = k$ . Further,  $K.(x_1, \dots, x_d; R)$  is a minimal free resolution of  $R/m$ . Thus  $\text{pd}_R k = d$ .

- (2) Let  $M$  be an arbitrary finitely generated  $R$ -module. Let  $F.$  be a minimal free resolution of  $M : \dots \rightarrow R^{\beta_n(M)} \rightarrow R^{\beta_{n-1}(M)} \rightarrow \dots \rightarrow R^{\beta_0(M)} \rightarrow M \rightarrow 0$ . Recall  $\beta_i(M) = \dim_k \text{Tor}_i^R(M, k)$ . Using  $\text{pd}_R k = d = \dim R$ , we see  $\text{Tor}_i^R(M, k) = 0$  for all  $i > d$ . Thus  $\beta_i(M) = 0$  for all  $i > d$  and so  $\text{pd}_R M \leq d$ .  $\square$

**Theorem 1.5** (Auslander, Buchsbaum, Serre '57).  *$(R, m)$  is a RLR if and only if  $\text{pd}_R k < \infty$ .*

When this theorem was proved, it lead the way to the proofs of the following then-unknown results:

- (1) If  $(R, m)$  is a RLR, then  $R_p$  is a RLR for all  $p \in \text{Spec } R$ .
- (2) RLRs are UFDs.

**Theorem 1.6.** *If  $(R, m)$  is a RLR, then  $R_p$  is a RLR for all  $p \in \text{Spec } R$ .*

*Proof.* As  $R/p$  is a finitely generated  $R$ -module,  $\text{pd}_R R/p \leq \infty$ . Let  $0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_0 \rightarrow R/p \rightarrow 0$  be a free resolution of  $R/p$ . Localize at  $p$  to get  $0 \rightarrow (F_n)_p \rightarrow \dots \rightarrow (F_0)_p \rightarrow R_p/pR_p \rightarrow 0$ , a free resolution of  $R_p/pR_p = k(p)$ . Thus  $\text{pd}_{R_p} k(p) < \infty$  and so  $R_p$  is a RLR.  $\square$

**Definition.** A local ring  $(R, m)$  is a **Complete Intersection** if  $R \cong S/(x_1, \dots, x_c)$  where  $S$  is a RLR and  $x_1, \dots, x_c$  is a regular sequence on  $S$ .

Since RLRs are CM, any complete intersection is CM.

**Definition.**  $(R, m)$  is **Gorenstein** if  $R$  has finite injective dimension, that is, there exists a complex  $I : 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \dots \rightarrow I^n \rightarrow 0$  where  $I^i$  is injective for all  $i$  and  $H^j(I) = \begin{cases} R & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$

Recall (we showed in Math 915) that  $\text{id}_R M \leq n$  if and only if  $\text{Ext}_R^i(N, M) = 0$  for all  $i > n$  and for all  $R$ -modules  $N$  which is if and only if (by Baer's Criterion)  $\text{Ext}_R^i(R/J, M) = 0$  for all  $i > n$  and for all ideals  $J$ .

**Proposition 1.7.** RLRs are Gorenstein.

*Proof.* We will show  $\text{id}_R R \leq \dim R$ . Let  $J$  be an ideal of  $R$  and consider  $\text{Ext}_R^i(R/J, R)$ . As  $\text{pd}_R R/J \leq \dim R$ , we see  $\text{Ext}_R^i(R/J, R) = 0$  for all  $i > \dim R$ . Therefore  $\text{id}_R R \leq \dim R$ . So  $R$  is Gorenstein.  $\square$

We now have the following inclusions of various types of rings:

$$\text{Local Rings} \supseteq \text{CM Rings} \supseteq \text{Gorenstein Rings} \supseteq \text{Complete Intersections} \supseteq \text{RLRs}$$

**1.1. Proving the Auslander-Buchsbaum-Serre Theorem.** We wish to prove Theorem 1.5, but in order to do so, we must recall some results from Homological Algebra.

**Proposition 1.8.** Let  $R$  be a commutative ring,  $M$  an  $R$ -module. TFAE

- (1)  $\text{pd}_R M \leq n$
- (2)  $\text{Ext}_R^{n+1}(M, N) = 0$  for all  $R$ -modules  $N$
- (3)  $\text{Ext}_R^i(M, N) = 0$  for all  $R$ -modules  $N$  and  $i > n$

The essence of that result comes from the fact that  $M$  is projective if and only if  $\text{Ext}_R^0(M, -) = \text{Hom}_R(M, -)$  is exact. Recall there is an analogous result for the injective dimension which comes from the fact that  $N$  is injective if and only if  $\text{Hom}_R(-, N)$  is exact.

**Proposition 1.9.** Let  $R$  be commutative,  $N$  an  $R$ -module. TFAE

- (1)  $\text{id}_R N \leq n$
- (2)  $\text{Ext}_R^{n+1}(M, N) = 0$  for all  $R$ -modules  $M$
- (3)  $\text{Ext}_R^{n+1}(R/I, N) = 0$  for all ideals  $I$
- (4)  $\text{Ext}_R^i(M, N) = 0$  for all  $i > n$  and  $R$ -modules  $M$

We have yet another analogous result but this time for the flat dimension.

**Proposition 1.10.** Let  $R$  be commutative,  $M$  an  $R$ -module. TFAE

- (1)  $\text{fd}_R M \leq n$
- (2)  $\text{Tor}_{n+1}^R(M, N) = 0$  for all  $R$ -modules  $N$
- (3)  $\text{Tor}_i^R(M, N) = 0$  for all  $R$ -modules  $N$  and  $i > n$

Recall that finitely presented flat modules are projective and in a Noetherian ring, finitely generated flats are projective. Thus the above proposition boils down to the following.

**Proposition 1.11.** Let  $R$  be Noetherian and  $M$  a finitely generated  $R$ -module. TFAE

- (1)  $\text{pd}_R M \leq n$
- (2)  $\text{Tor}_{n+1}^R(M, N) = 0$  for all  $R$ -modules  $N$
- (3)  $\text{Tor}_i^R(M, N) = 0$  for all  $R$ -modules  $N$  and  $i > n$

The proof relies on the fact that the flat dimension is the projective dimension in this case for when we consider  $0 \rightarrow K \rightarrow R_n \xrightarrow{\phi} M \rightarrow 0$  with  $K = \ker \phi$  we see  $K$  is also finitely generated as it is a submodule of  $R^n$ .

**Proposition 1.12.** *Let  $(R, m, k)$  be a Noetherian local ring,  $M$  a finitely generated  $R$ -module. TFAE*

- (1)  $\text{pd}_R M \leq n$
- (2)  $\text{Tor}_{n+1}^R(M, k) = 0$

**Fact.** If  $A, B$  are  $R$ -modules then  $\text{pd}_R A \oplus B = \max\{\text{pd}_R A, \text{pd}_R B\}$ . This comes from Proposition 1.8 and the fact that Ext commutes with direct sums.

**Lemma 1.13.** *Let  $(R, m, k)$  be local,  $M$  a finitely generated  $R$ -module,  $x \in m$  both  $R$ -regular and  $M$ -regular and suppose  $\text{pd}_R M < \infty$ . Then  $\text{pd}_{R/(x)} M/xM = \text{pd}_R M$ .*

*Proof.* Let  $F$  be a minimal free resolution of  $M$ . Say  $d = \text{pd}_R M$  which is the length of  $F$ . Consider the complex  $F \otimes_R R/(x)$ .

$$\begin{aligned} F : 0 \rightarrow R^{\beta_d} \rightarrow R^{\beta_{d-1}} \rightarrow \dots \rightarrow R^{\beta_0} \rightarrow 0 \text{ (acyclic with } H_0 = M) \\ F \otimes_R R/(x) : 0 \rightarrow (R/(x))^{\beta_d} \rightarrow (R/(x))^{\beta_{d-1}} \rightarrow \dots \rightarrow (R/(x))^{\beta_0} \rightarrow 0 \end{aligned}$$

Note that the entries of  $F \otimes_R R/(x)$  are still in the maximal ideal as they were in  $F$ . So if we can show  $F \otimes_R R/(x)$  is acyclic, it will be a minimal free resolution. Recall  $H_i(F \otimes_R R/(x)) = \text{Tor}_i^R(M, R/(x))$ . Now  $0 \rightarrow R \xrightarrow{x} R \rightarrow 0$  is a free resolution of  $R/(x)$  as  $x$  is  $R$ -regular. Applying  $M \otimes_R -$  gives us  $0 \rightarrow M \xrightarrow{x} M \rightarrow 0$ . So

$$\text{Tor}_i^R(M, R/(x)) = \begin{cases} 0 & \text{for } i > 2, \\ 0 = (0 :_M x) & \text{for } i = 1 \\ M/xM = M \otimes_R R/(x) & \text{for } i = 0 \end{cases}$$

Since  $H_i(F \otimes_R R/(x)) = \text{Tor}_i^R(M, R/(x))$ , we see  $F \otimes_R R/(x)$  is acyclic and is a minimal free resolution for  $M/xM$ . Thus  $\text{pd}_{R/(x)} M/xM = d = \text{pd}_R M$ .  $\square$

**Lemma 1.14.** *Let  $(R, m)$  be a local ring and  $x \in m \setminus m^2$ . Then  $m/xm \cong m/(x) \oplus (x)/xm$  as  $R/(x)$ -modules.*

*Proof.* Consider the short exact sequence  $0 \rightarrow (x)/xm \xrightarrow{i} m/xm \xrightarrow{j} m/(x) \rightarrow 0$  where  $i$  is the natural inclusion map. We want to show this map splits. Let  $n = \mu_R(m)$  and  $m = (x_1 := x, x_2, \dots, x_n)$ . Define  $\ell : m/xm \rightarrow (x)/xm$  by  $\ell(\overline{r_1 x_1 + \dots + r_n x_n}) = \overline{r_1 x_1}$ . Clearly this defines a homomorphism and  $\ell \oplus i = 1_{(x)/xm}$ . So its left only to show  $\ell$  is well-defined. So suppose  $\overline{r_1 x_1 + \dots + r_n x_n} = \overline{0}$  (as we can subtract, it is enough to show 0 maps to 0). Then  $r_1 x_1 + \dots + r_n x_n = u x_1$  where  $u \in m$ . So  $(r_1 - u)x_1 + r_2 x_2 + \dots + r_n x_n = 0$ . Note  $r_1 - u \in m$  else  $r_1 - u$  is a unit and  $x_1 \in (x_2, \dots, x_n)$  a contradiction. As  $u \in m$  this implies  $r_1 \in m$  and so  $\overline{r_1 x_1} = \overline{0}$ .  $\square$

**Theorem 1.15** (Theorem 1.5 restated). *Let  $(R, m, k)$  be local. TFAE*

- (1)  $R$  is regular
- (2)  $\text{pd}_R k < \infty$
- (3)  $\text{pd}_R M \leq \dim R$  for all finitely generated  $R$ -modules  $M$
- (4)  $\text{pd}_R M \leq \dim R$  for all  $R$ -modules  $M$

Moreover, if the above hold, then  $\dim R = \text{pd}_R k$ .

*Proof.* (1)  $\Rightarrow$  (2): Let  $m = (x_1, \dots, x_d)$  where  $d = \dim R$ . We have shown  $x_1, \dots, x_d$  is a regular sequence. Therefore, the Koszul complex  $K(x_1, \dots, x_d)$  is acyclic (that is, the homology is 0 except maybe in the 0<sup>th</sup> spot). Also,  $H_0(K(x_1, \dots, x_d)) = R/(x_1, \dots, x_d) = R/m = k$ . Thus the Koszul complex is a minimal free resolution of  $k$  (minimal because all of the entries are in the maximal ideal, as proved in an exercise in 915). Therefore,  $\text{pd}_R k = d = \dim R$ .

(2)  $\Rightarrow$  (3): From an above proposition  $\text{pd}_R M \leq \dim R$  if and only if  $\text{Tor}_{\dim R+1}^R(M, k) = 0$ . But  $\text{pd}_R k = \dim R$  and so  $\text{Tor}_{d+1}^R(M, k) = 0$ .

(3)  $\Rightarrow$  (4): Recall that  $\text{pd}_R M \leq d := \dim R$  for all  $R$ -modules  $M$  if and only if  $\text{Ext}_R^{d+1}(M, N) = 0$  for all  $R$ -modules  $M$  and  $N$  which is if and only if  $\text{Ext}_R^{d+1}(R/I, N) = 0$  for all  $R$ -modules  $N$  and ideals  $I$ . The last condition holds as  $\text{pd } R/I \leq d$  by (3) and thus  $\text{pd}_R M \leq \dim R$ .

(4)  $\Rightarrow$  (2): Clear.

(2)  $\Rightarrow$  (1): We use induction on  $\text{depth } R$ . For  $\text{depth } R = 0$ , the Auslander Buchsbaum Formula gives  $\text{pd}_R k + \text{depth } k = \text{depth } R = 0$ . So  $\text{pd}_R k = 0$ , which says  $k$  is a free  $R$ -module and thus  $m = (0)$  (by an 817-818 argument). Then  $R$  is a field and hence regular. Now suppose  $\text{depth } R > 0$  so that  $m \notin \text{Ass}_R R$ . Let  $\{p_1, \dots, p_\ell\} = \text{Ass}_R R$  and choose  $x \in m \setminus (m^2 \cup p_1 \cup \dots \cup p_\ell)$ . Then  $x$  is a non-zerodivisor on  $R$  and a minimal generator for  $m$ . As (2)  $\Rightarrow$  (3) we see  $\text{pd}_R m < \infty$ . Since  $x$  is  $R$  and  $m$ -regular, Lemma 1.13 tells us  $\text{pd}_{R/(x)} m/xm < \infty$ . By Lemma 1.14,  $\text{pd}_{R/(x)} m/xm = \sup\{\text{pd}_{R/(x)} m/(x), \text{pd}_{R/(x)}(x)/xm\}$ . So  $\text{pd}_{R/(x)}(x)/xm < \infty$ . As  $x$  is a non-zerodivisor on  $R$ ,  $(x)/xm \cong R/m$  (consider the map  $R/x \xrightarrow{x} (x)/xm$ ). Thus  $\text{pd}_{R/(x)} R/m = \text{pd}_{R/(x)}(R/(x))/(m/(x)) < \infty$ . Now  $\text{depth } R/(x) = \text{depth } R - 1$  and so by induction  $R/(x)$  is a RLR. As  $x$  is part of a minimal generating set for  $m$ ,  $R$  is also a RLR.  $\square$

**Theorem 1.16** (Rees). *Let  $R$  be a commutative ring,  $M, N$   $R$ -modules,  $x \in R$   $M$ -regular and  $R$ -regular such that  $xN = 0$ . Then*

- (1)  $\text{Hom}_R(N, M) = 0$
- (2) For all  $i \geq 1$   $\text{Ext}_R^i(N, M) \cong \text{Ext}_{R/(x)}^{i-1}(N, M/xM)$

*Proof.* Consider the exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$  and apply  $\text{Hom}_R(N, -)$ :

$$0 \rightarrow \text{Hom}_R(N, M) \xrightarrow{x} \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, M/xM) \rightarrow \text{Ext}_R^1(N, M) \xrightarrow{x} \text{Ext}_R^1(N, M) \rightarrow \dots$$

Since  $xN = 0$  we see  $x \text{Ext}_R^i(N, M) = 0$  for all  $i \geq 0$ . So multiplication by  $x$  is the zero map, which implies  $\text{Hom}_R(N, M) = 0$ . This also gives shortens our sequence above to

$$0 \rightarrow \text{Hom}_R(N, M/xM) \rightarrow \text{Ext}_R^1(N, M) \rightarrow 0$$

which implies  $\text{Ext}_R^i(N, M) \cong \text{Hom}_R(N, M/xM) \cong \text{Hom}_{R/(x)}(N, M/xM)$ . Thus we have proved (1) and the  $i = 1$  case of (2).

Now, let  $I : 0 \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$  be an injective resolution of  $M$ . So each  $I^i$  is injective and  $H^i(I) = \begin{cases} M & i = 0 \\ 0 & i > 0. \end{cases}$  Consider the cochain complex  $\text{Hom}_R(R/(x), I)$ :

$$E : 0 \rightarrow \underbrace{\text{Hom}_R(R/(x), I^0)}_{=: E^0} \rightarrow \underbrace{\text{Hom}_R(R/(x), I^1)}_{=: E^1} \rightarrow \underbrace{\text{Hom}_R(R/(x), I^2)}_{=: E^2} \rightarrow \dots$$

Then  $\text{Ext}_R^i(R/(x), M) = H^i(E)$ .

Take a projective resolution of  $R/(x) : 0 \rightarrow R \xrightarrow{x} R \rightarrow 0$ . Apply  $\text{Hom}_R(-, M)$  to get

$$0 \rightarrow \underbrace{\text{Hom}_R(R, M)}_{\cong M} \xrightarrow{x} \underbrace{\text{Hom}_R(R, M)}_{\cong M} \rightarrow 0.$$

So

$$H^i(E) = \text{Ext}_R^i(R/(x), M) = H^i(\text{above}) = \begin{cases} 0 & i \neq 1 \\ M/xM & i = 1 \end{cases}.$$

Consider  $E$  again as

$$0 \rightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \dots$$

Let  $K = \ker \partial^1$ . Then

$$(*)0 \rightarrow E^0 \rightarrow K \rightarrow H^1(E^\cdot) \rightarrow 0 \text{ and } (**)0 \rightarrow K \rightarrow E^1 \rightarrow E^2 \rightarrow \dots$$

are exact by definition (as  $H^1(E^\cdot) = \frac{\ker \partial^1}{\text{im } \partial^0}$ ). From 902, recall that since  $I^i$  is injective,  $E^i = \text{Hom}_R(R/(x), I^i)$  is an injective  $R/(x)$ -module. Since  $E^0$  is injective,  $(*)$  splits and so  $K \cong E^0 \oplus H^1(E^0) = E^0 \oplus M/xM$ . Now  $(**)$  is an injective resolution of  $K$  and so for  $i \geq 2$  we have

$$\begin{aligned} \text{Ext}_R^i(N, M) &= H^i(\text{Hom}_R(N, I^\cdot)) \\ &= H^i(\text{Hom}_R(N \otimes_{R/(x)} R/(x), I^\cdot)) \\ &= H^i(\text{Hom}_{R/(x)}(N, \text{Hom}_R(R/(x), I^\cdot))) \\ &= H^{i-1}(\text{Hom}_{R/(x)}(N, \tilde{E}^\cdot)) \text{ where } \tilde{E}^\cdot : 0 \rightarrow E^1 \rightarrow E^2 \dots \\ &= \text{Ext}_{R/(x)}^{i-1}(N, K) \text{ since } \tilde{E}^\cdot \text{ is an injective resolution of } K \\ &= \text{Ext}_{R/(x)}^{i-1}(N, E^0) \oplus \text{Ext}_{R/(x)}^{i-1}(N, M/xM) \\ &= \text{Ext}_{R/(x)}^{i-1}(N, M/xM) \end{aligned}$$

as an injective resolution for  $E^0$  is  $0 \rightarrow E \rightarrow 0$  and as  $i \geq 2$ . □

## 1.2. Serre's Condition, DVRs, and Normal Rings.

**Definition** (Serre's Condition). *Let  $R$  be a Noetherian ring,  $k \geq 0$  an integer.*

- (1)  *$R$  is said to satisfy  $S_k$  if  $\text{depth } R_p \geq \min\{\text{ht } p, k\}$  for all  $p \in \text{Spec } R$ .*
- (2)  *$R$  is said to satisfy  $R_k$  if  $R_p$  is a RLR for all  $p \in \text{Spec } R$  with  $\text{ht } p \leq k$ .*

Note that  $R$  is Cohen-Macaulay if and only if  $R$  is  $S_k$  for all  $k \geq 0$ . Similarly,  $(R, m)$  is a RLR if and only if  $R$  is  $R_k$  for all  $k \geq 0$ .

**Exercise.** A Noetherian ring is reduced if and only if  $R$  satisfies  $S_1$  and  $R_0$ .

*Proof.* For the forward direction, recall that in 902 we showed  $R$  reduced implies  $R_p$  is a field for all  $p \in \text{Min } R$  and hence  $R_p$  is a RLR for all  $p \in \text{Min } R$ . Thus  $R$  satisfies  $R_0$ . For  $S_1$ , let  $p \in \text{Spec } R$ . If  $\text{ht } p = 0$ , done. So assume  $\text{dim } R_p = \text{ht } p > 0$ . Then  $R_p$  is not a field which implies  $p \notin \text{Ass } R$  (recall for a Noetherian ring  $R$  that  $R$  is reduced if and only if  $R_p$  is a field for all  $p \in \text{Ass } R$ ). This says  $pR_p \notin \text{Ass } R_p$  and hence  $\text{depth } R_p > 0$  (as  $pR_p$  is the maximal ideal of  $R_p$ ). So  $\text{depth } R_p \geq \min\{\text{ht } p, 1\}$  for all  $p \in \text{Spec } R$ . Thus  $R$  satisfies  $S_1$ .

For the backward direction, note that  $S_1$  gives us that either  $\text{depth } R_p > 0$  or  $\text{depth } R_p = 0 = \text{ht } p$  for all  $p \in \text{Spec } R$ . Of course,  $\text{depth } R_p = 0$  if and only if  $pR_p \notin \text{Ass } R_p$  which is if and only if  $p \in \text{Ass } R$ . So for  $p \in \text{Spec } R$  with  $\text{depth } R_p > 0$ , we have  $p \notin \text{Ass } R$ . Thus for all  $p \in \text{Ass } R$ , we must have  $\text{depth } R_p = 0 = \text{ht } p$ , that is  $\text{Ass } R = \text{Min } R$ . By  $R_0$ , we know  $R_p$  is a RLR for all  $p \in \text{Min } R = \text{Ass } R$  and so  $R_p$  is a domain for all  $p \in \text{Ass } R$ . Since  $R_p$  is a domain of  $\text{dim } R_p = 0$  for all  $p \in \text{Ass } R$ , we have  $R_p$  is a field for all  $p \in \text{Ass } R$ . Thus  $R$  is reduced. □

**Definition.** A Noetherian ring is **normal** if  $R_p$  is an integrally closed domain for all  $p \in \text{Spec } R$  (equivalently,  $R$  is integrally closed in its total quotient ring).

**Exercise.** A Noetherian ring  $R$  is normal if and only if  $R$  satisfies  $S_2$  and  $R_1$ .

We will prove this after building a fair amount of machinery. First, we want to characterize one-dimensional RLRs (recall that zero-dimensional RLRs are just fields). Suppose  $R$  is a one-dimensional RLR and  $I \neq 0$  is an ideal of  $R$ .

*Claim.*  $I = (x^n)$  for some  $n \geq 0$ .

*Proof.* As  $R$  is a RLR, let  $m = (x)$ . By KIT, we know  $\cap m^n = (0)$ . Hence  $I \not\subseteq m^n$  for some  $n$ . Let  $n$  be the least such number. Then  $I \subset m^{n-1} = (x^{n-1})$ . So  $I = (x^{n-1})(I : x^{n-1})$ . If  $(I : x^{n-1}) \neq R$ , then  $(I : x^{n-1}) \subseteq (x) = m$ . Then  $I = (x^{n-1})(I : x^{n-1}) \subseteq (x^{n-1})(x) = m^n$ , a contradiction. Thus  $I = (x^{n-1})(I : x^{n-1}) = (x^{n-1})R = (x^{n-1})$ .

Hence a one-dimensional RLR is a PID.

**Corollary 1.17.** *Let  $(R, m)$  be a local ring of dimension 1. TFAE*

- (1)  $R$  is a RLR
- (2)  $m = (x)$
- (3) There exists  $x \in R$  such that every ideal  $I$  of  $R$  is  $(x^n)$  for some  $n$
- (4)  $R$  is a PID

Such rings satisfying any of the above conditions are called **discrete valuation rings (DVR)**.

**Remark.** Since PIDs are UFDs which are integrally closed, one-dimensional RLRs are integrally closed.

**Proposition 1.18** (9.2 in A&M). *Let  $(R, m)$  be a Noetherian local domain of dimension one. Then  $R$  integrally closed implies  $m$  is principal.*

*Proof.* If  $I \neq 0$ ,  $R$  is an ideal, then  $I$  is  $m$ -primary and  $I \supseteq m^n$  for some  $n$ . Let  $a \in m \setminus \{0\}$ . By above, there exists  $n$  such that  $m^n \subseteq (a)$  and  $m^{n-1} \not\subseteq (a)$ . Choose  $b \in m^{n-1}$  with  $b \notin (a)$  and let  $x = \frac{a}{b} \in K$ , the field of fractions. Note  $x^{-1} \notin R$  as  $b \notin (a)$ . So  $x^{-1}$  is not integral over  $R$  (as it is integrally closed). Recall that if  $S \supseteq R$  are rings, then  $u \in S$  is integral over  $R$  if and only if there exists a finitely generated  $R$ -module  $M \subseteq S$  such that  $uM \subseteq M$  and  $\text{Ann}_S M = 0$ . So  $x^{-1}$  not integral over  $R$  implies  $x^{-1}m \not\subseteq m$ . But  $x^{-1}m \subseteq R$  by construction (as  $b \in m^{n-1}$  and  $m^n \subseteq (a)$ ). So  $x^{-1}m = R$  which implies  $m = xR = (x)$ . So  $x$  is principal.  $\square$

Note that this shows that a one-dimensional Noetherian domain  $R$  is a RLR if and only if  $R$  is integrally closed.

**Exercise.** For a Noetherian ring  $R$ ,  $R$  is normal if and only if  $R$  satisfies  $S_2$  and  $R_1$ .

*Proof.* For the forward direction, suppose  $R$  is normal. Then  $R_p$  is an integrally closed domain for all  $p \in \text{Spec } R$ . To show  $R$  satisfies  $R_1$ , we want to show  $R_p$  is a RLR for all  $p \in \text{Spec } R$  with  $\text{ht } p \leq 1$ . So suppose  $p \in \text{Spec } R$  with  $\text{ht } p = 1$ . Then  $R_p$  is an integrally closed domain of  $\dim R_p = 1$ . By the proposition,  $m = (x)$  implies  $R$  is a RLR. If  $\text{ht } p = 0$ , then  $R_p$  is a zero-dimensional domain which implies it is a field and hence a RLR. To show  $S_2$ , we want to show  $\text{depth } R_p \geq \min\{\text{ht } p, 2\}$ . Suppose  $p \in \text{Spec } R$ . If  $\text{ht } p = 0$ , done. If  $\text{ht } p = 1$ , then  $R_p$  is a domain and so there exists a non-zero-divisor. So suppose  $\text{ht } p \geq 2$ . Then  $R_p$  is an integrally closed domain of  $\dim R_p \geq 2$ . Then there exist  $x \in P$  such that  $x$  is a non-zero-divisor in  $R_p$ .

*Claim.*  $pR_p \notin \text{Ass}(R_p/xR_p)$

*Proof.* If  $(R, m)$  is an integrally closed local domain of  $\dim R = 2$  and  $x \neq 0$  with  $m \in \text{Ass}(R/xR)$ , then  $m = ((x) : y)$  for some  $y \in R$ . This implies  $my \subseteq (x)$  and so  $m \frac{y}{x} \subseteq R$ . Suppose  $m \frac{y}{x} \subseteq m$ . Then  $\frac{y}{x}$  is integral over  $R$  which implies  $\frac{y}{x} \in R$  and  $y \in (x)$ , a contradiction as then  $m = R$ . If  $m \frac{y}{x} = R$ , then  $my = (x)$ . But  $m \cong my$  as  $y$  is a non-zero-divisor and hence  $m$  is principal, a contradiction to KPIT as  $\text{ht } m \geq 2$ .

Hence there exists  $y \in pR_p$  such that  $x, y$  is regular.

For the backward direction, suppose  $R$  satisfies  $S_2$  and  $R_1$ . Then  $R_p$  satisfies  $S_2$  and  $R_1$ . By the following proposition, we may assume  $R_p$  is a domain. We want to show  $R$  is integrally closed. Suppose  $\frac{a}{b} \in K$ , the total quotient field of  $R$ , is integral over  $R$ . Let  $p \in \text{Spec } R$  with  $\text{ht } p = 1$ . Then  $\frac{a}{b}$  is integral over  $R_p$ . Condition  $R_1$  says  $R_p$  is a RLR of dimension one and hence  $R$  is integrally closed. Thus  $\frac{a}{b} \in R_p$  for all  $p$  with  $\text{ht } p = 1$ . Now  $\frac{a}{b} \in R_p$  implies  $\frac{a}{b} = \frac{r}{s}$ . So there exists  $s \notin p$  such that  $sa \in (b)$  which is if and only if  $s \in ((b) : a)$ . Therefore  $((b) : a) \not\subseteq p$  with  $\text{ht } p = 1$ . Suppose  $((b) : a) \neq R$ . Let  $q \in \text{Spec } R$  be minimal over  $((b) : a)$ . Then  $\text{ht } q \geq 2$ . Localize at  $q$  so that  $qR_q$  is minimal over  $((b) :_{R_q} a)$ . As  $qR_q$  is maximal, we have  $q^n R_q \subseteq ((b) :_{R_q} a)$  for some  $n$  and hence  $q^n R_q a \subseteq (b)$  for some  $n$ . If  $a \in (b)R_q$ , then  $((b) :_{R_q} a) = R_q$ , a contradiction as  $qR_q$  is minimal over it. So  $q^n R_q$  consists of zero-divisors on  $R_q/(x)R_q$  which implies  $q^n R_q$  is contained in some associated prime of  $R_q/(b)R_q$ . Recall that in a local ring, the only prime containing  $m^n$  is  $m$  and hence  $qR_q \in \text{Ass}(R_q/(b)R_q)$ . So  $\text{depth } R_q/(b)R_q = 0$  and hence  $\text{depth } R_q = 1$  as  $b$  is a nonzerodivisor. This contradicts  $S_2$  and so  $((b) : a) = R$  which implies  $a \in (b)$  and so  $\frac{a}{b} \in R$ . Thus  $R$  is integrally closed.  $\square$

**Proposition 1.19.** *Suppose  $R$  is Noetherian satisfying  $S_2$  and  $R_1$ . Then  $R$  is a domain.*

*Proof.* Let  $W = \{\text{non-zero-divisors on } R\} = R \setminus \left( \bigcup_{p \in \text{Ass } R} p \right)$ . Now  $S_2$  implies  $S_1$  which implies  $\text{Min } R = \text{Ass } R$ . So

$W = R \setminus \bigcup_{p \in \text{Min } R} p$ . Let  $\{p_1, \dots, p_t\} = \text{Min } R$ . As  $R$  is reduced, it is enough to show  $t = 1$ . Note  $R \rightarrow R_W$  defined by  $r \mapsto \frac{r}{1}$  is injective. Now  $\text{Spec } R_W = \{(p_1)_W, \dots, (p_t)_W\} = \text{Min}_{R_W} R_W$ . Since  $p_1 \cap \dots \cap p_t = (0)$ ,  $(p_1)_W \cap \dots \cap (p_t)_W = 0$ . Also  $(p_i)_W + (p_j)_W = R_W$  for all  $i \neq j$  as the  $(p_i)_W$ 's are maximal. Thus by the Chinese Remainder Theorem, we have

$$\begin{aligned} R_W &\cong R_W / (p_1)_W \cap \dots \cap (p_t)_W \\ &\cong R_W / (p_1)_W \times \dots \times R_W / (p_t)_W \\ &\cong R_{p_1} / p_1 R_{p_1} \times \dots \times R_{p_t} / p_t R_{p_t} \text{ as } R \text{ is reduced, } R_p \text{ is a field for } p \in \text{Min } R \\ &\cong R_{p_1} \times \dots \times R_{p_t}. \end{aligned}$$

We wish to now show  $t = 1$ . Then,  $R_W \cong R_{p_1}$  and since  $R$  is a domain if and only if  $R_W$  is a field, we will be done. By the exercise,  $R$  is integrally closed in  $R_W$  (this did not use the assumption that  $R$  was a domain). So if  $\frac{r}{w} \in R_w$  is a nontrivial idempotent, then  $\left(\frac{r}{w}\right)^2 - \frac{r}{w} = 0$  and is integral over  $R$  (as it satisfies  $x^2 - x$ ). Then  $\frac{r}{w} \in R$ , that is, there is a nontrivial idempotent in  $R$ . Since  $R$  is local, this is a contradiction. Thus  $t = 1$  and  $R_W$  is a field. Therefore,  $R$  is a domain.  $\square$

**Proposition 1.20.** *Let  $R$  be Noetherian. Then TFAE*

- (1)  $R$  is  $S_2$  and  $R_1$
- (2)  $R_p$  is integrally closed for all  $p$
- (3)  $R_p$  is integrally closed in  $R_W$  for all  $p$ .

**Definition.** Let  $k$  be a field. An **affine  $k$ -algebra** is a ring of the form  $R = k[x_1, \dots, x_n]/I$  (that is, a finitely generated  $k$ -algebra). For  $I = (f_1, \dots, f_s)$ , the **Jacobian Matrix** of  $f_1, \dots, f_s$  is defined as  $J(\underline{f}) := \left( \frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq s, 1 \leq j \leq n}$ .

Note that the Jacobian matrix depends on the specific choices of  $f_i$  and  $x_j$ . However, the ideals of certain minors of  $J$  are invariant.

**Theorem 1.21.** *Suppose  $k$  is perfect. Let  $S = k[x_1, \dots, x_n]$  and  $I = (f_1, \dots, f_s) \subseteq S$ . Suppose  $\text{ht } p = \text{ht } I$  for all  $p \in \text{Min } S/I$ . Let  $c := \text{ht } I$  and  $I_c(J(\underline{f}))$  be the ideal of  $S$  generated by the  $c \times c$  minors of  $J(\underline{f})$ . Then*

- (1) *The image of  $I_c(J(\underline{f}))$  in  $S/I$  is dependent only on the isomorphism class of  $S/I$  (that is, if  $S/I \cong S'/I'$  then the images of the corresponding ideals are the same). This ideal is called the **Jacobian Matrix** of  $S/I$ . Denote this ideal by  $J(S/I)$ .*
- (2) *Let  $R = S/I$  with  $S, I$  as above. Let  $p \in \text{Spec } R$ . Then  $R_p$  is a RLR if and only if  $p \not\supseteq J(R)$  (that is, the regular locus of  $R$ ,  $\text{Reg } R = \{p \in \text{Spec } R \mid R_p \text{ is a RLR}\} = \text{Spec } R - V(J(R))$  is open).*

**Examples.** Let  $\text{char } k = 0$ . Let  $R = k[x, y, z]/(x^2 - yz)$  and  $S = k[x, y, z]/(x^3 - y^2z)$ . Then  $J(x^2 - yz) = (2x - z, -y)$ , which implies  $J(R) = (2\bar{x}, -\bar{z}, -\bar{y}) = (\bar{x}, \bar{y}, \bar{z})$ . Similarly,  $J(X) = (3\bar{x}^2, -2\bar{y}\bar{z}, -\bar{y}^2) = (\bar{x}^2, \bar{y}\bar{z}, \bar{y}^2)$ . Note that  $R$  and  $S$  are domains as we modded out by irreducible elements. Further  $\text{ht } p = \text{ht } I$  for all  $p \in \text{Min } R$  and for all  $p \in \text{Min } S$ . Both  $R$  and  $S$  are Cohen-Macaulay and hence  $S_2$ . By the above theorem,  $R_p$  is a RLR if and only if  $p \not\supseteq (\bar{x}, \bar{y}, \bar{z})$ . Since  $(\bar{x}, \bar{y}, \bar{z})$  is maximal in  $R$ ,  $R$  is  $R_1$  and hence normal. Note however that  $(\bar{x}, \bar{y}) \supseteq J(S)$  and has height 1 (it is height 2 in  $k[x, y, z]$  and we mod out by a non-zero-divisor to get  $S$ ). So  $S$  is not  $R_1$  which implies  $S$  is not normal.

## 2. INJECTIVE MODULES AND GORENSTEIN RINGS

Recall that for a commutative ring  $R$  and an  $R$ -module  $M$ ,

$$\text{id}_R M = \sup\{i \mid \text{Ext}_R^i(N, M) \neq 0 \text{ for some } R\text{-module } N\}.$$

**Corollary 2.1.** *Let  $R$  be commutative,  $M$  an  $R$ -module,  $x \in R$   $R$ -regular and  $M$ -regular. Then  $\text{id}_R(M/xM) \leq \text{id}_R M - 1$ .*

*Proof.* Let  $N$  be an  $R/(x)$ -module. Then for  $i + 1 > \text{id}_R M$

$$\text{Ext}_{R/(x)}^i(N, M/xM) \cong \text{Ext}_R^{i+1}(N, M) = 0. \quad \square$$

**Definition.** *A local Noetherian ring  $R$  is **Gorenstein** if  $\text{id}_R R < \infty$ .*

**Corollary 2.2.** *Suppose  $(R, m)$  is Gorenstein and  $x \in m$  is a non-zero-divisor. Then  $R/(x)$  is Gorenstein.*

*Proof.* Follows from corollary.  $\square$

Recall that for a commutative ring  $R$  and  $R$ -modules  $M, N$  with  $M$  finitely presented that  $\text{Hom}_R(M, N)_S \cong \text{Hom}_{R_S}(M_S, N_S)$  where  $S$  is a mcs. We want to know when we can do this localization with  $\text{Ext}$ , that is, what restrictions give us

$$\text{Ext}_R^i(M, N)_S \cong \text{Ext}_{R_S}^i(M_S, N_S).$$

It holds for all  $i \geq 0$  if there exists an exact sequence  $\cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$ . In particular, if  $R$  is Noetherian and  $M$  finitely generated, it holds. Thus if  $R$  is Noetherian and  $E$  an injective  $R$ -module, then  $E_S$  is an injective  $R_S$ -module for a multiplicatively closed set  $S$ .

*Proof.*  $E_S$  is an injective  $R_S$ -module if and only if for all ideals  $I_S$  of  $R_S$  we have  $\text{Ext}_R^1(R/I, E) \cong \text{Ext}_{R_S}^1(R_S/I_S) = 0$ . Of course,  $\text{Ext}_R^1(R/I, E)_S = 0$  as  $E$  is injective.  $\square$

**Corollary 2.3.** *If  $R$  is Noetherian,  $M$  an  $R$ -module, then  $\text{id}_{R_S} M_S \leq \text{id}_R M$ .*

*Proof.* Take an injective resolution of  $M$  and localize to get an injective resolution of  $M_S$ .  $\square$

**Corollary 2.4.** *If  $(R, m)$  is Gorenstein and  $p \in \text{Spec } R$ , then  $R_p$  is Gorenstein.*

**Corollary 2.5.** *Let  $(R, m)$  be a complete intersection, then  $R$  is Gorenstein.*

*Proof.* Follows from Corollary 2.2 and the fact that a RLR is Gorenstein.  $\square$

**Exercise.** Let  $(R, m)$  be local and suppose  $E$  is a nonzero finitely generated injective  $R$ -module. Prove that  $R$  is Artinian (i.e., that  $\dim R = 0$ ).

*Proof.* First we assume  $R$  is a domain. Then it is enough to show  $R$  is a field. Suppose  $m \neq 0$ . Then there exists  $x \in m \setminus \{0\}$ . Notice  $mE \subseteq E$ . If we show  $E \subseteq mE$ , NAK would imply  $E = 0$ , a contradiction. Let  $e \in E$ . Define  $\phi : R \rightarrow E$  via multiplication by  $e$ . By Baer's Criterion, there exists  $\psi : R \rightarrow E$  such that the following diagram commutes

$$\begin{array}{ccc} & E & \\ & \uparrow \phi & \swarrow \psi \\ 0 & \longrightarrow R & \xrightarrow{x} R \end{array}$$

Say  $\psi(1) = e'$ . Then we have  $e = \phi(1) = \psi x(1) = \psi(x) = x\psi(1) = xe'$ . Thus  $e \in mE$  which implies  $E = mE$ , a contradiction. So  $m = 0$  and thus  $R$  is a field.

For the general case, let  $p \in \text{Ass}_R E$  where  $p = (0 :_R e)$  for some  $e \in E \setminus \{0\}$ . Let  $E' = \text{Hom}_R(R/p, E)$ . As  $p \in \text{Ass}_R E$  we know  $E' \neq 0$ . Further  $E'$  is a finitely generated injective  $R/p$ -module. Since  $R/p$  is a domain, the domain case applies to give us  $R/p$  is a field and hence  $p = m$ . Thus  $\text{Ass}_R E = \{m\}$  and hence  $m = (0 :_R e)$ . If  $p \neq m$  for  $p \in \text{Spec } R$ , then  $pe = 0$ . So  $\text{Hom}_R(R/p, E) \neq 0$  (for consider  $R/p \rightarrow E$  defined by  $1 \mapsto e$ ). This again shows  $p = m$  and hence  $\dim R = 0$ .  $\square$

**2.1. Injective Modules and Essential Extensions.** For a family of  $R$ -modules  $\{E_i\}_{i \in I}$  it is easy to show  $\prod_{i \in I} E_i$  is injective if and only if  $E_i$  is injective for all  $i$ . This is also true for finite sums, but not, in general, for arbitrary direct sums.

**Proposition 2.6.** *Let  $\{E_i\}_{i \in I}$  be a family of injective  $R$ -modules. Assume  $R$  is Noetherian. Then  $\bigoplus_{i \in I} E_i$  is injective.*

*Proof.* We will use Baer's Criterion. Let  $I$  be an ideal of  $R$  and consider the following diagram

$$\begin{array}{ccc} & \bigoplus E_i & \\ & \uparrow \phi & \\ 0 & \longrightarrow I & \longrightarrow R \end{array}$$

Since  $I$  is finitely generated there exists  $E_{i_1}, \dots, E_{i_n} \in \{E_i\}$  such that  $im\phi \subseteq \bigoplus_{j=1}^n E_{i_j} \subseteq \bigoplus_{i \in I} E_i$ . As a finite direct sum of injective is injective, we may define  $f : R \rightarrow \bigoplus_{j=1}^n E_{i_j}$  making the outer square of the diagram commute.

$$\begin{array}{ccc} \bigoplus E_i & \xrightarrow{\pi} & \bigoplus_{j=1}^n E_{i_j} \\ \uparrow \phi & \xleftarrow{\ell} & \uparrow f \\ 0 & \longrightarrow I & \longrightarrow R \end{array}$$

Since  $\ell\pi|_{im\phi} = 1_{im\phi}$ , the diagram commutes. Further, the map  $\ell f$  makes  $\bigoplus_{i \in I} E_i$  injective.  $\square$

**Exercise.** If  $R$  is a commutative ring such that the direct sum of every family of injectives is injective, then  $R$  is Noetherian (Bass).

*Proof.* If  $R$  is not Noetherian, there exists an infinite ascending chain

$$(0) = I_0 \subsetneq I_1 \subsetneq \dots \subsetneq I_i \subsetneq \dots$$

Define  $J := \bigcup_{i=0}^{\infty} I_i$  and let  $E_i$  be an injective  $R$ -module containing  $R/I_i$ . Set  $G = \bigoplus_{i=1}^{\infty} E_i$ .

For each  $a \in J \setminus \{0\}$ , there exists a unique  $n \geq 1$  such that  $a \in I_{n+1} \setminus I_n$ . Hence  $a + I_{n+k} = \bar{0} \in E_{n+k}$  for all  $k \geq 1$ . Map  $\phi : J \rightarrow G$  by  $a \mapsto (a + I_1, a + I_2, \dots)$ . Since  $G$  is injective, there exists  $y \in G$  such that  $\phi(b) = yb$  for all  $b \in J$ . Since  $y \in \bigoplus_{i=1}^N E_i$  for some  $N$ , we have  $\phi(J) \subseteq \bigoplus_{i=1}^N E_i$ . But we know there exists  $c \in I_{N+2} \setminus I_{N+1}$  and thus the  $N+1$  coordinate of  $\phi(c)$  is  $c + I_{N+1} \neq 0 + I_{N+1}$ . Hence  $\phi(c) \notin \bigoplus_{i=1}^N E_i$ , a contradiction. Thus  $R$  is Noetherian.  $\square$

**Definition.** Let  $N \subseteq M$  be  $R$ -modules. We say  $M$  is an **essential extension** of  $N$  if for all  $x \in M \setminus \{0\}$  we have  $Rx \cap N \neq (0)$ .

**Example.** Let  $R$  be a domain,  $Q$  its field of fractions. Then  $R \subseteq Q$  is essential.

Note that if  $L \subseteq M \subseteq N$ , then  $L \subseteq N$  is essential if and only if  $L \subseteq M$  and  $M \subseteq N$  are essential.

**Proposition 2.7.** *Let  $N$  be an  $R$ -module. Then  $N$  is injective if and only if  $N$  has no proper essential extensions.*

*Proof.* For the forward direction, suppose  $N \subseteq M$  is essential. As  $N$  is injective,  $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$  splits. Thus  $M \cong N \oplus M/N$ . Therefore  $Rx \cap N = (0)$  for any  $x \in M/N \subseteq M$ . So  $x = 0$  for all  $x \in M/N$ , which implies  $M = N$ .

For the backward direction, suppose there exists an injective  $R$ -modules  $E$  such that  $N \subseteq E$ . Let  $\Gamma = \{K \subseteq E \mid K \cap N = (0)\}$ . Zornify to get  $K \subseteq \Gamma$  maximal. Consider the map  $0 \rightarrow N \xrightarrow{f} E/K$  defined by  $n \mapsto n + K$ . If we consider  $N \subseteq E/K$ , then this is an essential extension by maximality of  $K$  (suppose  $e + K \neq 0$  in  $E/K$  and  $R(e + K) \cap N = (0)$ , then  $K' = K + Re$  has trivial intersection with  $N$  and  $K' \supseteq K$ ). By hypothesis, we must have  $0 \rightarrow N \xrightarrow{f} E/K$  bijective and hence  $E = N + K$ . But  $K \cap N = (0)$  and so  $E = N \oplus K$ . Since direct summands of injectives are injective,  $N$  is injective.  $\square$

**Theorem 2.8.** Let  $M \subseteq E$  be  $R$ -modules. TFAE

- (1)  $E$  is a maximal essential extension of  $M$ .
- (2)  $E$  is a minimal injective containing  $M$ .
- (3)  $E$  is injective and essential over  $M$ .

Moreover, such an  $E$  exists and is unique up to isomorphism. Call  $E$  the **injective hull** or **injective envelope** of  $M$ , denoted  $E_R(M)$ .

*Proof.* (1)  $\Rightarrow$  (3): Let  $N \supseteq E$  where  $N$  is essential over  $E$ . Then  $N$  is essential over  $M$  which implies  $N = E$ .

Thus  $E$  has no proper essential extensions, which implies  $E$  is injective.

(3)  $\Rightarrow$  (2): Let  $E'$  be injective such that  $M \subseteq E' \subseteq E$ . As  $E/M$  is essential, so is  $E/E'$ . But  $E'$  is injective and so there does not exist any proper extension. Thus  $E = E'$ .

(2)  $\Rightarrow$  (1): By Zorn's Lemma let  $N$  be a maximal essential extension of  $M$  contained in  $E$ .

*Claim.*  $N$  is injective.

*Proof.* Suppose  $N \subseteq K$  where  $K/N$  is essential. Consider the diagram

$$\begin{array}{ccc} & E & \\ & \uparrow \psi & \nearrow \phi \\ 0 & \longrightarrow N & \xrightarrow{ess} K \end{array}$$

As  $E$  is injective, there exists  $\phi : K \rightarrow E$  making the diagram commute. Note that  $\phi$  is one-to-one as  $\psi$  is one-to-one and  $N \subseteq K$  is essential (if  $x \in \ker \phi \setminus \{0\}$ , then  $Rx \cap N \neq (0)$ . Say  $rx \in Rx \cap N \setminus \{0\}$ . So  $\psi(rx) = \phi(rx) = r\phi(x) = 0$ . So  $rx \in \ker \psi = \{0\}$  a contradiction). So  $K \cong K' \subseteq E$  with  $N \subseteq K' \subseteq E$  and  $K'$  essential over  $M$ . By maximality, we have  $K' = N$  and hence  $K \cong N$ . Therefore  $N$  has no proper essential extensions and is injective.

So  $E = N$  which implies  $E$  is essential over  $M$ . As  $E$  is injective,  $E$  is a maximal essential extension of  $M$ .

For existence, we know there exists an injective  $R$ -module  $E \supseteq M$ . Let  $E'$  be a maximal essential extension of  $M$  contained in  $E$  (by Zorn). By the proof of (2)  $\Rightarrow$  (1), we know  $E'$  is injective and essential over  $M$ .

For uniqueness, suppose  $E$  and  $E'$  satisfy conditions (1) – (3) for  $M$ . Then we have the diagram

$$\begin{array}{ccc} & E & \\ & \uparrow 1-1 & \nearrow \\ 0 & \longrightarrow M & \xrightarrow{ess} E' \end{array}$$

As before, we get the map from  $E' \rightarrow E$  is injective and hence  $M \subseteq E' \subseteq E$ . As  $E$  is a minimal injective by (2), we see  $E \cong E'$ . □

In practice, we let  $I$  be an injective containing  $M$ . Take a maximal essential extension of  $M$  in  $I$  to get  $M \subseteq E_R(M) \subseteq I$ . Then the map  $0 \rightarrow E_R(M) \rightarrow I$  splits and so  $I = E_R(M) \oplus I'$ .

**Exercises.**

- (1) Let  $R$  be Noetherian,  $M$  an  $R$ -module, and  $S$  a mcs. Then  $E_R(M)_S \cong E_{R_S}(M_S)$ .

*Proof.* We will show  $E_R(M)_S$  is the injective hull of  $M_S$  as an  $R_S$ -module. As  $E_R(M)$  is injective, we know  $E_R(M)_S$  is as localization preserves injective modules over a Noetherian ring. Thus it is enough to show  $E_R(M)_S$  is essential over  $M_S$ . Let  $\frac{x}{s} \in E_R(M)_S \setminus \{0\}$ . Since  $\frac{x}{s} \neq 0$ ,  $\text{Ann}(x) \cap S = \emptyset$ . Let  $I = (M :_R x) \neq 0$  (since  $x \in E_R(M)$  implies  $xR \cap M \neq 0$ ). Let  $N = Rx \cap M = Ix$ .

We want to show  $N_S \neq 0$ . Note  $N_S = (Ix)_S = I_S \frac{x}{1}$ . If  $r \in R$ , then  $I_S \frac{rx}{s} \neq 0$  implies  $I_S \frac{x}{s} \neq 0$ . As  $x \in E_R(M)$ , we know  $Rx \neq 0$ . Now  $(Rx)_S = R_S \frac{x}{s} \neq 0$  implies it has an associated prime. Let  $p \in \text{Ass}_R(Rx)$  such that  $p \cap S = \emptyset$ . Then  $p = (0 : rx)$ . Replace  $rx$  with  $x$  and assume  $p = \text{Ann } x$  is prime with  $p \cap S = \emptyset$ .

Say  $I = (a_1, \dots, a_n)$ . Then  $Ix = Ra_1x + \dots + Ra_nx$ . Now  $N_S = 0$  if and only if there exists  $t \in S$  such that  $tIx = 0$  which is if and only if  $tI \subseteq \text{Ann}(x) = p$ . Thus if  $N_S = 0$ , then  $tI \subseteq p$  which implies  $I \subseteq \text{Ann } x$  as  $t \in S$  does not intersect  $p$ . So  $Ix = 0$ , a contradiction as  $Rx \cap M \neq 0$ . So  $N_S \neq 0$  and so  $R_S \frac{x}{s} \cap M_S \neq 0$ . Thus  $E_R(M)_S$  is an essential injective module over  $M_S$ .  $\square$

- (2) Let  $R$  be Noetherian,  $M$  an  $R$ -module. Then  $\text{Ass}_R(M) = \text{Ass}_R E_R(M)$ . [We do not assume  $M$  is finitely generated. For an arbitrary  $R$ -module  $N$ , we say  $p \in \text{Ass}_R N$  if and only if  $p \in \text{Spec } R$  and  $p = (0 :_R x)$  for some  $x \in N \setminus \{0\}$ ].

*Proof.* The forward containment is trivial as  $M \subseteq E_R(M)$ . To show  $\text{Ass}_R(M) \supseteq \text{Ass}_R E_R(M)$ , let  $p \in \text{Ass}_R E_R(M)$ . Then  $R/p \hookrightarrow E_R(M)$ . Since  $E_R(M)$  is essential over  $M$ , we know  $R/p \cap M \neq 0$ . Let  $x \in R/p \cap M$ . We will show  $p = (0 :_R x)$ . Let  $s \in R$  with  $sx = 0$ . Then  $\bar{s}x = 0$  in  $R/p$ , which is a domain, thus  $\bar{s} = 0$ . Thus  $s \in p$  and  $p = (0 :_R x)$ . Therefore,  $p \in \text{Ass}_R M$ .  $\square$

The second exercise is in fact true for any essential extension of the module  $M$ . We did not use the fact that  $E_R(M)$  was injective anywhere in the proof. Furthermore, the exercise gives us the following result.

**Corollary 2.9.** *Let  $R$  be Noetherian,  $p, q \in \text{Spec } R$ . Then  $E_R(R/p) \cong E_R(R/q)$  if and only if  $p = q$ .*

*Proof.*  $\{p\} = \text{Ass}_R R/p = \text{Ass}_R E_R(R/p) = \text{Ass}_R E_R(R/q) = \text{Ass}_R R/q = \{q\}$ .  $\square$

**Definition.** *Let  $R$  be a ring,  $M$  an  $R$ -module. Say  $M$  is a **divisible**  $R$ -module if for all  $x \in R$  a non-zerodivisor and for all  $m \in M$  there exists  $m'$  such that  $xm' = m$ .*

**Facts.** (Proven in 902)

- (1) Injective Modules are divisible.
- (2) If  $R$  is a domain,  $M$  torsion-free and divisible, then  $M$  is injective.
- (3) If  $R$  is a PID then every divisible module is injective.

**Examples.**

- (1) Let  $R$  be a domain,  $Q$  its field of fractions. Then  $Q$  is torsion free and divisible, hence injective. Also  $Q$  is an essential extension of  $R$ . Thus  $Q$  is the injective hull of  $R$ .
- (2) Let  $R = k[x]$  for  $k$  a field. Then  $K \cong k[x]/(x)$  is an  $R$ -module. What is  $E_R(K)$ ? Let  $R = \bigoplus_1^\infty kx^i$  as a  $k$ -vector space. Give  $F$  an  $R$ -module structure via the action  $x^n \cdot x^{-i} = 0$  if  $n - i < 0$  and  $x^n \cdot x^{-i} = 0$  if  $n - i \geq 0$ . Identify  $K$  with  $kx^{-1} \subseteq F$  (can do so by defining  $\phi : R \rightarrow F$  by  $1 \mapsto x^{-1}$ . Here  $(x) = \ker \phi$  and so  $K \cong kx^{-1}$ )

*Claim.*  $F \cong E_R(K)$ .

*Proof.* Note  $kx^{-1} \subseteq F$  is essential. Let  $\alpha = \sum_1^N x_i x^{-1}, c_N \neq 0$ . Then  $x^{N-1} \alpha = c_N x^{-1} \neq 0 \in kx^{-1}$ . So  $R\alpha \cap kx^{-1} \neq (0)$ . It is enough to show  $F$  is injective. Since  $R = k[x]$  is a PID, we can show  $F$  is divisible. Let  $p(x) \in k[x] \setminus \{0\}$  and  $u = \sum_1^N c_i x^{-i} \in F$ . Write  $p(x) = x^n q(x)$  where  $q(0) \neq 0$ . Consider  $q(x) \in k[x]$  and so there exists  $\alpha_i \in k$  for  $i = 0, 1, \dots$  such that  $q(x) \left( \sum_0^\infty \alpha_i x^i \right) = 1$ . Let  $\ell(x) = \sum_{i=0}^{n+N-1} \alpha_i x^i \in k[x]$ . Then  $q(x)\ell(x) \cong 1 \pmod{x^{n+N}}$ , that is  $q(x)\ell(x) - 1 = s(x)x^{n+N}$  for some  $s(x) \in k[x]$ . But  $x^{n+N}x^{-n}u = 0$  (as all the exponents are  $\geq 0$ ). Thus  $(q(x)\ell(x) - 1)(x^{-n}u) = 0$  and so  $x^{-n}uq(x)\ell(x) = x^{-n}u$ , that is,  $\underbrace{p(x)\ell(x)x^{-n}u}_{\in F} = u$ . So  $F$  is divisible and hence injective.

**Remark.** If  $R = k[x_1, \dots, x_d]$  for  $k$  a field, then  $E_R(k) = \bigoplus_{(i_1, \dots, i_d) \in \mathbb{N}^d} kx_1^{-i_1} \cdots x_d^{-i_d}$  where  $(x_1^{j_1} \cdots x_d^{j_d})(x_1^{-i_1} \cdots x_d^{-i_d}) = 0$  if  $j_k - i_k \geq 0$  for some  $k$  and  $= x_1^{j_1 - i_1} \cdots x_d^{j_d - i_d}$  otherwise.

**Proposition 2.10.** *Let  $R$  be Noetherian,  $E \neq 0$  an injective  $R$ -module. Then  $E$  is indecomposable if and only if  $E \cong E_R(R/p)$  for some  $p \in \text{Spec } R$ .*

*Proof.* For the backward direction, suppose  $E := E_R(R/p) = M_1 \oplus M_2$  for  $M_1, M_2 \neq 0$ . Now  $E$  is essential over  $R/p$  and hence  $J_1 := M_1 \cap R/p \neq (0)$  and  $J_2 := M_2 \cap R/p \neq (0)$ . So  $J_1, J_2$  are nonzero ideals of  $R/p$ . Note that  $J_1 \cap J_2 \subseteq M_1 \cap M_2 = (0)$  as  $E = M_1 \oplus M_2$ . This contradicts the fact that  $R$  is a domain (as if  $i \in J_1, j \in J_2$ , then  $ij \in J_1 \cap J_2 \setminus \{0\}$ ).

For the forward direction, let  $p \in \text{Ass}_R E$ . Then  $R/p \hookrightarrow E$ . Consider the diagram

$$\begin{array}{ccc} & E & \\ & \uparrow & \swarrow \\ 0 & \longrightarrow R/p & \xrightarrow{\text{ess}} E_R(R/p) \end{array}$$

By injectivity, there exists a map  $E_R(R/p) \rightarrow E$  and moreover it is one-to-one. We get  $R/p \subseteq E_R(R/p) \subseteq E$ . So  $E = E_R(R/p) \oplus I$  for some  $I$ . As  $E$  is indecomposable,  $I = 0$ . Thus  $E = E_R(R/p)$ .  $\square$

**Theorem 2.11.** *Let  $R$  be Noetherian. Then every nonzero injective module is a direct sum of indecomposable injectives.*

*Proof.* Let  $E \neq 0$  be injective. Let  $\Lambda = \{\{E_\lambda\}_{\lambda \in J} \mid E_\lambda \subseteq E \text{ are indecomposable injectives with } \sum E_\lambda = \oplus E_\lambda\}$ . Note  $\Lambda \neq \emptyset$  as  $\emptyset \in \Lambda$ . Zornify to get a maximal element  $S = \{E_\lambda\}_{\lambda \in J}$ . Let  $N = \sum_{\lambda \in J} E_\lambda = \oplus_{\lambda \in J} E_\lambda$ . This is injective as  $R$  is Noetherian.

We wish to show  $E = N$ . We have the short exact sequence  $0 \rightarrow N \rightarrow E \rightarrow E/N \rightarrow 0$ . As  $N$  is injective, the sequence splits. So  $E = N \oplus E/N$ , which implies  $E/N$  is injective as it is a direct summand of an injective module. If  $E/N \neq 0$ , it contains an indecomposable injective module  $E'$ . Then  $E'$  (as a submodule of  $E$ ) has trivial intersection with  $N$ . Thus  $S \cap \{E'\} \in \Lambda$ , a contradiction to maximality. Thus  $E/N = 0$  and  $E = N$ .  $\square$

Combining the above two results, we see  $E \cong \bigoplus_{p \in \text{Spec } R} E_R(R/p)^{\mu(p, E)}$  for any injective  $R$ -module  $E$  where  $\mu(p, E)$  is some cardinal number.

**Remarks.**

- (1) Let  $R$  be Noetherian. If  $M \subseteq N$  is essential, then  $M_S \subseteq N_S$  is essential as  $R_S$ -modules. [Follows from proof of exercise which shows you can localize injective hulls.]
- (2) If  $M \subseteq N$  is essential and  $I$  an ideal, then  $(0 :_M I) = \text{Hom}_R(R/I, M) \subseteq \text{Hom}_R(R/I, N) = (0 :_N I)$  is essential as  $R/I$ -modules.
- (3)  $E_{R/I}(\text{Hom}_R(R/I, M)) = \text{Hom}_R(R/I, E_R(M))$ .

*Proof.* Want to show the right hand side is an injective hull of  $\text{Hom}_R(R/I, M)$  as an  $R/I$ -module. We know  $M \subseteq E_R(M)$  is essential. By Remark 2,  $\text{Hom}_R(R/I, M) \subseteq \text{Hom}_R(R/I, E_R(M))$  and the right hand side is an injective  $R/I$ -module.  $\square$

- (4) If  $IM = 0$ , then  $E_{R/I}M = \text{Hom}_R(R/I, E_R(M))$ . [Follows from Remark 3.]

- (5) Let  $p, q \in \text{Spec } R$ . Then  $\text{Hom}_R(R/p, E_R(R/q)) = \begin{cases} E_{R/p}(R/q) & \text{if } p \subseteq q \\ 0 & \text{otherwise} \end{cases}$

*Proof.* If  $p \subseteq q$ , then  $p \cdot R/q = 0$ . Apply remark 4. If  $p \not\subseteq q$ , let  $e \in E_R(R/q) \setminus \{0\}$ . Then  $\text{Ass}_R Re = \{q\}$  (by previous exercise). Hence if  $re = 0$ , then  $r \in q$ . In particular,  $\text{Hom}_R(R/p, E) = (0 :_{E_R(R/q)} p) = 0$  as  $p \not\subseteq q$ .  $\square$

- (6) Let  $p, q \in \text{Spec } R$ . Then  $\text{Hom}_{R_p}(k(p), E_R(R/q)_p) = \begin{cases} k(p) & \text{if } q = p \\ 0 & \text{otherwise} \end{cases}$ , where  $k(p) = R_p/pR_p$ , the residue field of  $R/p$ .

*Proof.* As  $R/p$  is finitely presented

$$\begin{aligned} \text{Hom}_{R_p}(k(p), E_R(R/q)_p) &= \text{Hom}_R(R/p, E_R(R/q))_p \\ &= \begin{cases} E_{R/p}(R/q)_p & \text{if } p \subseteq q \\ 0 & \text{if } p \not\subseteq q \end{cases} \text{ by Remark 5} \\ &= \begin{cases} E_{k(p)}(k(p)) & \text{if } p \subseteq q \\ 0 & \text{otherwise} \end{cases} \text{ by previous Exercise.} \end{aligned}$$

Now  $k(p)$  is its own injective hull as it is a field. Thus  $E_{k(p)}(k(p)) = k(p)$ .  $\square$

(7) Let  $E = \bigoplus_{p \in \text{Spec } R} E(R/p)^{\mu(p,E)}$  be injective. Let  $p \in \text{Spec } R$ . Then

$$\begin{aligned} \text{Hom}_{R_p}(k(p), E) &= \text{Hom}_{R_p}(k(p), \bigoplus_q E_R(R/q)_p^{\mu(q,E)}) \\ &= \prod_q \text{Hom}_{R_p}(k(p), E_R(R/q)_p^{\mu(q,E)}) \\ &\quad \text{where the summands are 0 unless } p = q \\ &= \text{Hom}_{R_p}(k(p), E_R(R/p)_p)^{\mu(p,E)} \\ &= k(p)^{\mu(p,E)}. \end{aligned}$$

Thus, if  $\mu(p,E) < \infty$ , then  $\mu(p,E) = \dim_{k(p)} \text{Hom}_{R_p}(k(p), E)$ . [Note that this does not depend on the decomposition of  $E$ .]

**Theorem 2.12.** *Let  $R$  be Noetherian. Let  $i : M \hookrightarrow E$  where  $E$  is injective. Then  $E = E_R(M)$  if and only if for all  $p \in \text{Spec } R$  the map  $\hat{i}_p : \text{Hom}_{R_p}(k(p), M_p) \hookrightarrow \text{Hom}_{R_p}(k(p), E_p)$  is an isomorphism.*

*Proof.* For the forward direction, note  $\hat{i}_p$  is injective as we have  $i$  was injective and injectivity is preserved under localizations and applying  $\text{Hom}_{R_p}(k(p), -)$ . Now,  $M \subseteq E$  essential implies  $M_p \subseteq E_p$  is essential as  $R_p$ -modules. Then  $\text{Hom}_{R_p}(k(p), M_p) \hookrightarrow \text{Hom}_{R_p}(k(p), E_p)$  is essential as  $k(p)$ -modules. Since  $k(p)$  is a field, these are essential as vector space and so  $\hat{i}_p$  is an isomorphism.

For the backward direction, it is enough to show  $M \subseteq E$  is essential. Let  $x \in E \setminus \{0\}$  and  $p \in \text{Ass}_R Rx$ . Then  $R/p \hookrightarrow Rx$  via  $\bar{1} \mapsto rx =: y$ . Now it is enough to show  $Ry \cap M \neq (0)$ . Let  $\psi : R/p \rightarrow E$  be defined by  $\bar{1} \mapsto y$ . This is injective. Localize at  $p$  to get  $\frac{\psi}{1} : k(p) \rightarrow E_p$  defined by  $\bar{1} \mapsto \frac{y}{1}$ . This is also injective. We have the following commutative diagram.

$$\begin{array}{ccc} k(p) & \xrightarrow{\phi} & M_p \\ \downarrow \frac{\psi}{1} & \swarrow i_p & \\ E_p & & \end{array}$$

There exists a map  $\phi : k(p) \rightarrow M_p$  such that  $i_p(\phi) = \frac{\psi}{1}$  as  $\hat{i}_p$  is surjective. Say  $\phi(\bar{1}) = \frac{m}{s} \in M_p$ . Then  $\frac{m}{s} = \frac{y}{1}$  in  $E_p$  by commutativity of the diagram. So there exists  $t \notin p$  such that  $tsy = tm$ . So  $tm \in Ry \cap M$ . If  $tm = 0$ , then  $\frac{m}{s} = 0$  and so  $\frac{y}{1} = 0$ , a contradiction as  $\frac{\psi}{1}$  is injective. Thus  $Ry \cap M \neq (0)$ .  $\square$

**Corollary 2.13.** *Let  $M$  be a finitely generated  $R$ -module. Then  $E_r(M) = \bigoplus_{p \in \text{Ass}_R M} E_R(R/p)^{\mu(p,M)}$  where  $\mu(p,M) = \dim_{k(p)} \text{Hom}_{R_p}(k(p), M_p) < \infty$ .*

*Proof.* Let  $E = E_R(M)$ . Then  $E = \bigoplus_{p \in \text{Ass}_R M} E_R(R/p)^{\mu(p,E)}$ . By the theorem and remark 7, for all  $p \in \text{Spec } R$ , we have  $\text{Hom}_{R_p}(k(p), M_p) \cong \text{Hom}_{R_p}(k(p), E_p) \cong k(p)^{\mu(p,E)}$ . Take the dimension of both sides over  $k(p)$ .  $\square$

**Lemma 2.14.** *Let  $R$  be Noetherian,  $M$  an  $R$ -module,  $N$  a finitely generated  $R$ -module and  $n \in N$ . Suppose  $\text{Ext}_R^n(R/p, M) = 0$  for all  $p \in \text{Supp } N$ . Then  $\text{Ext}_R^n(N, M) = 0$ .*

*Proof.* Consider a prime filtration for  $N : N = N_t \supseteq N_{t-1} \supseteq \dots \supseteq N_1 \supseteq N_0 = (0)$  where  $N_{i+1}/N_i \cong R/p_i$  where  $p_i \in \text{Supp } N$  for all  $i = 0, \dots, t-1$ . Induct on  $t$ . For  $t = 1$ , we have  $N \cong R/p$  for some  $p \in \text{Supp } N$  which implies

$\text{Ext}_R^n(N, M) = 0$  by assumption. For  $t > 1$ , we have  $0 \rightarrow N_{t-1} \rightarrow N_t \rightarrow R/p_{t-1} \rightarrow 0$ . Apply  $\text{Hom}_R(-, M)$  to get

$$\cdots \rightarrow \text{Ext}_R^n(R/p_{t-1}, M) \rightarrow \text{Ext}_R^n(N_t, M) \rightarrow \text{Ext}_R^n(N_{t-1}, M) \rightarrow \cdots$$

where the first and last terms are 0 by induction and the  $t = 1$  case. Thus  $\text{Ext}_R^n(N_t, M) = 0$ .  $\square$

**Corollary 2.15.** *Let  $R$  be Noetherian,  $M$  an  $R$ -module. TFAE*

- (1)  $\text{id}_R M \leq n$ .
- (2)  $\text{Ext}_R^{n+1}(R/p, M) = 0$  for all  $p \in \text{Spec } R$ .

*Proof.* For (1)  $\Rightarrow$  (2), we know  $\text{Ext}_R^{n+1}(N, M) = 0$  for all  $R$ -modules  $N$  and so in particular for  $N = R/p$ . For (2)  $\Rightarrow$  (1), if  $\text{Ext}_R^{n+1}(R/p, M) = 0$  for all  $p \in \text{Spec } R$ , then  $\text{Ext}_R^{n+1}(N, M) = 0$  for all finitely generated  $R$ -modules  $N$  by the lemma. Then  $\text{Ext}_R^{n+1}(R/J, M) = 0$  for all ideals  $J \subseteq R$  which implies  $\text{id}_R M \leq n$ .  $\square$

**Proposition 2.16.** *Let  $(R, m, k)$  be Noetherian,  $p \neq m$  prime,  $M$  a finitely generated  $R$ -module. If  $\text{Ext}_R^{n+1}(R/q, M) = 0$  for all  $q \in \text{Spec } R$  with  $q \supseteq p$ , then  $\text{Ext}_R^n(R/p, M) = 0$ .*

*Proof.* Choose  $x \in m \setminus p$ . Then  $x$  is  $R/p$ -regular. Consider the short exact sequence  $0 \rightarrow R/p \xrightarrow{x} R/p \rightarrow R/(x, p) \rightarrow 0$  and the induced long exact sequence on  $\text{Ext}$  :

$$\cdots \rightarrow \text{Ext}_R^n(R/p, M) \xrightarrow{x} \text{Ext}_R^n(R/p, M) \rightarrow \text{Ext}_R^{n+1}(R/(x, p), M) \rightarrow \cdots$$

Since  $\text{Supp } R/(x, p) = \{q \in \text{Spec } R \mid q \supset (x, p)\} \subset \{q \in \text{Spec } R \mid q \supseteq p\}$  we have  $\text{Ext}_R^{n+1}(R/q, M) = 0$  for all  $q \in \text{Supp } R/(x, p)$ . By the Lemma,  $\text{Ext}_R^{n+1}(R/(x, p), M) = 0$ . Thus  $\text{Ext}_R^n(R/p, M) \xrightarrow{x} \text{Ext}_R^n(R/p, M)$  is surjective, which implies  $x(\text{Ext}_R^n(R/p, M)) = \text{Ext}_R^n(R/p, M)$ . By Nakayama's Lemma, done.  $\square$

**Note.** Here we use the fact that  $\text{Ext}_R(R/p, M)$  was finitely generated because both arguments were. To see this, take a free resolution of  $R/p$ . Say  $F : \cdots \rightarrow R^{n_2} \rightarrow R^{n_1} \rightarrow R \rightarrow 0$ . Apply  $\text{Hom}_R(F, M)$  and get  $\text{Hom}_R(R^{n_i}, M) \cong M^{n_i}$  where  $M$  is finitely generated.

**Proposition 2.17.** *Let  $(R, m, k)$  be Noetherian and  $M$  a finitely generated  $R$ -module. Then*

$$\text{id}_R M = \sup\{i : \text{Ext}_R^i(k, M) \neq 0\}.$$

*Proof.* Let  $t = \sup\{i : \text{Ext}_R^i(k, M) \neq 0\}$ . Since for an  $R$ -module  $N$ , we know  $\text{id}_R M \leq n$  if and only if  $\text{Ext}_R^{n+1}(K, N) = 0$  for all  $R$ -modules  $K$ , we have  $\text{id}_R M \geq t$ . So it is left to show  $\text{id}_R M \leq t$ . Clearly,  $\text{Ext}_R^i(R/m, M) = 0$  for all  $i > t$ . Thus repeated applications of the proposition implies  $\text{Ext}_R^i(R/p, M) = 0$  for all  $p \in \text{Spec } R$  and for all  $i > t$ . By the above corollary, we have  $\text{id}_R M \leq t$ .  $\square$

**Corollary 2.18.** *Let  $(R, m, k)$  be Noetherian,  $M$  a finitely generated  $R$ -module. If  $x \in m$  is  $R$ - and  $M$ -regular, then  $\text{id}_{R/(x)} M/xM = \text{id}_R M - 1$ .*

*Proof.* Notice that

$$\begin{aligned} \text{id}_R(M) &= \sup\{i : \text{Ext}_R^i(k, M) \neq 0\} \\ &= \sup\{i : \text{Ext}_{R/(x)}^{i-1}(k, M/xM) \neq 0\} \text{ by Theorem 1.16} \\ &= \sup\{i + 1 : \text{Ext}_{R/(x)}^i(k, M/xM) \neq 0\} \\ &= \text{id}_{R/(x)}(M/xM) + 1 \end{aligned} \quad \square$$

With the same assumptions, it is also the case that  $\text{pd}_R M = \sup\{i \mid \text{Ext}_R^i(M, k) \neq 0\}$ .

**Definition.** *Let  $R$  be Noetherian,  $M$  an  $R$ -module. Let  $I : 0 \rightarrow M \xrightarrow{\phi^0} I^0 \xrightarrow{\phi^1} I^1 \xrightarrow{\phi^2} \cdots$  be an injective resolution of  $M$ . We say  $I$  is minimal if  $I^i = E_R(\text{im } \phi^i) = E_R(\text{coker } \phi^{i-1})$  for all  $i \geq 0$ .*

**Remark.** Minimal injective resolutions always exist. Let  $I^0 = E_R(M)$  :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M & \xrightarrow{\phi^0} & I^0 & \overset{\phi^1}{\dashrightarrow} & I^1 = E_R(C^0) \\
 & & & & \searrow & & \searrow \\
 & & & & & & C^1 = \text{coker } \phi^1 \\
 & & & & \swarrow & & \swarrow \\
 & & & & C^0 = \text{coker } \phi^0 & & 
 \end{array}$$

**Theorem 2.19.** Let  $R$  be Noetherian,  $M$  an  $R$ -module and  $I : 0 \rightarrow I^0 \xrightarrow{\phi^1} I^1 \xrightarrow{\phi^2} \dots$  an injective resolution of  $M$ . Then  $I$  is minimal if and only if for all  $p \in \text{Spec } R$  the maps  $\text{Hom}_{R_p}(k(p), I^i) \rightarrow \text{Hom}_{R_p}(k(p), I_p^{i+1})$  are zero for all  $i \geq 0$ .

*Proof.* Let  $Z_i = \ker \phi^{i+1} = \text{im } \phi^i$  for all  $i \geq 0$ . Then  $0 \rightarrow Z_i \xrightarrow{\psi_i} I^i \xrightarrow{\phi^{i+1}} I^{i+1}$  is exact. Localize at  $p \in \text{Spec } R$  and apply  $\text{Hom}_{R_p}(k(p), -)$  to get the exact sequence

$$0 \rightarrow \text{Hom}_{R_p}(k(p), (Z_i)_p) \xrightarrow{(\hat{\psi}_i)_p} \text{Hom}_{R_p}(k(p), I_p^i) \xrightarrow{(\phi^{i+1})_p} \text{Hom}_{R_p}(k(p), I_p^{i+1})$$

Now  $I$  is minimal if and only if  $I^i = E_R(Z_i)$  for all  $i$ . By Theorem 2.12, this is if and only if for all  $p \in \text{Spec } R$  we have  $(\hat{\psi}_i)_p$  is an isomorphism, which is if and only if  $(\phi^{i+1})_p = 0$  as the sequence is exact.  $\square$

**Corollary 2.20.** Let  $R$  be Noetherian,  $M$  an  $R$ -module. If  $I$  is a minimal injective resolution of  $M$ , then  $(I)_S$  is a minimal  $R_S$ -injective resolution of  $M_S$ .

**Theorem 2.21.** Let  $R$  be Noetherian,  $M$  a finitely generated  $R$ -module,  $I$  a minimal injective resolution of  $M$ . Let  $p \in \text{Spec } R$ . The number of copies of  $E_R(R/p)$  in a decomposition of  $I^i$  is equal to  $\dim_{k(p)} \text{Ext}_{R_p}^i(k(p), M_p)$ , which is finite. Hence the number of copies of  $E_R(R/p)$  in  $I^i$  (denoted  $\mu_i(p, M)$ ) does not depend on the choice of minimal injective resolution of  $M$ . The set  $\{\mu_i(p, M) | p \in \text{Spec } R\}$  is called the  $i^{\text{th}}$  **Bass Numbers** of  $M$ .

*Proof.* First notice  $\mu_i(p, M) < \infty$  as  $\text{Ext}_{R_p}^i(k(p), M_p)$  is a finitely generated  $k(p)$ -module and hence a finite dimensional  $k(p)$  vector space. By remark 7, if  $I = \bigoplus_{p \in \text{Spec } R} E_R(R/p)^{\alpha_p}$ , then  $\text{Hom}_{R_p}(k(p), I_p) = k(p)^{\alpha_p}$ . Fix  $p \in \text{Spec } R$ . For all  $i$  let  $I^i = E_R(R/p)^{\alpha_{p,i}} \oplus A^i$  (where  $p \notin \text{Ass}_R A^i$ , that is,  $E(R/p)$  is not a summand of  $A^i$ ). Take a minimal injective resolution of  $M$ , localize, and apply  $\text{Hom}_{R_p}(k(p), -)$  to get

$$0 \rightarrow \underbrace{\text{Hom}_{R_p}(k(p), I_p^i)}_{k(p)^{\alpha_{p,i}}} \rightarrow \underbrace{\text{Hom}_{R_p}(k(p), I_p^{i+1})}_{k(p)^{\alpha_{p,i+1}}} \rightarrow \dots$$

where all of the maps are zero by the theorem. As  $I_p$  is an injective resolution of  $M_p$ , we see  $\text{Ext}_{R_p}^i(k(p), M_p) \cong k(p)^{\alpha_{p,i}}$ . Now take the dimension over  $k(p)$  of both sides.  $\square$

**Corollary 2.22.** Let  $(R, m)$  be Noetherian,  $M$  finitely generated. Then for all  $p \in \text{Spec } R$

$$\text{id}_R M = \sup\{i | \mu_i(m, M) \neq 0\} = \sup\{i | \mu_i(p, M) \neq 0\}.$$

*Proof.* For the first equality, note that  $\mu_i(m, M) = \dim_{R/m} \text{Ext}_R^i(R/m, M)$ . Apply Proposition 2.17. For the second equality, let  $s$  be the middle term and  $t$  the right hand side. Let  $p \in \text{Spec } R$  such that  $\mu_t(p, M) \neq 0$ . Then  $\text{Ext}_{R_p}^t(k(p), M_p) \neq 0$  which implies  $\text{Ext}_R^t(R/p, M) \neq 0$ . If  $p \neq m$ , then there exists  $n > t$  such that  $\text{Ext}_R^n(R/m, M) \neq 0$ , that is  $\mu_n(m, M) \neq 0$ , a contradiction. Thus  $s = t$ .  $\square$

**Corollary 2.23.** If  $(R, m)$  is local,  $M$  finitely generated and  $\text{id}_R M = n$ , then  $I^n = E_R(R/m)^{\mu_n(m, M)}$  where  $I$  is a minimal injective resolution for  $M$ .

*Proof.* If  $\mu_n(p, M) \neq 0$  where  $p \neq m$ , there exists  $s > n$  such that  $\mu_s(m, M) \neq 0$ , a contradiction as  $\text{id}_R M = n$ .  $\square$

**Remark.**  $\mu_i(p, M) = \mu_i(pR_S, M_S)$  for any multiplicatively closed set  $S$  such that  $S \cap p = \emptyset$ .

The following result is a generalization of Proposition 2.16.

**Proposition 2.24.** *Let  $p \subset q$  be primes. Suppose  $\text{ht}(q/p) = \ell$ . Then  $\mu_i(p, M) \neq 0$  implies  $\mu_{i+\ell}(q, M) \neq 0$ .*

*Proof.* By induction, it is enough to show the case where  $\text{ht}(q/p) = 1$ . As localizations preserve Bass Numbers (see remark), we may localize at  $q$  and therefore assume  $q = m$ . Now  $\mu_i(p, M) \neq 0$  implies  $\text{Ext}_R^i(R/p, M) \neq 0$ . By Proposition 2.16, since  $m$  is the only prime containing  $p$ , we see  $\text{Ext}_R^{i+1}(R/m, M) \neq 0$  and so  $\mu_{i+1}(m, M) \neq 0$ .  $\square$

**Corollary 2.25.**  $\sup\{i \mid \mu_i(p, M) \neq 0, p \in \text{Spec } R\} = \sup\{i \mid \mu_i(m, M) \neq 0\} = \text{id}_R M$ .

**Theorem 2.26.** *Let  $(R, m, k)$  be local,  $M$  a finitely generated  $R$ -module of finite injective dimension. Then  $\dim M \leq \text{id}_R M = \text{depth } R$ .*

*Proof.* For the first inequality, let  $p_0 \subset p_1 \subset \dots \subset p_d = m$  be a maximal chain of primes in  $\text{Supp } M$  (so that  $d = \dim M$ ). We want to show  $\text{Ext}_{R_{p_i}}^i(k(p_i), M_{p_i}) \neq 0$  for all  $i$ . As then, we will have  $\text{Ext}_{R_m}^d(k(m), M_m) \neq 0$  and thus  $\text{Ext}_R^d(k, M) \neq 0$ . By Proposition 2.17, we will then have  $\text{id}_R M \geq d = \dim M$ . So induct on  $i$ . If  $i = 0$ , then  $p_0 R_{p_0} \in \text{Ass } M_{p_0}$  (since it is minimal in the support). Therefore  $\text{Ext}_{R_{p_0}}^0(k(p_0), M_{p_0}) = \text{Hom}_{R_{p_0}}(k(p_0), M_{p_0}) \neq 0$ . For  $i > 0$ , let  $B = R_{p_i}$ . Then  $\text{Ext}_B^{i-1}(B/p_{i-1}B, M_{p_i})_{p_{i-1}} \cong \text{Ext}_{R_{p_{i-1}}}^{i-1}(k(p_{i-1}), M_{p_{i-1}}) \neq 0$  by induction. By Proposition 2.16, we see  $\text{Ext}_{R_{p_i}}^i(k(p_i), M_{p_i}) \neq 0$ .

For the equality, let  $r = \text{id}_R M$  and  $t = \text{depth } R$ . Let  $\bar{x} = x_1, \dots, x_t$  be a maximal regular  $R$ -sequence. Then the Koszul Complex  $K(\bar{x})$  is a minimal free resolution of  $R/(\bar{x})$  which implies  $\text{pd}_{R/(\bar{x})} R/(\bar{x}) = t$ .

*Claim.*  $\text{Ext}_R^t(R/(\bar{x}), M) \neq 0$ .

*Proof.* Consider  $K : 0 \rightarrow R \xrightarrow{\alpha} R^t \rightarrow \dots \rightarrow R^t \rightarrow R \rightarrow 0$  where  $\alpha$  is defined by 1 maps to some permutation of  $\pm x_i$ . Apply  $\text{Hom}_R(-, M)$  to get

$$0 \rightarrow \text{Hom}_R(R, M) \rightarrow \dots \rightarrow \underbrace{\text{Hom}_R(R^t, M)}_{\cong M^t} \xrightarrow{\alpha^T} \underbrace{\text{Hom}_R(R, M)}_{\cong M}.$$

Now  $\text{Ext}_R^t(R/(\bar{x}), M) = H^t(\text{Hom}(K(\bar{x}), M)) = \ker 0/\text{im } \alpha^T = M/\bar{x}M \neq 0$  by NAK.

Then  $r \geq t$  by Proposition 2.17. Now since  $\text{depth } R/(\bar{x}) = 0$ , there exists an embedding  $K \hookrightarrow R/(\bar{x})$  which induces a surjection

$$\text{Ext}_R^r(R/(\bar{x}), M) \rightarrow \text{Ext}_R^r(k, M) \rightarrow 0$$

where the last term is 0 as  $\text{Ext}_R^{r+1}(N, M) = 0$  for all  $N$  as  $\text{id}_R M = r$ . But  $\text{Ext}_R^r(k, M) \neq 0$  by Proposition 2.17 and so  $\text{Ext}_R^r(R/(\bar{x}), M) \neq 0$ . So  $t = \text{pd}_R R/(\bar{x}) \geq r$ .  $\square$

**Corollary 2.27.**  *$R$  Gorenstein implies  $R$  is Cohen Macaulay.*

*Proof.* From the theorem, we have  $\dim R \leq \text{id } R = \text{depth } R$ .  $\square$

**Definition.** *A (not necessarily local) ring  $R$  is Gorenstein if  $R_m$  is Gorenstein for all maximal ideals  $m$ .*

**Proposition 2.28.** *Suppose  $R$  is Noetherian.*

- (1) *Suppose  $R$  is Gorenstein. Then for all multiplicatively closed subsets  $S$  of  $R$ , we have  $R_S$  is also Gorenstein. In particular,  $R_p$  is Gorenstein for all  $p \in \text{Spec } R$ .*
- (2) *Suppose  $\bar{x} = (x_1, \dots, x_n)$  is an  $R$ -regular sequence. If  $R$  is Gorenstein, then so is  $R/(\bar{x})$ . The converse holds if  $R$  is local.*
- (3) *Suppose  $R$  is local. Then  $R$  is Gorenstein if and only if  $\hat{R}$  is Gorenstein.*

*Proof.* (1) Let  $q \subseteq R_S$  be maximal. Then  $q = pR_S$  for some  $p \in \text{Spec } R$ . Then  $(R_S)_q \cong R_p$ . Let  $m \subseteq R$  be maximal with  $p \subseteq m$ . Then  $R_p$  is a localization of  $R_m$  which is Gorenstein by assumption. Recall  $\text{id}_{R_p} R_p \leq \text{id}_{R_m} R_m < \infty$ . So  $R_p$  is Gorenstein.

(2) WLOG, assume  $R$  is local. Apply Corollary 2.18 repeatedly to get  $\text{id}_{R/(\bar{x})} R/(\bar{x}) = \text{id}_R R - n$ .

(3) We will use the following facts

- $M$  finitely generated implies  $\hat{M} = M \otimes_R \hat{R}$ .
- $\hat{R}/\hat{m} \cong R/m$  ( and in particular  $\hat{k} = k$ .)
- $\hat{R}$  is a faithfully flat  $R$ -module.

So  $\text{Ext}_R^i(\hat{k}, \hat{R}) \cong \text{Ext}_R^i(k, R) \otimes_R \hat{R}$  because  $\hat{R}$  is flat. Since  $\hat{R}$  is faithful, the left hand side is 0 if and only if  $\text{Ext}_R^i(k, R) = 0$ . By Proposition 2.17, done.  $\square$

**Exercise.** Let  $(R, m, k)$  be Noetherian. If  $\text{id}_R k < \infty$ , then  $R$  is regular.

*Proof.* First recall  $\text{id}_R M = \sup\{i \mid \text{Ext}_R^i(k, M) \neq 0\}$  and  $\text{pd}_R M = \{i \mid \text{Ext}_R^i(M, k) \neq 0\}$ . If  $M = k$ , we have  $\text{id}_R k = \text{pd}_R k$ . Done by Theorem 1.5.  $\square$

**Exercise.** Let  $(R, m, k)$  be Gorenstein,  $M$  a finitely generated  $R$ -module. Then  $\text{pd}_R M < \infty$  if and only if  $\text{id}_R M < \infty$ .

*Proof.* For the forward direction, let  $n := \text{pd}_R M$  and induct on  $n$ . If  $n = 0$ , then  $M$  is projective. Since  $M$  is finitely generated and  $R$  is local,  $M$  is free. Then  $\text{id}_R M = \text{id}_R R < \infty$  as  $R$  is Gorenstein. So assume  $n > 0$ . Consider the short exact sequence  $0 \rightarrow K \rightarrow R^t \xrightarrow{\phi} M \rightarrow 0$  where  $K = \ker \phi$ . As  $M$  is finitely generated, we see  $K$  is. Further,  $\text{pd}_R K = \text{pd}_R M - 1 < n$ . Thus  $\text{id}_R k < \infty$  by induction. Now  $\text{id}_R K < \infty$  and  $\text{id}_R R^t < \infty$  implies  $\text{id}_R M < \infty$  (In fact it is the case that whenever two terms of a short exact sequence have finite injective or projective dimension, then the third does by examining the vanishing of Ext in the induced long exact sequence).

For the backward direction, assume  $\text{id}_R M < \infty$  and induct on  $\text{depth}_R R = \dim R$  (these are equal by the corollary). If  $\dim R = 0$ , then  $E_R(R/m)$  is the only indecomposable injective  $R$ -module. By the Theorem,  $\text{id} R = \text{depth} R = 0$  implies  $R$  is injective. Since  $R$  is local, it is indecomposable [If  $R = I \oplus J$  for  $I, J \subset m$ , then  $1 = d + e$  for  $d \in I, e \in J$  implies  $1 \in m$ , contradiction]. So  $R = E(R/m)$ . Now  $\text{id}_R M = \text{depth} R = \text{id} R = 0$  by the Theorem. So  $M$  injective implies  $M = E(R/m)^t = R^t$ . Thus  $M$  is projective which implies  $\text{pd}_R M = 0 < \infty$ . Now suppose  $\text{depth} R = \dim R > 0$ . There there are two cases.

- If  $\text{depth} M > 0$ , there exists  $x \in m$  a non-zero-divisor on  $M$  and  $R$ . Then  $\text{depth} R/x = \text{depth} R - 1$  and  $R/x$  is Gorenstein. By Corollary 2.18  $\text{id}_R R/xM/xM < \infty$ . By induction and Lemma 1.13 (which is also true in the infinite case) we have  $\text{pd}_R M = \text{pd}_{R/x} M/xM < \infty$ .
- If  $\text{depth} M = 0$ , consider the short exact sequence  $0 \rightarrow K \rightarrow R^t \rightarrow M \rightarrow 0$ . Now  $M$  is not free as  $\text{depth} R \neq \text{depth} M$  and so  $K \neq 0$ . Since  $\text{depth} R^t > 0$  we have  $\text{depth} K > 0$ . Since  $R$  is Gorenstein  $\text{id} R^t < \infty$  and  $\text{id} M < \infty$  by assumption. Thus  $\text{id}_R K < \infty$ . By case 1,  $\text{pd}_R K < \infty$ . Now since  $\text{pd}_R K < \infty$  and  $\text{pd}_R R^t < \infty$ , we must also have  $\text{pd}_R M < \infty$ .  $\square$

**Definition.** If  $(R, m)$  is local,  $M$  a finitely generated  $R$ -module, say the **socle** of  $M$  is  $\text{soc}_R M := (0 :_M m) \cong \text{Hom}_R(R/m, M)$ . The socle is an  $R/m$ -module.

**Remark.** Let  $(R, m, k)$  be a zero-dimensional local ring. Then  $\text{soc} R \neq 0$  as  $m^t = 0$  for some  $t$ . Further, we have that  $(0)$  is an irreducible ideal in  $R$  if and only if  $\dim_k \text{soc} R = 1$ .

*Proof.* First recall that an ideal  $I$  is irreducible if whenever  $I = J_1 \cap J_2$  we have  $I = J_1$  or  $I = J_2$ . Now, for the forward direction, suppose  $\dim_k \text{soc} R = 1$ . Let  $x, y \in \text{soc} R$  be linearly independent over  $R/m$  (since the socle is an  $R/m$  vector space). Then  $Rx \cap Ry = (R/m)x \cap (R/m)y = (0)$ . Thus  $(0)$  is reducible.

For the backward direction, suppose  $I \cap J = (0)$  where  $I, J \neq (0)$ . Let  $x \in I \setminus \{0\}, y \in J \setminus \{0\}$ . Then  $(x) \cap (y) = (0)$ . Since  $R$  is zero-dimensional, there exists  $n$  such that  $m^n = 0$ . Thus choose  $r$  such that  $m^r x \neq (0)$  but  $m^{r+1} x = (0)$  and choose  $t$  such that  $m^t y \neq (0)$  but  $m^{t+1} y = (0)$ . Let  $u \in m^r x \setminus \{0\}$  and  $v \in m^t y \setminus \{0\}$ . Then  $u, v \in \text{soc} R$ . However  $(u) \cap (v) \subseteq (x) \cap (y) = (0)$ , making  $u$  and  $v$  linearly independent in  $\text{soc} R$ . Thus  $\dim_k \text{soc} R > 1$ .  $\square$

**Proposition 2.29.** Let  $(R, m, k)$  be zero-dimensional. TFAE

- (1)  $R$  is Gorenstein.

- (2)  $\dim_k \text{soc } R = 1$ .
- (3)  $R = E_R(R/m)$ .

*Proof.* We proved (1)  $\Leftrightarrow$  (3) in the proof of the exercise. For (1)  $\Rightarrow$  (2), note that  $\text{soc } R \cong \text{Hom}_R(R/m, R) \cong \text{Hom}_R(R/m, E_R(R/m)) \cong E_{R/m}(R/m) = R/m$  and  $R/m$  is a field and hence injective (and its own injective hull). Thus  $\dim_k \text{soc } R = 1$ .

For (2)  $\Rightarrow$  (3), we will use the fact that for a zero-dimensional local ring  $\lambda_R(R) = \lambda_R(E_R(R/m))$  (which is finite as  $R$  is Artinian). Note that  $\text{soc } R \subseteq R$  is essential (take  $x \in R \setminus \{0\}$ . Then there exists  $r$  such that  $m^r x \neq 0$  but  $m^{r+1}x = 0$ . Then  $m^r x \subseteq \text{soc } R$  and hence  $Rx \cap \text{soc } R \neq 0$ ). Hence  $R/m \cong \text{soc } R \subseteq R$  is essential. Consider the following diagram.

$$\begin{array}{ccccc}
 & & E_R(R/m) & & \\
 & & \uparrow & \swarrow & \\
 0 & \longrightarrow & R/m & \xrightarrow{\text{ess}} & R \\
 & & \uparrow & & \\
 & & 0 & & 
 \end{array}$$

This says the dotted map must be injective and so we have  $0 \rightarrow R \rightarrow E_R(R/m) \rightarrow C \rightarrow 0$ . Since  $\lambda_R(R) = \lambda_R(E_R(R/m))$ , we see  $C = 0$ . Thus  $R \cong E_R(R/m)$ . □

**Examples.**

- Let  $R = k[x, y]/(x^2, xy, y^2)$ . Since  $R$  is modded out by  $m^2$ , we see  $\text{soc } R = m = (\bar{x}, \bar{y})$ . This is two-dimensional and hence  $R$  is not Gorenstein.
- Let  $R = k[x, y]/(x^2, y^2)$ . Here  $\text{soc } R = (xy)$ . Hence  $\dim \text{soc } R = 1$  and so  $R$  is Gorenstein.

**Definition.** Let  $(R, m)$  be Cohen Macaulay of dimension  $d$ . Then the **(CM) type** of  $R$  is

$$r(R) := \mu_d(m, R) = \dim_{R/m} \text{Ext}_R^d(R/m, R).$$

**Remark.**

- (1) If  $d = 0$ , then  $r(R) = \dim_{R/m} \text{Hom}_R(R/m, R) = \dim_k \text{soc } R$ .
- (2) Let  $R$  be Cohen Macaulay of dimension  $d$  and  $x_1, \dots, x_d$  a system of parameters for  $R$ . By Theorem 1.16, we see

$$r(R) = \dim_{R/m} \text{Ext}_R^d(R/m, R) = \dim_{R/m} \text{Ext}_{R/(x_1, \dots, x_d)}^0(R/m, R/(x_1, \dots, x_d)) = \dim_{R/m} \text{soc } R/(x_1, \dots, x_d).$$

**Theorem 2.30.** Let  $(R, m)$  be local. TFAE

- (1)  $R$  is Gorenstein.
- (2)  $R$  is Cohen Macaulay and  $r(R) = 1$ .
- (3)  $R$  is Cohen Macaulay and every (equivalently, some) ideal generated by a system of parameters is an irreducible ideal.

*Proof.* For (2)  $\Leftrightarrow$  (3), let  $x_1, \dots, x_d$  be a system of parameters. Then  $r(R) = 1$  if and only if  $\dim \text{soc } R/(x_1, \dots, x_d) = 1$  (by remark) if and only if  $(\bar{0})$  is irreducible in  $R/(x_1, \dots, x_d)$  if and only if  $(x_1, \dots, x_d)$  is irreducible in  $R$ .

For (1)  $\Rightarrow$  (2), we know  $R$  is Cohen Macaulay (as it is Gorenstein). Let  $x_1, \dots, x_d$  be a system of parameters. Then  $R/(x_1, \dots, x_d)$  is a zero-dimensional Gorenstein ring. Thus by the second remark and the proposition, we have  $r(R) = \dim \text{soc } R/(x_1, \dots, x_d) = 1$ .

For (2)  $\Rightarrow$  (1), let  $x_1, \dots, x_d$  be a system of parameters. Then  $\dim \text{soc } R/(x_1, \dots, x_d) = r(R) = 1$ . Thus  $R/(x_1, \dots, x_d)$  is Gorenstein by the proposition and hence  $R$  is Gorenstein by Corollary 2.18 as  $x_1, \dots, x_d$  is a regular sequence. □

**Theorem 2.31.** Let  $(R, m)$  be local. TFAE

(1)  $R$  is Gorenstein

(2) For all  $p \in \text{Spec } R$  we have  $\mu_i(p, R) = \begin{cases} 1 & \text{if } i = \text{ht } p \\ 0 & \text{if } i \neq \text{ht } p \end{cases}$

*Proof.* For (2)  $\Rightarrow$  (1), we have  $\mu_i(m, R) = 0$  for all  $i > \text{ht } m$ . Thus  $\text{id}_R R \leq \text{ht } m = \dim R$ . Thus  $R$  is Gorenstein.

For (1)  $\Rightarrow$  (2), it is enough to show the case where  $p = m$  by localizing (as  $R_p$  is Gorenstein and the Bass numbers are preserved). By Theorem 2.26, we have  $\text{id}_R R = \text{depth } R = \dim R = d$ . Thus  $\mu_i(m, R) = 0$  for all  $i > d = \dim R$ . Since  $R$  is Cohen Macaulay,  $\text{depth } R = d$ . So  $\text{Ext}_R^i(R/m, R) = 0$  for all  $i < d$  (and hence  $\mu_i(m, R) = 0$  for all  $i < d$ ). Lastly, we see  $\mu_d(m, R) = \dim_{R/m} \text{Ext}_R^d(R/m, R) = r(R) = 1$  as  $R$  is Gorenstein.  $\square$

The theorem shows that if  $I$  is a minimal injective resolution of  $R$ , then  $I^i = \bigoplus_{\text{ht } p=i} E_R(R/p)$ .

**Definition.** Let  $(R, m)$  be local and  $E = E_R(R/m)$ . For an  $R$ -module  $M$ , we define the **Matlis Dual** of  $M$  to be  $M^v = \text{Hom}_R(M, E)$ .

As  $E$  is injective, we see  $(\cdot)^v$  is an exact covariant functor.

**Proposition 2.32.** If  $\lambda(M) < \infty$  then  $\lambda(M^v) = \lambda(M)$ .

*Proof.* Induct of  $\lambda(M)$ . If  $\lambda(M) = 1$ , then  $M \cong R/m$ . So  $M^v = \text{Hom}_R(R/m, E) = E_{R/m}(R/m) = R/m$ . Thus  $\lambda(M^v) = 1$ . Suppose  $\lambda(M) > 1$ . Choose a proper submodule  $N$  of  $M$  and get  $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$  exact. Apply  $(\cdot)^v$  to get a short exact sequence  $0 \rightarrow N^v \rightarrow M^v \rightarrow (M/N)^v \rightarrow 0$ . By induction  $\lambda(M^v) = \lambda(N^v) + \lambda((M/N)^v) = \lambda(N) + \lambda(M/N) = \lambda(M)$ .  $\square$

**Corollary 2.33.** For an Artinian ring  $R$ ,  $\lambda(R) = \lambda(E_R(R/m))$ .

**Exercise.** Let  $(R, m)$  be an Artinian local ring. TFAE

- (1)  $R$  is Gorenstein
- (2) All finitely generated  $R$ -modules are reflexive
- (3)  $I = \text{Ann Ann } I$  for all ideals  $I$  of  $R$
- (4) For all  $I, J \neq (0)$ , we have  $I \cap J \neq (0)$ .

*Proof.* (1)  $\Rightarrow$  (2) If  $R$  is Gorenstein with dimension 0, then  $R \cong E$ . So  $M^* = \text{Hom}_R(M, R) \cong \text{Hom}_R(M, E) = M^v$ . We will induct on  $\lambda(M)$ . If  $\lambda(M) = 1$ , then  $M \cong R/m$ . We want to show  $\Phi : R/m \rightarrow (R/m)^{**}$  defined by  $\bar{1} \mapsto ev_{\bar{1}}$  is an isomorphism where  $ev_{\bar{1}} : \text{Hom}_R(R/m, R) \rightarrow R$  is defined by  $\phi \mapsto \phi(\bar{1})$ . Since  $\Phi$  is a map of one dimensional vector spaces (We know  $(R/m)^{**} \cong R/m$  as  $\text{Hom}_R(R/m, E) \cong R/m$ ), it is enough to show  $\Phi \neq 0$ . Of course,  $\phi \in \text{Hom}_R(R/m, R)$  is defined by  $\phi(\bar{1}) = x \in (0 :_R m)$ . Since  $(0 :_R m) \neq 0$ , we see  $\Phi \neq 0$  and so  $M \cong R/m$  is reflexive. Assume  $\lambda(M) > 1$ . Choose a proper submodule  $N$  of  $M$  to get

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & N^{**} & \longrightarrow & M^{**} & \longrightarrow & (M/N)^{**} & \longrightarrow & 0 \end{array}$$

where the outer two downward maps are isomorphisms by induction. The rows are both exact as  $(\cdot)^* = (\cdot)^v$ . By the Five Lemma, we get  $M \xrightarrow{\cong} M^{**}$ .

(2)  $\Rightarrow$  (3) Let  $I$  be an ideal of  $R$ . Clearly  $I \subseteq \text{Ann Ann } I$ . By assumption, the map  $R/I \rightarrow \text{Hom}_R(\text{Hom}_R(R/I, R), R)$  defined by  $\bar{x} \mapsto ev_{\bar{x}}$  is an isomorphism. So let  $y \in \text{Ann Ann } I$ . Then  $\bar{y} \in R/I$ . It is enough to show  $\bar{y} \mapsto ev_{\bar{y}}$  as then  $\bar{y} = 0$  and then  $y \in I$ . Let  $\phi \in \text{Hom}_R(R/I, R) \cong \text{Ann } I$ . Then  $\phi(\bar{y}) = y\phi(\bar{1}) = 0$  as  $\bar{1} \in \text{Ann } I$ . Thus  $ev_{\bar{y}} = 0$  and  $\bar{y} \mapsto 0$ .

(3)  $\Rightarrow$  (1) Let  $x \in \text{soc } R \setminus \{0\}$ . Then  $(0 :_R x) = m$ . So  $(0 :_R m) = (0 :_R (0 :_R x)) = (x)$  (by assumption). Thus  $\text{soc } R = (x)$ , which implies  $\dim_k \text{soc } R = 1$ . Hence  $R$  is Gorenstein.

(1)  $\Leftrightarrow$  (4) Since  $\dim R = 0$ , we have proven that  $0$  is an irreducible ideal if and only if  $\dim_k \text{soc } R = 1$  which is if and only if  $R$  is Gorenstein.  $\square$

**Proposition 2.34.** *For  $(R, m)$  local,  $M = 0$  if and only if  $M^v = 0$ .*

*Proof.* If  $M$  is finitely generated, consider the exact sequence  $M \rightarrow M/mM \rightarrow 0$ . Apply  $\text{Hom}_R(-, E)$  to get  $0 \rightarrow \text{Hom}_R(M/mM, E) \rightarrow M^v$ . If  $M = 0$ , then certainly  $M^v = 0$ . If  $M^v = 0$ , then  $\text{Hom}_R(M/mM, E) = (M/mM)^v = 0$ . Since  $\lambda(M/mM) = \lambda((M/mM)^v) = 0$ , we have  $M/mM = 0$  and hence  $M = 0$  by NAK.

If  $M \neq 0$  is any  $R$ -module, let  $N \subseteq M$  be a nonzero finitely generated submodule. Then  $0 \rightarrow N \rightarrow M$  exact implies  $M^v \rightarrow N^v \rightarrow 0$  is exact. If  $M^v = 0$ , then  $N^v = 0$  and so  $N = 0$  as it is finitely generated, a contradiction. Hence  $M^v \neq 0$ . If  $M = 0$  then clearly  $M^v = 0$ .  $\square$

This says  $(-)^v$  is a faithful contravariant functor.

**Corollary 2.35.** *If  $\lambda(M) = \infty$ , then  $\lambda(M^v) = \infty$ .*

*Proof.* Let  $t \in \mathbb{N}$ . Then there exists a sequence of submodules  $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_t = M$ .

*Claim.*  $\lambda(N_i^v) \geq i$  for all  $i$ .

*Proof.* If  $i = 0$ , done. If  $i = 1$ , then  $N_1 \neq 0$  implies  $N_1^v \neq 0$ . Thus  $\lambda(N_1^v) \geq 1$ . Suppose  $\lambda(N_{i-1}^v) \geq i-1$  and consider the short exact sequence  $0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow N_i/N_{i-1} \rightarrow 0$ . Apply  $(-)^v$  to get  $0 \rightarrow (N_i/N_{i-1})^v \rightarrow N_i^v \rightarrow N_{i-1}^v \rightarrow 0$ . Since  $N_i/N_{i-1} \neq 0$ , we know  $\lambda((N_i/N_{i-1})^v) \geq 1$ . By induction and additivity of length, get  $\lambda(N_i^v) \geq i$ .  $\square$

**2.2. Completions.** Let  $(R, m)$  be local,  $M$  a finitely generated  $R$ -modules. By Krull's Intersection Theorem  $\bigcap_{n=1}^{\infty} m^n M = (0)$ . For  $x \in M \setminus \{0\}$ , define  $\|x\| = 2^{-n}$  where  $x \in m^n M \setminus m^{n+1} M$ . Let  $\|0\| = 0$ . Then  $\|\cdot\|$  defines a norm. Let  $\widetilde{M} = \{\{x_i\}_{i=1}^{\infty} | \{x_i\} \text{ is a Cauchy Sequence}\}$ . [Note that  $\widetilde{M}$  has the structure of an abelian group]. For  $\{x_i\}, \{y_j\} \in \widetilde{M}$ , say  $\{x_i\} \sim \{y_j\}$  if  $\lim_{i \rightarrow \infty} x_i - y_i = 0$ . This forms an equivalence relation. Define the **completion**  $\hat{M} = \widetilde{M} / \sim$ . Then  $\hat{M}$  has an abelian group structure. Also, there exists a natural injective group homomorphism  $M \rightarrow \hat{M}$  defined by  $x \mapsto \hat{x} := [\{x\}]$ . In particular, this says  $M \neq 0$  implies  $\hat{M} \neq 0$ , that is, the completion is a faithful operation. In the case of the ring, we see  $\hat{R}$  has a ring structure and the natural map  $\phi : R \rightarrow \hat{R}$  defined by  $r \mapsto \hat{r}$  is an injective ring map. Further,  $\hat{M}$  has a natural  $\hat{R}$ -module structure.

We can also define the completion in terms of inverse limits. Consider the inverse system  $\{M/m^i M, \psi_i^j\}_{i,j \in \mathbb{Z}}$  where for  $i \leq j$  we have  $\psi_i^j : M/m^j M \rightarrow M/m^i M$  is defined by  $x + m^j M \mapsto x + m^i M$  (note these are onto). Then the inverse limit

$$\varprojlim M/m^i M = \{(x_i + m^i M) \in \prod_{j=i}^{\infty} M/m^j M | \text{for } i \leq j, x_i + m^i M = x_j + m^i M \Leftrightarrow x_i - x_j \in m^i M\}$$

From here one can show this is just  $\hat{M}$ .

**Fact.** The inverse limit is a left exact covariant functor, that is, if  $0 \rightarrow \{A_i, \psi_i^j\} \xrightarrow{\alpha_i} \{B_i, \phi_i^j\} \xrightarrow{\beta_i} \{C_i, \delta_i^j\}$  is exact then  $0 \rightarrow \varprojlim A_i \xrightarrow{\hat{\alpha}} \varprojlim B_i \xrightarrow{\hat{\beta}} \varprojlim C_i$  is exact. Furthermore, if  $\beta_i$  is onto for all  $i$  and  $\delta_i^j$  are surjective for all  $i \leq j$ , then  $\hat{\beta}$  is onto.

**Corollary 2.36.** *If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence of finitely generated  $R$ -modules, then  $0 \rightarrow \hat{A} \rightarrow \hat{B} \rightarrow \hat{C} \rightarrow 0$  is exact.*

*Proof.* Since  $- \otimes R/m^n$  is right exact, we have  $A/m^n A \rightarrow B/m^n B \rightarrow C/m^n C \rightarrow 0$  is exact. In fact,  $0 \rightarrow A/m^n B \cap A \rightarrow B/m^n B \rightarrow C/m^n C \rightarrow 0$  is exact for all  $n$  (where here we assume  $A \subseteq B$ ). Thus apply the inverse limit to get the exact sequence

$$0 \rightarrow \varprojlim A/m^n B \cap A \rightarrow \hat{B} \rightarrow \hat{C} \rightarrow 0.$$

Notice now  $\cap(m^n B \cap A) \subseteq \cap m^n B = (0)$ . Define  $\|x\|' = 2^{-n}$  if  $x \in (m^n B \cap A) \setminus (m^{n+1} B \cap A)$  and  $\|0\|' = 0$ . Then  $\|\cdot\|$  and  $\|\cdot\|'$  define the same topology (recall the Artin Rees Lemma which says there exists  $k$  such that for all  $n \geq k$  we have  $m^n A \subseteq m^n B \cap A = m^{n-k}(m^k B \cap A) \subseteq m^{n-k} A$ ). Thus we have  $\varprojlim A/m^n B \cap A = \hat{A}$ .  $\square$

It is easy to see that  $\widehat{A \oplus B} = \hat{A} \oplus \hat{B}$  and thus  $\widehat{R^n} \cong \hat{R}^n \cong R^n \otimes_R \hat{R}$ . By the Five Lemma, if  $M$  is finitely generated then  $M \otimes_R \hat{R} \cong \hat{M}$ . Hence  $- \otimes \hat{R}$  is exact (for finitely generated modules) and  $\hat{R}$  is a flat  $R$ -algebra.

**Definition.** We say an  $R$ -module is **complete** if the natural map  $M \rightarrow \hat{M}$  defined by  $x \mapsto \hat{x}$  is an isomorphism.

If  $m^n M = 0$  for some  $n$ , then  $M$  is complete (since Cauchy Sequences must be eventually constant). In particular, this says  $\widehat{R/m^n} \cong R/m^n$  for all  $n$ . Consider  $0 \rightarrow m^n \rightarrow R \rightarrow R/m^n \rightarrow 0$  and apply  $- \otimes_R \hat{R}$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & m^n \otimes_R \hat{R} & \longrightarrow & R \otimes_R \hat{R} & \longrightarrow & R/m^n \otimes_R \hat{R} \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \widehat{m^n} & \longrightarrow & \hat{R} & \longrightarrow & \widehat{R/m^n} = R/m^n \longrightarrow 0 \end{array}$$

From the diagram, we see  $\widehat{m^n} = m^n \hat{R}$ . Also,  $\hat{R}/\widehat{m^n} \cong \hat{R}/m^n \hat{R} \cong R/m^n$ . In particular,  $\hat{m} = m \hat{R}$  is a maximal ideal (since  $\hat{R}/\hat{m} \cong R/m$  is a field). One can show

- (1)  $\hat{m}$  is the only maximal ideal of  $\hat{R}$  (i.e.,  $x \notin \hat{m}$  is a unit)
- (2)  $\hat{R}$  is complete with respect to  $\hat{m}$ .
- (3)  $\hat{R}$  is Noetherian.
- (4)  $\dim R = \dim \hat{R}$
- (5) Modding out preserves completeness, but localizing does not.

[Items 3 and 4 use the associated graded ring and the fact that  $\dim gr_I R = \dim R$  (since  $gr_m R \cong gr_{\hat{m}} \hat{R}$ ). The important things to remember about the completion are

- $R \rightarrow \hat{R}$  is faithfully flat.
- $\hat{R}/\hat{m} \cong R/m$ .

**Theorem 2.37 (Cohen Structure Theorem).** Any complete local ring is the homomorphic image of a complete regular local ring, that is  $R = S/J$  where  $S$  is a complete local ring and  $J \subseteq S$  is an ideal.

In particular, this shows that any complete local ring is universally catenary.

Suppose  $\phi : R \rightarrow S$  is a flat ring extension. Then for all  $R$ -modules  $M, N$  we have  $\text{Tor}_i^S(M \otimes_R S, N \otimes_R S) \cong \text{Tor}_i^R(M, N) \otimes_R S$  and  $\text{Ext}_S^i(M \otimes_R S, N \otimes_R S) \cong \text{Ext}_R^i(M, N) \otimes_R S$ .

“Proof”. If  $C$  is a chain complex, then  $C \otimes_R S$  is a chain complex and  $H_i(C \otimes_R S) \cong H_i(C) \otimes_R S$  (by flatness). Let  $P$  be a projective resolution of  $M$ . Then  $P \otimes_R S$  is a projective  $S$ -resolution of  $M \otimes_R S$ . So

$$\text{Tor}_i^S(M \otimes_R S, N \otimes_R S) = H_i((P \otimes_R S) \otimes_S (N \otimes_R S)) = H_i(P \otimes_R N \otimes_R S) \cong H_i(P \otimes_R N) \otimes_R S = \text{Tor}_i^R(M, N) \otimes_R S.$$

Similarly for Ext.  $\square$

In particular, this says we can commute Ext and Tor with  $- \otimes_R \hat{R}$ . So  $\text{Ext}_R^i(R/m, R) \otimes_R \hat{R} \cong \text{Ext}_{\hat{R}}^i(\hat{R}/\hat{m}, \hat{R})$ . Thus  $\text{Ext}_R^i(R/m, R) = 0$  if and only if  $\text{Ext}_{\hat{R}}^i(\hat{R}/\hat{m}, \hat{R}) = 0$ . Thus  $\text{id}_R R = \text{id}_{\hat{R}} \hat{R}$ . So  $R$  is Gorenstein if and only if  $\hat{R}$  is Gorenstein. This also shows  $\text{depth } R = \text{depth } \hat{R}$  (as it is characterized by the first non-vanishing of Ext) and since  $\dim R = \dim \hat{R}$ , we see  $R$  is Cohen-Macaulay if and only if  $\hat{R}$  is Cohen-Macaulay. We can similarly show now  $\text{Tor}_i^R(R/m, R/m) = 0$  if and only if  $\text{Tor}_i^{\hat{R}}(\hat{R}/\hat{m}, \hat{R}/\hat{m}) = 0$ . Thus  $\text{pd}_R R/m = \text{pd}_{\hat{R}} \hat{R}/\hat{m}$  and so  $R$  is regular if and only if  $\hat{R}$  is regular. One can also show  $\lambda(M) = \lambda(\hat{M})$  by induction. Thus taking the completion preserves several nice properties and thus makes certain situations easier to deal with.

Note that for all  $\hat{r} \in \hat{R}$  and  $n \in \mathbb{N}$  there exists  $r \in R$  such that  $\hat{r} - r \in m^n \hat{R} = \widehat{m^n} = \hat{m}^n$ . Since  $\phi : R \rightarrow \hat{R}$  is faithfully flat, for any ideal  $I$  of  $R$ , we have  $I \hat{R} \cap R = I$ .

- It is clear that  $I\hat{R} \cap R \supseteq I$ . For the other direction, consider the natural injection  $R/I \rightarrow \widehat{R/I}$ . Since  $R/I$  is finitely generated, we know  $\widehat{R/I} \cong R/I \otimes_R \hat{R} = \hat{R}/I\hat{R}$ . Thus we have  $R/I \rightarrow \hat{R}/I\hat{R}$  is injective. Of course, this just says if  $y \in R \setminus I$ , then  $y \in \hat{R} \setminus I\hat{R}$ , that is,  $y \notin I$  implies  $y \notin I\hat{R} \cap R$ . Thus  $I\hat{R} \cap R = I$ .

In particular, this says  $m^n \hat{R} \cap R = m^n$ .

**Proposition 2.38.** *Let  $R$  be Noetherian,  $p \in \text{Spec } R$ . Then  $E_R(R/p)$  is an  $R_p$ -module. Therefore,  $E_R(R/p) \cong E_R(R/p)_p \cong E_{R_p}(R_p/pR_p)$ .*

*Proof.* Let  $S$  be a multiplicatively closed subset and  $M$  an  $R$ -module. Suppose for all  $x \in M$  and  $s \in S$  there exists a unique  $x' \in M$  such that  $sx' = x$ . Then  $M \cong M_S$  via  $\frac{x}{s} = sx'$ . So it is enough to show there exists such an  $x'$ . Let  $s \in R \setminus p$  and  $x \in E_R(R/p)$ . Since  $\text{Ass } E_R(R/p) = \text{Ass } R/p = \{p\}$ , we see  $s$  is a non-zero-divisor on  $E_R(R/p)$ . Consider the diagram

$$\begin{array}{ccc} & E_R(R/p) & \\ & \uparrow 1-1 & \swarrow \exists f \\ 0 & \longrightarrow Rx & \xrightarrow{s} Rx \end{array}$$

Since  $E_R(R/p)$  is injective, there exists  $f$  making the diagram commute, that is,  $sf(x) = f(sx) = x$ . Let  $x' := f(x)$ . Then there exists  $x' \in E$  such that  $sx' = x$ . As  $x$  is a non-zero-divisor on  $E$ , we see  $x'$  is unique. By exercise, done.  $\square$

**Proposition 2.39.** *Every element of  $E_R(R/p)$  is annihilated by a power of  $p$ .*

*Proof.* Let  $x \in E_R(R/p) \setminus \{0\}$ . Then  $\text{Ass } Rx \subseteq \text{Ass}_R E_R(R/p) = \{p\}$ . Thus  $\text{Ass}_R Rx = \{p\}$ . Therefore,  $\sqrt{\text{Ann}_R Rx} = \text{Ann}_R Rx = p$  and  $p^n \subseteq \text{Ann}_R Rx$ .  $\square$

**Proposition 2.40.** *Let  $(R, m)$  be local,  $E = E_R(R/m)$ . Then  $\text{Ann}_R E = 0$ .*

*Proof.* Let  $r \in \text{Ann}_R E \setminus \{0\}$ . By Krull's Intersection Theorem, there exists  $n$  such that  $r \notin m^n$ . So  $0 = r \text{Hom}_R(R/m^n, E) \cong \bar{r} E_{R/m^n}(R/m)$  where  $\bar{r} \in R/m^n \setminus \{0\}$ . Since  $\dim R/m^n = 0$ , it is enough to show  $\text{Ann}_R E = 0$  when  $\dim R = 0$ . Suppose  $r \in \text{Ann}_R E$ . Then  $E \cong \text{Hom}_R(R/(r), E) = E_{R/(r)}(R/m)$ . If  $R$  is zero dimensional, then  $\lambda(R) < \infty$  and we have see  $\lambda(E) = \lambda(R^v) = \lambda(R) < \infty$ . So

$$\lambda_R(R) = \lambda_R(E) = \lambda_R(E_{R/(r)}(R/m)) = \lambda_{R/(r)}(E_{R/(r)}(R/m)) = \lambda_R(R/(r)).$$

Then, the short exact sequence  $0 \rightarrow (r) \rightarrow R \rightarrow R/(r) \rightarrow 0$  and the additivity of length give  $\lambda((r)) = 0$ . Thus  $r = 0$ .  $\square$

In particular, the proposition shows  $E$  is not finitely generated, except in the case that  $\dim R = 0$ .

**Remark.** Let  $R$  be Noetherian,  $p \in \text{Spec } R$ . Let  $r \in R$  such that there exists  $s \in R \setminus p$  with  $sr = 0$ . Then  $r \in \text{Ann}_R E_R(R/p)$ . In fact,  $\text{Ann}_R E_R(R/p) = \{r \in R \mid \frac{r}{1} = 0 \text{ in } R_p\}$ .

*Proof.* Suppose  $\frac{r}{1} = 0$  in  $R_p$ . As  $E_R(R/p)$  is an  $R_p$ -module, we see  $\frac{r}{1} \in \text{Ann}_{R_p} E_R(R/p)$ . So  $r \in \text{Ann}_R E_R(R/p)$ .  $\square$

Let  $(R, m)$  be local,  $E = E_R(R/m)$  and  $(-)^v = \text{Hom}_R(-, E)$ . For any  $R$ -module, there exists a natural map  $\theta_M : M \rightarrow M^{vv}$  defined by  $x \mapsto ev_x : M^v \rightarrow E$  where  $ev_x(\phi) = \phi(x)$ . For any  $R$ -module, we see  $\theta_M$  is injective: Let  $x \in M \setminus \{0\}$  so that  $Rx \cong R/(0 :_R x) \rightarrow R/m \hookrightarrow E_R(R/m)$ . So there exists a nonzero map  $\psi : Rx \rightarrow E$  and we can consider the diagram below

$$\begin{array}{ccc} & E_R(R/p) & \\ & \uparrow \psi & \swarrow \exists f \\ 0 & \longrightarrow Rx & \longrightarrow Rx \end{array}$$

As  $E_R(R/p)$  is injective, there exists  $f : M \rightarrow E$  such that  $f(x) = \psi(x) \neq 0$ . Hence  $ev_x(f) = f(x) \neq 0$  and  $\theta_M : M \rightarrow M^{vv}$  is injective.

**Proposition 2.41.** *Let  $\dim R = 0$ . Then  $\theta_R : R \rightarrow R^{vv}$  defined by  $r \mapsto ev_r$  is an isomorphism.*

*Proof.* By the above, it is enough to show  $\theta_R$  is surjective. We know  $0 \rightarrow R \xrightarrow{\theta_R} R^{vv} \rightarrow R/R^{vv} \rightarrow 0$  is exact. Of course,  $\lambda(R) = \lambda(R^v) = \lambda(R^{vv})$  and so the additivity of length says  $\lambda(R/R^{vv}) = 0$ . Hence  $\theta_R$  is an isomorphism.  $\square$

**Corollary 2.42.** *Let  $\dim R = 0$ . Then for any finitely generated  $R$ -module  $M$ ,  $\theta_M$  is an isomorphism.*

*Proof.* Since  $\theta_R$  is an isomorphism, we know  $\theta_{R^n}$  is an isomorphism for all  $n$ . Since  $R$  is Noetherian,  $M$  is finitely presented and so  $R^m \rightarrow R^n \rightarrow M \rightarrow 0$  is exact. Apply  $(-)^{vv}$  and the Five Lemma to get the desired result.  $\square$

Consider  $\theta_R : R \rightarrow R^{vv} = \text{Hom}_R(E, E)$  (since  $E = R^v$ ) defined by  $r \mapsto ev_r : E \rightarrow E$ . Let  $x \in E = \text{Hom}_R(R, E)$ . Then  $x := \psi_x : R \rightarrow E$  is defined by  $1 \mapsto x$ . Thus  $ev_r(x) = \psi_x(r) = rx$ . Hence  $\theta_r : R \rightarrow \text{Hom}_R(E, E)$  is given by multiplication by  $r$ .

**Proposition 2.43.** *Let  $(R, m, k)$  be local and  $E = E_R(k)$ . Then  $E$  is an  $\hat{R}$ -module and  $E = E_{\hat{R}}(k)$ .*

*Proof.* Let  $\hat{r} \in \hat{R}$  and  $x \in E$ . Then there exists  $n$  such that  $m^n x = 0$ . So there exists  $r \in R$  such that  $r - \hat{r} \in m^n \hat{R}$ . Define  $\hat{r}x := rx$ .

- This is well-defined: Suppose  $m^\ell x = 0$  and  $\hat{r} - r' \in m^\ell \hat{R}$ . We want to show  $r'x = rx$ . WLOG assume  $\ell \geq n$ . Then  $r - r' = (r - \hat{r}) + (\hat{r} - r') \in m^n \hat{R} \cap R = m^n$ . Thus  $(r - r')x = 0$  and so  $rx = r'x$ .

Hence  $E$  is an  $\hat{R}$ -module (and the action is compatible with the  $R$ -module structure). Now  $k \subseteq E$  is essential as  $R$ -modules and hence as  $\hat{R}$ -modules. We have

$$\begin{array}{ccc} & E_{\hat{R}}(k) & \\ & \uparrow 1-1 & \nearrow \\ 0 & \longrightarrow k & \xrightarrow{ess} E \end{array}$$

Thus  $k \subseteq E \subseteq E_{\hat{R}}(k)$ . If we show  $k \subseteq E_{\hat{R}}(k)$  is essential as  $R$ -modules, then the maximality of  $k \subseteq E_R(k)$  implies  $E_R(k) = E_{\hat{R}}(k)$ . Let  $y \in E_{\hat{R}}(k) \setminus \{0\}$ . Then  $\hat{R}y \cap k \neq \{0\}$ . Therefore  $\hat{r} \in \hat{R}$  such that  $\hat{r}y \in k \setminus \{0\}$ . Now there exists  $n$  such that  $m^n y = 0$ . Also there exists  $r \in R$  such that  $r - \hat{r} \in m^n \hat{R}$ . Then  $(r - \hat{r})y = 0$  implies  $ry = \hat{r}y \in k \setminus \{0\}$ . Hence  $Ry \cap k \neq \{0\}$ .  $\square$

**Theorem 2.44.** *Let  $(R, m)$  be local,  $E = E_R(k)$ . Then the map  $\theta : \hat{R} \rightarrow \text{Hom}_R(E, E) = R^{vv}$  given by  $\theta(\hat{r}) = \mu_{\hat{r}}$  is an isomorphism.*

*Proof.* We know  $\hat{r} \in \ker \theta$  if and only if  $\hat{r} \in \text{Ann}_{\hat{R}} E = \text{Ann}_{\hat{R}} E_{\hat{R}}(\hat{k}) = 0$ . Thus  $\theta$  is injective. Let  $E_n = (0 :_E m^n) \cong \text{Hom}_R(R/m^n, E) \cong E_{R/m^n}(k)$ . Then  $E_n \supseteq E_{n+1}$  for all  $n$  and  $f(E_n) \subseteq E_n$  where  $f \in \text{Hom}_R(E, E)$ . Let  $f_n = f|_{E_n}$  so  $f_n : E_n \rightarrow E_n$  is an  $R/m^n$  homomorphism. Now  $\dim R/m^n = 0$  and so  $f_n$  is given by multiplication by  $\hat{r}_n \in R/m^n$  by some  $r_n \in R$ .

*Claim.*  $\{r_n\}$  is a Cauchy Sequence.

*Proof.* Note  $f_n|_{E_{n+1}} = f_{n+1}$ . Let  $x \in E_{n+1}$ . Then  $(r_n - r_{n+1})x = r_n x - r_{n+1} x = f_n(x) - f_{n+1}(x) = 0$ . Thus  $f_n - f_{n+1} \in \text{Ann}_{E_{n+1}} = \text{Ann}_R E_{R/m^{n+1}}(k) = m^{n+1}$  (since  $\text{Ann}_{R/m^{n+1}} E_{R/m^{n+1}}(k) = 0$  as we are local).

Let  $\hat{r} \in \hat{R}$  be defined by  $\hat{r} = \lim_{n \rightarrow \infty} r_n$ , that is,  $\hat{r} - r_n \in m^n \hat{R}$  for all  $n$ .

*Claim.*  $f = \mu_{\hat{r}}$ .

*Proof.* Let  $x \in E$ . Then  $m^n x = 0$  for some  $n$  which implies  $x \in E_n$  for some  $n$ . Then  $f(x) = f_n(x) = r_n x = \hat{r}x$ .

Thus  $E^v = \hat{R}$ , which implies  $R^{vv} = \hat{R}$  and so  $\hat{R}^v = \hat{R}$ .  $\square$

We've shown  $E_R(k) = E_{\hat{R}}(k)$ . Let  $(R, m)$  be local,  $M$  an  $R$ -module. Then  $\text{Supp}_R M \subseteq \{m\}$  (if and only if for all  $x \in M$  there exists  $n$  such that  $m^n x = 0$ ) and in the same way  $M$  is an  $\hat{R}$ -module (If  $x \in M$ ,  $\hat{r} \in \hat{R}$ , and  $m^n x = 0$  there exists  $r \in R$  such that  $\hat{r} - r \in m^n \hat{R}$ . Define  $\hat{r}x := rx$ ). Therefore, the  $R$ -submodules of  $M$  are precisely the  $\hat{R}$ -submodules of  $M$ . Hence  $M$  has ACC or DCC as an  $\hat{R}$ -module if and only if  $M$  has ACC or DCC as an  $\hat{R}$ -module.

Notice, if we apply the above theorem to  $\hat{R}$ , we get (since  $\hat{\hat{R}} = \hat{R}$ ) that  $\hat{R} \rightarrow \text{Hom}_{\hat{R}}(E, E)$  defined by  $\hat{r} \mapsto \mu_{\hat{r}}$  is an isomorphism. Hence, if  $R$  is complete, we see by the commutivity of the diagram

$$\begin{array}{ccc} R & \xrightarrow{\cong} & \text{Hom}_R(E, E) \\ \downarrow = & & \downarrow \cong \\ R & \longrightarrow & \text{Hom}_R(\text{Hom}_R(R, E), E) = R^{vv} \end{array}$$

that the bottom row is an isomorphism and hence  $R = R^{vv}$ .

**Proposition 2.45.** *If  $(R, m, k)$  is local and  $E = E_R(k)$ , then  $E$  is Artinian.*

*Proof.* Since  $\text{Supp } E \subseteq \{m\}$ , have  $E$  is Artinian as an  $R$ -module if and only if  $E$  is Artinian as an  $\hat{R}$ -module. WLOG, since  $E_R(k) = E_{\hat{R}}(k)$ , assume  $R$  is complete. Consider the descending chain  $\cdots \subseteq N_{i+2} \subseteq N_{i+1} \subseteq N_i \subseteq \cdots \subseteq N_0 = E$ . Then we have injections  $N_{i+2} \xrightarrow{\psi_{i+2}} N_{i+1} \xrightarrow{\psi_{i+1}} N_i \cdots$ . Apply  $(-)^v = \text{Hom}_R(-, E)$  to get surjections  $E^v \xrightarrow{\psi_1^v} N_1^v \xrightarrow{\psi_2^v} N_2^v \cdots$ . Now  $E^v = \text{Hom}_R(E, E) = R$  as  $R$  is complete. Let  $I_i = \ker(\psi_i^v \circ \cdots \circ \psi_1^v : R \rightarrow N_i^v)$ . Then  $I_i \subseteq I_{i+1} \subseteq \cdots$  is an ascending chain in  $R$  which is Noetherian. Thus there exists  $i$  such that  $I_i = I_{i+1} = \cdots$ . therefore  $N_k^v \rightarrow N_{k+1}^v$  is an isomorphism for all  $k \geq i$ . Let  $C_k = \text{coker } \psi_k$ . This gives us the exact sequence  $0 \rightarrow N_{k+1} \rightarrow N_k \rightarrow C_k \rightarrow 0$ . Apply  $(-)^v$  to get  $0 \rightarrow C_k^v \rightarrow N_k^v \xrightarrow{\cong} N_{k+1}^v \rightarrow 0$ . Thus  $C_k^v = 0$  which implies  $C_k = 0$  for all  $k \geq i$ . So  $\psi_k$  is an isomorphism for all  $k \geq i$ . Thus  $N_k = N_{k+1}$  for all  $k \geq i$ .  $\square$

Let  $M \neq 0$  be an Artinian  $R$ -module for  $(R, m)$  local. Let  $p \in \text{Ass}_R M$ . Then  $0 \rightarrow R/p \rightarrow M$  is exact, which implies  $R/p$  is Artinian (as it is a submodule of an Artinian module) and so  $R/p$  is Artinian as a ring. Thus  $\dim R/p = 0$  and  $p = m$ . Thus  $\text{Ass}_R M = \{m\} = \text{Supp}_R M$ .

If  $M$  is Artinian, then  $\text{soc}_R M$  is an Artinian  $R$ -module and hence  $\text{soc}_R M$  is an Artinian  $R/m$ -module. Thus  $\dim_k \text{soc}_R M < \infty$ .

**Theorem 2.46.** *Let  $(R, m)$  be local,  $E = E_R(R/m)$ ,  $M$  an  $R$ -module. TFAE*

- (1)  $M$  is Artinian
- (2)  $\text{Supp}_R M \subseteq \{m\}$  and  $\text{soc}_R M$  is finitely generated.
- (3) There exists  $n$  such that  $0 \rightarrow M \rightarrow E^n$  is exact.

*Proof.* (1)  $\Rightarrow$  (2) is proved above. For (3)  $\Rightarrow$  (1), we proved  $E$  is Artinian and so  $E^n$  is. Since submodules of Artinian modules are Artinian, done.

For (2)  $\Rightarrow$  (3), we have that every element of  $M$  is annihilated by a power of the maximal ideal. This means  $\text{soc}_R M \subseteq M$  is essential. Now  $\text{soc}_R M$  is finitely generated and hence  $\text{soc}_R M \cong (R/m)^n$  for some  $n$ . Since  $R/m \hookrightarrow E$ , we see  $\text{soc}_R M \hookrightarrow E^n$  and so we have the diagram below:

$$\begin{array}{ccc} & & E^n \\ & \uparrow & \swarrow \\ & 1-1 & \\ 0 & \longrightarrow & \text{soc}_R M \xrightarrow{\text{ess}} M \end{array}$$

Thus there exists an injective map  $M \hookrightarrow E^n$ .  $\square$

**Theorem 2.47 (Matlis Duality).** *Let  $(R, m)$  be complete,  $E = E_R(R/m)$ , and  $(-)^v = \text{Hom}_R(-, E)$ . Then  $(-)^v$  gives a bijective contravariant correspondence*

$$\langle\langle \text{Noetherian } R\text{-Modules} \rangle\rangle \leftrightarrow \langle\langle \text{Artinian } R\text{-modules} \rangle\rangle$$

where a Noetherian module  $M \mapsto M^v$  and an Artinian module  $N \mapsto N^v$  with  $M^{vv} \cong M$  and  $N^{vv} \cong N$  via the natural maps that send  $x \mapsto ev_x$ .

*Proof.* Let  $M$  be Noetherian. Then it is finitely generated and there exists  $n$  such that  $R^n \rightarrow M \rightarrow 0$  is exact. Apply  $(-)^v$  to get  $0 \rightarrow M^v \rightarrow (R^n)^v \cong E^n$ , which is artinian. Thus  $M^v$  is a submodule of an artinian module and therefore artinian. If  $N$  is artinian, then the previous theorem gives  $0 \rightarrow N \rightarrow E^n$ . Apply  $(-)^v$  to get  $R^n = (E^n)^v \rightarrow N^v \rightarrow 0$ . Thus  $N^v$  is finitely generated and hence Noetherian.

To show the natural map  $M \rightarrow M^{vv}$  defined by  $x \mapsto ev_x$  is an isomorphism for  $M$  finitely generated. We have shown this for  $R$ , and hence it holds for direct sums, and for finitely presented modules by the Five Lemma. Similarly, if  $N$  is Artinian, we have  $0 \rightarrow N \rightarrow E^n \rightarrow \text{coker}$  is exact. Now the cokernel is Artinian as it is the homomorphic image of an Artinian module. Apply the theorem again to get  $0 \rightarrow N \rightarrow E^n \rightarrow E^m$ . Apply the double dual and the Five Lemma since  $E^n \cong (E^n)^{vv}$ .  $\square$

For noncomplete rings, there is a bijective correspondence

$$\langle\langle \text{Noetherian } \hat{R}\text{-modules} \rangle\rangle \leftrightarrow \langle\langle \text{Artinian } R\text{-modules} \rangle\rangle .$$

### 2.3. Canonical Modules.

**Definition.** *Let  $(R, m, k)$  be local,  $M \neq 0$  a finitely generated  $R$ -module with  $\text{depth } M = t$ . Then  $r(M) = \dim_k \text{Ext}_R^t(k, M)$  is the **type** of  $M$ .*

**Definition.** *Let  $(R, m, k)$  be CM. A finitely generated maximal Cohen-Macaulay module  $C$  of type 1 of finite injective dimension is called a **canonical module** of  $R$ .*

**Remark.**  $C$  is a canonical module of  $R$  if and only if  $\dim_k \text{Ext}_R^i(k, C) = \delta_{i,d}$ .

*Proof.* For the backward direction,  $\dim_k \text{Ext}_R^i(k, C) = 0$  for  $i \neq d$  implies  $\text{depth } C = \inf\{i \mid \text{Ext}_R^i(k, C) \neq 0\} = d$  and so  $C$  is maximal Cohen-Macaulay. It also shows  $\text{id}_R C = \sup\{i \mid \text{Ext}_R^i(k, C) \neq 0\} = d$  and so  $\text{id}_R C < \infty$ . Finally, since  $\dim_k \text{Ext}_R^d(k, C) = 1$ ,  $C$  is of type 1.

For the forward direction, if  $C$  is finitely generated and of finite injective dimension, then  $\text{depth } C \leq \text{id}_R C \leq \dim R$ . Since  $R$  is Cohen Macaulay, these are all equal.  $\square$

#### Examples.

- (1) If  $\dim R = 0$ , then  $E_R(R/m)$  is injective, MCM (since  $\dim R = 0$ ), finitely generated (it has finite length), and  $r(E) = \dim_k \text{Hom}_R(R/m, E) = \dim_k E_{R/m}(R/m) = \dim_k R/m = 1$ . Thus  $E_R(R/m)$  is a canonical module. If  $C$  is another canonical module, then  $\text{id}_R C = 0$  implies  $C$  is injective. Then  $C$  must be a sum of indecomposables but since  $E$  is the only one,  $C \cong E$ .
- (2) If  $R$  is Gorenstein, then  $R$  is the canonical module (in fact this is an if and only if!).

**Lemma 2.48.** *Let  $(R, m, k)$  be local,  $\phi : M \rightarrow N$  a homomorphism of finitely generated  $R$ -modules and  $\underline{x}$  an  $N$ -regular sequence. If  $\phi \otimes 1_{R/(\underline{x})}$  is an isomorphism, then so is  $\phi$ .*

*Proof.* First we show  $\phi$  is onto. Let  $C = \text{coker } \phi$ . Then  $M \rightarrow N \rightarrow C \rightarrow 0$  is exact. Apply  $- \otimes R/(\underline{x})$  to get  $M \otimes R/(\underline{x}) \xrightarrow{\phi \otimes 1} N \otimes R/(\underline{x}) \rightarrow C \otimes R/(\underline{x}) \rightarrow 0$  is exact. Since  $\phi \otimes 1_{R/(\underline{x})}$  is an isomorphism,  $C \otimes R/(\underline{x}) = 0$ . Since  $R$  is local, either  $C = 0$  or  $R/(\underline{x}) = 0$ . Of course the latter is not, and so  $C = 0$ .

To show  $\phi$  is injective, let  $K = \ker \phi$  and first suppose  $\underline{x} = x$ . Then  $0 \rightarrow K \xrightarrow{i} M \xrightarrow{\phi} N \rightarrow 0$  is exact. Since  $\underline{x}$  is  $N$ -regular,  $0 \rightarrow K/\underline{x}K \rightarrow M/\underline{x}M \rightarrow N/\underline{x}N \rightarrow 0$  is exact [Apply  $- \otimes R/(\underline{x})$  to the above sequence and get a long

exact sequence on Tor. Of course,  $\text{Tor}_1^R(R/(\underline{x}), N) = 0$  as if we take  $R^\alpha \rightarrow R \xrightarrow{x} R \rightarrow 0$  and tensor with  $N$ , we get  $R^\alpha \otimes N \rightarrow N \xrightarrow{x} N \rightarrow 0$  and thus the kernel is 0 as  $x$  is  $N$ -regular]. Now,  $K/\underline{x}K \cong K \otimes_R R/(\underline{x}) = 0$  as  $\phi \otimes 1$  is an isomorphism. By NAK,  $K = 0$ . Continue inductively.  $\square$

**Proposition 2.49.** *Let  $(R, m, k)$  be CM, dimension  $d$ , and  $C$  a maximal Cohen-Macaulay  $R$ -module. Then*

- (1) *Suppose  $M$  is MCM with  $\text{Ext}_R^j(M, C) = 0$  for all  $j > 0$ . Then  $\text{Hom}_R(M, C)$  is maximal Cohen-Macaulay and for any  $R$ -sequence  $\underline{x}$ , we have  $\text{Hom}_R(M, C) \otimes_R R/\underline{x}R = \text{Hom}_{R/\underline{x}R}(M/\underline{x}M, C/\underline{x}C)$ .*
- (2) *Assume  $\text{id}_R C < \infty$  and  $M$  is a Cohen Macaulay  $R$ -module of dimension  $t$ . Then*
  - (a)  $\text{Ext}_R^j(M, C) = 0$ , for  $j \neq d - t$
  - (b)  $\text{Ext}_R^{d-t}(M, C)$  is Cohen Macaulay of  $\dim t$ .

*Proof.* (1) Let  $x \in m$  be  $R$ -regular.

*Claim.*  $x$  is  $C$ -regular.

*Proof.* We know  $\text{depth } C \leq \dim R/p$  for all  $p \in \text{Ass } C$ . Since  $\text{depth } C = \dim R$ , such  $p$  must be minimal primes. But  $x$  is  $R$ -regular and so  $x$  is not in any minimal primes. Thus  $x$  is  $C$ -regular.

So the sequence  $0 \rightarrow C \xrightarrow{x} C \rightarrow C/xC \rightarrow 0$  is exact. Apply  $\text{Hom}_R(M, -)$  to get  $0 \rightarrow \text{Hom}_R(M, C) \xrightarrow{x} \text{Hom}_R(M, C) \rightarrow \text{Hom}_R(M, C/xC) \rightarrow 0$ , which is exact as  $\text{Ext}_R^j(M, C) = 0$  for all  $j > 0$ . Then  $x$  is regular on  $\text{Hom}_R(M, C)$ .

*Claim.* If  $R$  is a commutative ring,  $M, N$   $R$ -modules and  $I \subseteq R$  an ideal, then  $\text{Hom}_{R/I}(M/IM, N/IN) \cong \text{Hom}_R(M, N/IN)$ .

*Proof.* If  $\phi \in \text{Hom}_{R/I}(M/IM, N/IN)$ , then  $\bar{\phi} : M \rightarrow M/IM \rightarrow N/IN$  is in  $\text{Hom}_R(M, N/IN)$ .

If  $\psi \in \text{Hom}_R(M, N/IN)$ , then  $\psi(IM) \subseteq I\psi(M) = 0$ . Thus  $\psi \in \text{Hom}_{R/I}(M/IM, N/IN)$ .

Now, we have

$$\begin{aligned} \text{Hom}_{R/xR}(M/xM, C/xC) &\cong \text{Hom}_R(M, C/xC) \\ &\cong \text{Hom}_R(M, C)/x \text{Hom}_R(M, C) \\ &\cong \text{Hom}_R(M, C) \otimes_R R/xR \end{aligned}$$

Now  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$  exact implies

$$\cdots \rightarrow \underbrace{\text{Ext}_R^i(M, C)}_{=0} \rightarrow \text{Ext}_R^{i+1}(M/xM, C) \rightarrow \underbrace{\text{Ext}_R^{i+1}(M, C)}_{=0} \rightarrow \cdots$$

is exact. Thus  $\text{Ext}_R^j(M/xM, C) = 0$  for all  $j \geq 2$  and by Rees' isomorphism  $\text{Ext}_{R/(\underline{x})}^i(M/xM, C/xC) = 0$  for all  $i \geq 1$ . Thus we can use induction. In general, let  $\underline{x} = x_1, \dots, x_n$  for  $n > 1$ . Then for  $\bar{R} = R/(x_1, \dots, x_{n-1})R$  we have

$$\begin{aligned} \text{Hom}_R(M, C) \otimes R/\underline{x}R &\cong \text{Hom}_R(M, C) \otimes_R \bar{R} \otimes_{\bar{R}} \bar{R}/x_n \bar{R} \\ &\cong \text{Hom}_{\bar{R}}(M/(x_1, \dots, x_{n-1})M, C/(x_1, \dots, x_{n-1})C) \otimes_{\bar{R}} \bar{R}/x_n \bar{R} \\ &\cong \text{Hom}_{R/(\underline{x})}(M/\underline{x}M, C/\underline{x}C) \text{ by the base case.} \end{aligned}$$

Note that this shows  $\text{Hom}_R(M, C)$  is maximal Cohen Macaulay as  $x_1, \dots, x_n$  is a regular sequence.

- (2) (a) Recall  $\text{grade}(I, C) = \dim C - \dim C/IC$  for all ideals  $I$ . Let  $I = \text{Ann}_R M$ . Then  $I \subseteq \text{Ann}_R C/IC$ . We have  $\dim C/IC \leq \dim R/I = \dim M = t$ . Thus  $\text{grade}(I, C) \geq d - t$ . Therefore  $\text{Ext}_R^j(R/I, C) = 0$  for  $j < d - t$  (in particular, this says there exists a  $C$ -regular sequence in  $I$  of length  $d - t$ ).

Consider a prime filtration of  $M : 0 = N_k \subsetneq N_{k-1} \subsetneq \cdots \subsetneq N_0 = M$  where  $(N_i/N_{i+1})_{p_i} \cong (R/p_i)_{p_i} \cong k(p_i) \neq 0$ . Then  $M_{p_i} \neq 0$  and thus  $p_i \subseteq \text{Ann}_R M = I$  for all  $i$ . Thus  $\text{Ext}_R^j(R/p_i, C) = 0$  for  $j < d - t$  and for all  $i$  since a  $C$ -regular sequence in  $I$  is also in  $p_i$ . Finally, to show  $\text{Ext}_R^j(M, C) = 0$  we use induction on  $k$ , the length of our filtration. If  $k = 1$ , then  $M \cong R/p$  for some  $p \in \text{Spec } R$  and we are done. Assume  $k > 1$ . Then we have an exact sequence  $0 \rightarrow N_1 \rightarrow M \rightarrow M/N_1 \rightarrow 0$ . Since the length of a filtration for  $N_1$  is  $< k$

we have  $\text{Ext}_R^j(N, C) = 0$  for  $j < d - t$  by induction and since  $M/N_1 \cong R/p$  we have  $\text{Ext}_R^j(M, N_1, C) = 0$  for  $j < d - t$  by the base case. Apply  $\text{Hom}(-, C)$  to get  $\text{Ext}_R^j(M, C) = 0$  for  $j < d - t$ .

For  $j > d - t$ , we will induct on  $t = \dim M$ . For  $t = 0$ , we have  $\text{id}_R C = \text{depth } R = d$  by Theorem 2.26, that is,  $\text{Ext}_R^j(M, C) = 0$  for  $j > d$ . Now suppose  $t > 0$  and let  $x \in m$  by  $M$ -regular. Then the exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$  induces the exact sequence

$$\text{Ext}_R^j(M, C) \xrightarrow{x} \text{Ext}_R^j(M, C) \rightarrow \text{Ext}_R^{j+1}(M/xM, C).$$

We know  $M/xM$  is a  $(t - 1)$ -dimensional Cohen Macaulay module as  $x$  is  $M$ -regular. By induction,  $\text{Ext}_R^{j+1}(M/xM, C) = 0$  for  $j > d - t$  (i.e.,  $j + 1 > d - (t - 1)$ ). By NAK,  $\text{Ext}_R^j(M, C) = 0$  for  $j > d - t$ .

(b) We induct on  $t = \dim M$ . If  $t = 0$  then  $\text{Ann Ext}_R^d(M, C) \subseteq \text{Ann } M$ . So as long as  $\text{Ext}_R^d(M, C) \neq 0$ , we have  $t = 0 = \dim(\text{Ext}_R^d(M, C))$ . To show it is not 0, we induct on  $\lambda(M)$ . If  $\lambda(M) = 1$ , then  $M \cong k$  and then of course  $\text{Ext}_R^d(k, C) \neq 0$  as  $d = \text{depth } R$ . For  $\lambda(M) > 1$ , consider  $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ . Thus  $\dim \text{Ext}_R^d(M, C) = 0$ , and so it is Cohen Macaulay. If  $t > 1$ , let  $x \in m$  be  $M$ -regular. Then by part (1), the short exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$  induces the short exact sequence

$$0 \rightarrow \text{Ext}_R^{d-t}(M, C) \xrightarrow{x} \text{Ext}_R^{d-t}(M, C) \rightarrow \text{Ext}_R^{d-(t-1)}(M/xM, C) \rightarrow 0.$$

Now the last term is Cohen Macaulay of dimension  $t - 1$  by induction and of course

$$\text{Ext}_R^{d-(t-1)}(M/xM, C) \cong \text{Ext}_R^{d-t}(M, C)/x \text{Ext}_R^{d-t}(M, C).$$

Since  $x$  is  $\text{Ext}_R^{d-t}(M, C)$ -regular (multiplication by  $x$  is injective by the above short exact sequence), we see  $\text{Ext}_R^{d-t}(M, C)$  is Cohen Macaulay of dimension  $t$ . □

**Proposition 2.50.** *Let  $(R, m, k)$  be Cohen-Macaulay. Let  $C$  and  $C'$  be canonical modules. Then*

- (1) *For all maximal  $R$ -sequences  $\underline{x}$ , we have  $C/\underline{x}C \cong E_{R/(\underline{x})}(k)$ .*
- (2)  *$C \cong C'$*
- (3)  *$\text{Hom}_R(C, C') \cong R$ . Moreover, if  $\phi \in \text{Hom}_R(C, C')$  generates it, then  $\phi$  is an isomorphism.*
- (4) *The canonical map  $R \rightarrow \text{End}_R(C)$  defined by  $r \mapsto \mu_R$  is an isomorphism.*

*Proof.* (1) Let  $\underline{x}$  be a maximal  $R$ -sequence. By Corollary 2.18, since  $\underline{x}$  is  $C$ -regular, we have  $\text{id}_R C/\underline{x}C = 0$  and so  $C/\underline{x}C$  is injective. Thus it is a sum of indecomposable injectives, that is,  $C/\underline{x}C = E_{R/(\underline{x})}(k)^t$ . By Rees' Isomorphism, we see  $r(C/\underline{x}C) = 1$ . Thus  $t = 1$  and  $C/\underline{x}C \cong E_{R/(\underline{x})}(k)$ .

(2) By (1) and Lemma 2.48.

(3) By (1),  $C/\underline{x}C \cong E_{R/(\underline{x})}(k) \cong C'/\underline{x}C'$ . Now

$$\begin{aligned} \text{Hom}_R(C, C') \otimes_R R/(\underline{x}) &\cong \text{Hom}_{R/(\underline{x})}(C/\underline{x}C, C'/\underline{x}C') \text{ by Proposition 2.49} \\ &\cong \text{Hom}_{R/(\underline{x})}(E_{R/(\underline{x})}(k), E_{R/(\underline{x})}(k)) \\ &\cong R/(\underline{x}) \end{aligned}$$

By Nakayama's Lemma, as  $\underline{x} \subseteq m$

$$\begin{aligned} \mu_R(\text{Hom}_R(C, C')) &= \mu_R(\text{Hom}_R(C, C')/m \text{Hom}_R(C, C')) \\ &= \mu_R(\text{Hom}_R(C, C')/\underline{x} \text{Hom}_R(C, C')) \\ &= \mu_R(\text{Hom}_R(C, C') \otimes R/(\underline{x})) \\ &= \mu_R(R/(\underline{x})) = 1 \end{aligned}$$

Let  $\phi$  generated  $\text{Hom}_R(C, C')$ . By Proposition 2.49,  $\text{Hom}_R(C, C')$  is maximal Cohen Macaulay and so  $\underline{x}$  is regular. Consider the isomorphisms  $\text{Hom}_R(C, C') \otimes R/(\underline{x}) \cong \text{Hom}_{R/(\underline{x})}(C/\underline{x}C, C'/\underline{x}C') \cong R/(\underline{x})$  defined by  $\phi \otimes \bar{1} \mapsto \psi \mapsto \bar{1}$ . Notice that the latter isomorphism is given by  $\mu_r \leftrightarrow r$ . As  $\bar{1}$  generates  $R/(\underline{x})$ , we see  $\mu_1$  generates  $\text{Hom}_{R/(\underline{x})}(C/\underline{x}C, C'/\underline{x}C')$  and thus  $\psi$  is an isomorphism. So  $\phi \otimes \bar{1}$  is and by Lemma 2.48 we see

$\phi$  is an isomorphism. Let  $\psi : R \rightarrow \text{Hom}_R(C, C')$  be defined by  $1 \mapsto \phi$ . Then  $R/(\underline{x}) \cong R \otimes R/(\underline{x}) \xrightarrow{\psi \otimes 1} \text{Hom}_R(C, C') \otimes R/(\underline{x})$  is defined by  $\bar{1} \mapsto \phi \otimes \bar{1}$ . Since  $\underline{x}$  is regular on  $\text{Hom}_R(C, C')$ , Lemma 2.48 tells us  $\psi$  is an isomorphism.

- (4) Let  $\phi$  generate  $\text{End}_R(C)$ . Define  $\psi : R \rightarrow \text{End}_R(C) = R\phi$  by  $r \mapsto \mu_r$ . In particular,  $1 \mapsto \mu_1 = 1_C = r\phi$  for some  $r$ . So  $C = 1_C(C) = r\phi(C) \subseteq rC \subseteq C$ . Thus  $C = rC$ . If  $r \in \mathfrak{m}$ , then NAK implies  $C = 0$ , a contradiction. Thus  $r$  is a unit and thus  $\psi = r^{-1}\mu_1$  is onto. To show injectivity, say  $0 = \mu_s = s\mu_1 = s(r\phi) = (s\bar{1})\phi$ . Then  $sr = 0$  as  $R = R\phi$ . Of course,  $r$  is a unit and so  $s = 0$ .  $\square$

By Proposition 2.50, the canonical module is unique up to isomorphism. Thus we shall define  $\omega_R$  as **the canonical module** of  $R$ .

**Theorem 2.51.** *Let  $(R, \mathfrak{m}, k)$  be Cohen Macaulay.*

- (1) For all  $R$ -sequences  $\underline{x}$ , we have  $\omega_{R/\underline{x}R} \cong \omega_R/\underline{x}\omega_R$ .
- (2) For all  $p \in \text{Spec } R$ , we have  $\omega_{R/p} \cong (\omega_R)_p$ .
- (3)  $\omega_{\hat{R}} \cong \widehat{\omega_R}$ .

*Proof.* (1) By Rees' Isomorphism, we have  $\text{Ext}_{R/(\underline{x})}^{i-1}(k, \omega_{R/\underline{x}R}) = \text{Ext}_R^i(k, \omega_R) = k_{i,d}^\delta$  as  $\omega_R$  is the canonical module for  $R$ . Thus  $\dim_k \text{Ext}_{R/(\underline{x})}^i(k, \omega_{R/\underline{x}R}) = \delta_{i,d-1}$  and so  $\omega_{R/\underline{x}R}$  is the canonical module for  $R/\underline{x}R$ .  
(2) As  $\text{id}_{R_p}(\omega_R)_p < \text{id}_R \omega_R < \infty$ , we see  $(\omega_R)_p$  has finite injective dimension. Let  $\underline{x}$  be an  $R$ -sequence whose image in  $R_p$  is a maximal  $R_p$ -sequence. Set  $M = (\omega_R)_p/\underline{x}(\omega_R)_p$ . By Corollary 2.18,  $M$  is injective over  $R_p/\underline{x}R_p$ . Set  $r = r(M)$ . Then  $M = E_{R_p/\underline{x}R_p}(k(p))^r$ . We want to show  $r = 1$ . Notice

$$\text{Hom}_{R_p/\underline{x}R_p}(M, M) = \text{Hom}_{R_p/\underline{x}R_p}(E_{R_p/\underline{x}R_p}(k(p))^r, E_{R_p/\underline{x}R_p}(k(p))^r) = (R_p/\underline{x}R_p)^{r^2}.$$

On the other hand,

$$\text{Hom}_{R_p/\underline{x}R_p}(M, M) = \text{Hom}_{R/\underline{x}R}(\omega_{R/\underline{x}R}, \omega_{R/\underline{x}R})_p = \text{Hom}_{R/\underline{x}R}(\omega_{R/\underline{x}R}, \omega_{R/\underline{x}R})_p = (R/\underline{x}R)_p \cong (R_p/\underline{x}R_p).$$

Thus  $r^2 = 1$ , which says  $r = 1$ . Thus  $M$  is type 1 over  $R_p/\underline{x}R_p$ , which implies  $(\omega_R)_p$  has type 1 by Rees' isomorphism.

To show  $(\omega_R)_p$  is maximal Cohen Macaulay, we first prove the following claim.

*Claim.*  $\text{grade}(p, \omega_R) = \text{grade}(p, R)$ .

*Proof.* We know if  $x$  is  $R$ -regular then it is  $\omega_R$ -regular, so we need to show the opposite.

Suppose  $x$  is  $\omega_R$ -regular. Then  $R \cong \text{Hom}_R(\omega_R, \omega_R)$ . Suppose  $xr = 0$ . Then  $0 = x\mu_r$  implies  $x(r\omega_R) = 0$ . By  $r\omega_R \subseteq \omega_R$  and  $x$  is  $\omega_R$ -regular. So  $r\omega_R = 0$  which implies  $r = 0$  as  $\text{Ann } \omega_R = 0$ . Use induction to get the result for an  $\omega_R$ -sequence  $\underline{x}$  as  $\text{grade}(p/(x), \omega_{R/x\omega_R}) = \text{grade}(p/x, R/x)$ .

Thus we have  $\text{depth}(\omega_R)_p = \text{grade}(p, \omega_R) = \text{grade}(p, R) = \text{ht } p = \dim R_p$  as  $R$  is Cohen Macaulay.

- (3) We have  $\text{Ext}_{\hat{R}}^i(k, \widehat{\omega_R}) \cong \widehat{\text{Ext}_R^i(k, \omega_R)} \cong \text{Ext}_R^i(k, \omega_R) \otimes \hat{R} \cong k^{\delta_{id}} \otimes \hat{R} \cong k^{\delta_{id}}$ . Thus  $\dim_k \text{Ext}_{\hat{R}}^i(k, \widehat{\omega_R}) = \delta_{id}$  and so  $\widehat{\omega_R}$  is the canonical module for  $\hat{R}$ .  $\square$

**Definition.** *The trivial extension of  $R$  by  $M$  is denoted  $R * M = R \oplus M$  as  $R$ -modules with multiplication defined by  $(a, x)(b, y) = (ab, ay + bx)$  for  $a, b \in R, x, y \in M$ .*

**Facts.**

- (1)  $R * M$  is a ring.
- (2) If  $M$  is finitely generated, then  $(R, \mathfrak{m})$  Noetherian (resp Artinian) implies  $R * M$  is with maximal ideal  $\mathfrak{m} * M$ . In fact, primes of  $R * M$  are of the form  $p * M$  where  $p \in \text{Spec } R$ . So  $\dim R = \dim R * M$ .

**Theorem 2.52.** *If  $(R, \mathfrak{m}, k)$  is Cohen Macaulay, then TFAE*

- (1)  $R$  admits a canonical module.

(2)  $R$  is the homomorphic image of a Gorenstein local ring.

*Proof.* For (1)  $\Rightarrow$  (2),  $R$  is the homomorphic image of  $R * \omega_R$ . So it is enough to show  $R * \omega_R$  is Gorenstein, that is, Cohen Macaulay of type 1. Let  $\underline{x} = x_1, \dots, x_d$  be an  $R$ -sequence of maximal length. Then  $\underline{x}$  is an  $R * \omega_R$  sequence ( $(x, 0)(a, y) = (xa, xy) = (0, 0)$  if and only if  $a = 0, y = 0$ ). Thus  $R * \omega_R$  is Cohen Macaulay. Notice  $(R * \omega_R) / \underline{x}(R * \omega_R) \cong (R / \underline{x}R) * (\omega_R / \underline{x}\omega_R)$ . Also, by Proposition 2.50,  $\omega_R / \underline{x}\omega_R \cong \omega_{R / (\underline{x})} \cong E_{R / (\underline{x})}$ . So we may assume  $R$  is Artinian (since our original ring is Gorenstein if and only if it is after modding by a regular sequence). It's left to show  $R' = R * E_R(k)$  is type 1. Let  $(a, x) \in \text{soc } R'$ . So  $(b, 0)(a, x) = (ba, bx)$  for any  $b \in m$  implies  $a \in \text{soc } R$  and  $x \in \text{soc } E_R(k)$ . Assume  $a \neq 0$ . Then the exact sequence  $R \xrightarrow{a} R \rightarrow R / (a) \rightarrow 0$  induces the exact sequence  $E_{R / (a)}(k) \rightarrow E_R(k) \xrightarrow{a} E_R(k) \rightarrow 0$ . Recall  $\lambda(M) = \lambda(M^v)$  and so we have  $\lambda(E_{R / (a)}(k)) = \lambda(R / (a)) < \lambda(R) = \lambda(E_R(k))$ . So additivity of length says multiplication by  $a$  is not the zero map. So  $ay \neq 0$  for some  $y \in E_R(k)$ . Then  $(0, y)(a, x) = (0, ay) \neq (0, 0)$ , a contradiction as  $(a, x) \in \text{soc } R'$ . Thus  $a = 0$  and  $\text{soc } R' = \{(0, x) | x \in \text{soc } E_R(k)\}$ . Thus  $\text{soc } R * E_R(k) \cong \text{soc } E_R(k)$ . Thus  $r(R * E_R(k)) = r(E_R(k)) = 1$ . For (2)  $\Rightarrow$  (1), we use the following theorem.  $\square$

**Theorem 2.53.** *Let  $(R, m)$  be Cohen Macaulay.*

- (1)  $R$  is Gorenstein if and only if  $\omega_R$  exists and is isomorphic to  $R$ .
- (2) Let  $\psi : (R, m) \rightarrow (S, n)$  be a local homomorphism of Cohen Macaulay rings such that  $S$  is a finite  $R$ -module. If  $\omega_R$  exists then  $\omega_S$  exists and  $\omega_S \cong \text{Ext}_R^t(S, \omega_R)$  where  $t = \dim R - \dim S$ .

*Proof.* (1) For the forward direction,  $R$  Gorenstein implies  $R$  is Cohen Macaulay of type 1. So  $R$  is a maximal Cohen Macaulay  $R$ -module of type 1 with  $\text{id}_R R < \infty$ . Thus  $R \cong \omega_R$ . For the backward direction,  $\omega_R \cong R$  says  $\text{id}_R R < \infty$  and so  $R$  is Gorenstein.

- (2) Recall  $\text{grade}_I(M) = \dim M - \dim M/IM$  for all  $I \subseteq m$  with  $M \neq 0$  Cohen Macaulay. Note that  $\dim S = \dim(R / \ker \phi)$  (as  $S/R$  finite implies  $S/R / \ker \phi$  is finite and hence integral). So there exists an  $R$ -sequence  $\underline{x} = x_1, \dots, x_t$  with  $x_i \in \ker \phi$  and  $t = \dim R - \dim S$  (by the recall statement with  $I = \ker \phi$ ). Let  $\bar{R} = R / (\underline{x})$ . Since  $\omega_R / (\underline{x})\omega_R \cong \omega_{\bar{R}}$ , we have  $\text{Ext}_R^t(S, \omega_R) \cong \text{Hom}_{\bar{R}}(S, \omega_{\bar{R}})$  by Rees' Isomorphism. Thus we may assume  $\dim R = \dim S$  and without loss of generality replace  $R$  with  $\bar{R}$ . Let  $d = \dim R$  and  $\underline{y} = y_1, \dots, y_d$  be an  $R$ -sequence. Then  $\underline{y}$  is  $\omega_R$ -regular and  $\text{Hom}_R(S, \omega_R)$ -regular as both are maximal Cohen Macaulay. It follows from Proposition 2.49 that  $\text{Hom}_R(S, \omega_R) \otimes_R R' \cong \text{Hom}_{R'}(S', \omega_{R'})$  where  $R' = R / \underline{y}R$  and  $S' = S / \underline{y}S$ .

*Claim.* For a Cohen Macaulay local ring  $R$ ,  $C$  a maximal Cohen Macaulay  $R$ -module, and  $\underline{x}$  a maximal  $R$ -sequence,  $C / \underline{x}C$  is the canonical module of  $R / \underline{x}R$  if and only if  $C = \omega_R$ .

*Proof.*  $\text{Ext}_R^d(k, C) \cong \text{Hom}_{R / (\underline{x})R}(k, C / \underline{x}C)$ . Using the Ext definition of type and injective dimension, done.

By the Claim, it is enough to show  $\text{Hom}_{R'}(S', \omega_{R'}) = \omega_{S'}$ . By Proposition 2.50, since  $R'$  is Artinian,  $\omega_{R'} \cong E_{R'}(k)$ . So  $\text{Hom}_{R'}(S', \omega_{R'})$  is an injective  $S'$ -module. So  $\text{Hom}_{R'}(S', E_{R'}(k)) \cong E_{S'}(k)^r$  for some  $r$ . Thus by 3.2.12 in BH  $\lambda(E_{S'}(k)) = \lambda(S') = \lambda(\text{Hom}_{R'}(S', E_{R'}(k))) = r\lambda(E_{S'}(k))$ . Thus  $r = 1$ . So  $\text{Hom}_{R'}(S', \omega_{R'}) \cong E_{S'}(k) \cong \omega_{S'}$ . Therefore,  $\text{Hom}_R(S, \omega_R) \cong \omega_S$ .  $\square$

Note that a Noetherian complete local ring is the homomorphic image of a regular local ring, which is Gorenstein. So Theorem 2.53 implies the following corollary.

**Corollary 2.54.** *A complete Cohen Macaulay local ring admits a canonical module.*

**Corollary 2.55.** *Let  $(R, m, k)$  be a regular local ring and  $I \subseteq m$  an ideal of height  $g$  such that  $R/I =: S$  is Cohen Macaulay. Let  $F : 0 \rightarrow F_g \rightarrow \dots \rightarrow F_0 \rightarrow 0$  be the minimal free resolution of  $S$  and let  $G : 0 \rightarrow G_g \rightarrow \dots \rightarrow G_0 \rightarrow 0$  where  $G_i = F_{g-i}^* = \text{Hom}_R(F_{g-i}, R)$  for  $0 \leq i \leq g$ . Then  $G$  is the minimal free resolution of  $\omega_S$ .*

*Proof.* First note  $g$  is the length of the minimal free resolution of  $S$  as  $\text{pd}_R R/I = \text{depth } R - \text{depth } R/I = \dim R - \dim R/I = g$  since Cohen Macaulay. Now  $\text{Ext}_R^i(S, R) \cong H^i(F^*)$  for all  $i \geq 0$ . By Proposition 2.50 we have

$\text{Ext}_R^j(S, R) = 0$  for  $j \neq d - (d - g) = g$ . So Theorem 2.53 implies  $\omega_S \cong \text{Ext}_R^g(S, R) = H^g(F^*)$  as  $R \cong \omega_R$ . Thus  $G$  is a minimal free resolution of  $\omega_S$ .  $\square$

**Remark.** If  $S = R/I$  were also Gorenstein in the corollary, then  $S \cong \omega_S$  and the corollary would imply dualizing must give the same resolution. Thus  $\beta_i(R/I) = \beta_{g-i}(R/I)$  for all  $0 \leq i \leq g$ .

**Theorem 2.56.** *Let  $(R, m, k)$  be Cohen Macaulay of dimension  $d$  and  $C$  a finitely generated  $R$ -module. TFAE*

- (1)  $C$  is the canonical module of  $R$ .
- (2)  $\mu_i(p, C) = \delta_{ih}$  for all  $i \geq 0$  and  $p \in \text{Spec } R$  with  $\text{ht } p = h$ .
- (3) For all  $t = 0, \dots, d$  and all Cohen Macaulay  $R$ -modules  $M$  of dimension  $t$  one has
  - (a)  $\text{Ext}_R^{d-t}(M, C)$  is a Cohen Macaulay  $R$ -module of dimension  $t$
  - (b)  $\text{Ext}_R^i(M, C) = 0$  for all  $i \neq d - t$
  - (c) There exists  $M \xrightarrow{\cong} \text{Ext}_R^{d-t}(\text{Ext}_R^{d-t}(M, C), C)$  which in the case of  $d = t$  is just the natural homomorphisms from  $M$  into the bidual of  $M$  with respect to  $C$ .
- (4) For all maximal Cohen Macaulay  $R$ -modules  $M$  one has
  - (a)  $\text{Hom}_R(M, C)$  is a maximal Cohen Macaulay  $R$ -module
  - (b)  $\text{Ext}_R^i(M, C) = 0$  for all  $i > 0$
  - (c) The natural map  $M \rightarrow \text{Hom}_R(\text{Hom}_R(M, C), C)$  is an isomorphism.

*Proof.* For (1)  $\Rightarrow$  (2), Theorem 2.51 says  $C_p \cong \omega_{R_p}$ . Thus  $\mu_i(p, C) = \dim_{k(p)} \text{Ext}_{R_p}^i(k(p), C_p) = \delta_{ih}$  since  $h = \dim R_p$ .

For (2)  $\Rightarrow$  (1), letting  $p = m$  we get  $\mu_i(m, C) = \delta_{id}$ , that is  $\dim_k \text{Ext}_R^i(k, C) = \delta_{id}$  which implies  $C$  is the canonical module for  $R$ .

For (3)  $\Rightarrow$  (4), note that (4) is a special case of (3) as  $M$  maximal Cohen Macaulay implies  $t = d$ .

For (4)  $\Rightarrow$  (3), note that by choosing  $M = R$  we have  $C$  is maximal Cohen Macaulay. Recall we showed in the proof of Proposition 2.49 that for all Cohen Macaulay  $R$ -modules  $N$  that  $\text{grade}(\text{Ann } M, N) \geq d - t$ . Thus there exists an  $R$ -sequence  $\underline{x} \subseteq \text{Ann}_R M$  (which is also a  $C$  sequence as  $C$  is maximal Cohen Macaulay. Applying Rees' isomorphism iteratively gives us

$$\text{Ext}_R^{d-t}(\text{Ext}_R^{d-t}(M, C), C) \cong \text{Hom}_{R/(\underline{x})}(\text{Hom}_{R/(\underline{x})}(M, C/\underline{x}C), C/\underline{x}C).$$

Thus we may assume  $d = t$  in which case the result holds.

For (1)  $\Rightarrow$  (4), Parts (a) and (b) follow from Proposition 2.49. For part (c), let  $\underline{y}$  be a maximal  $R$ -sequence. Then Proposition 2.49 gives

$$\text{Hom}_R(\text{Hom}_R(M, C), C) \otimes R/(\underline{y}) \cong \text{Hom}_{R/(\underline{y})}(\text{Hom}_{R/(\underline{y})}(M/\underline{y}M, C/\underline{y}C), C/\underline{y}C).$$

Thus by Lemma 2.48, it is enough to show

$$\phi_{\otimes 1} : M/\underline{y}M \rightarrow \text{Hom}_{R/(\underline{y})}(\text{Hom}_{R/(\underline{y})}(M/\underline{y}M, C/\underline{y}C), C/\underline{y}C)$$

is an isomorphism. Thus we may further assume  $\dim R = 0$ . Of course, now  $C \cong E_R(k)$  and we've already shown  $M \cong M^{vv}$ .

For (4)  $\Rightarrow$  (1), choosing  $M = R$  gives us  $C$  is maximal Cohen Macaulay. As (4)  $\Leftrightarrow$  (3), we see (3)(b) implies  $\text{id}_R(C) < \infty$  by choosing  $M = k$ . Thus we need only show  $r(C) = 1$ . As above, we may again reduce to the case that  $\dim R = 0$  by Proposition 2.49. Then  $C$  injective implies  $C \cong E_R(k)^r$  for  $r = r(C)$ . Thus  $\text{Hom}_R(\text{Hom}_R(R, E^r), E^r) = \text{Hom}_R(\text{Hom}_R(R, E), E)^{r^2} = R^{r^2}$ . By (c), we see  $r^2 = 1$  and hence  $r(C) = 1$ . Thus  $C$  is the canonical module.  $\square$

**Proposition 2.57.** *Let  $R$  be a Cohen Macaulay local ring of dimension  $d$  and  $M$  a Cohen Macaulay  $R$ -module of dimension  $t$ . Then*

- (1)  $\mu(M) = r(\text{Ext}_R^{d-t}(M, \omega_R))$
- (2)  $r(M) = \mu(\text{Ext}_R^{d-t}(M, \omega_R))$

- (3)  $\omega_R$  is a faithful  $R$ -module and
- (a)  $r(\omega_R) = 1$  and  $\mu(\omega_R) = r(R)$
  - (b)  $\text{End}(\omega_R) = R$ .

*Proof.* Let  $\underline{x}$  be an  $R$ -sequence (and hence an  $\omega_R$ -sequence) of length  $d-t$  in  $\text{Ann } M$ . Then Rees' isomorphism and Theorem 2.51 give  $\text{Ext}_R^{d-t}(M, \omega_R) \cong \text{Hom}_{R/(\underline{x})}(M, \omega_{R/(\underline{x})})$ . So as before we may assume  $d = t$ . By Proposition 2.49 we may further assume  $\dim R = 0$ . Then  $\omega_R \cong E_R(k)$ . Now  $\text{Ext}_R^{d-t}(M, \omega_R) = \text{Hom}_R(M, E) = M^v$ . So

$$\mu(M) = \dim_k M/mM = \dim_k(M/mM)^v = \dim_k \text{soc } M^v = r(M^v).$$

Thus (1) holds. Since  $M \cong M^{vv}$ , we have  $r(M) = r(M^{vv}) = \mu(M^v)$  which proves (2). For (3), since  $\dim R = 0$  and  $\omega_R = E$ , we have shown  $R \cong R^{vv} = \text{Hom}_R(E, E) = \text{End}_R(E) = \text{End}_R(\omega_R)$ . This also shows  $\omega_R$  is faithful as multiplication by different elements of  $R$  give different homomorphisms. By the definition of the canonical module, we know  $r(\omega_R) = 1$ . Also by (1) we see  $\mu(\omega_R) = r(\text{End}(\omega_R)) = r(R)$   $\square$

**Corollary 2.58.** *Let  $R$  be a Cohen Macaulay local ring such that  $R \cong S/I$  for a Gorenstein ring  $S$  and  $I \subseteq S$ . Let  $M$  be a Cohen Macaulay  $R$ -module and  $p \in \text{Supp } M$ . Then  $r(M_p) \leq r(M)$ .*

*Proof.* Note that since  $\text{Ann}_R M = \text{Ann}_S M/I$ , we have  $\dim_S M = \dim_R M$ . Further  $\text{depth}_S M = \text{depth}_R M$  (send  $\underline{x} \leftrightarrow \underline{\bar{x}}$ ). Thus  $M$  is also a Cohen Macaulay  $S$ -module.

*Claim.*  $\text{Ext}_R^t(k, M) \cong \text{Hom}_R(k, M/\underline{x}M) \cong \text{Hom}_{R/(\underline{x})}(k, M/xM)$  for  $t = \dim M$ .

*Proof.* The second isomorphism comes from the fact that  $k$  and  $M/\underline{x}M$  are both  $R$  and  $R/(\underline{x})$  modules and the module structures are compatible. We will show the first isomorphism via induction on  $t$ . Consider the short exact sequence  $0 \rightarrow M \xrightarrow{x_1} M \rightarrow M/x_1M \rightarrow 0$ . For  $t = 1$ , applying  $\text{Hom}_R(k, -)$  gives us

$$0 \rightarrow \text{Hom}_R(k, M/x_1M) \rightarrow \text{Ext}_R^1(k, M) \xrightarrow{x_1} 0$$

where the first term is zero as  $\text{depth } M > 0$  and the last term is zero because multiplication by  $x_1 \in m$  is the zero map. For  $t > 1$ , applying  $\text{Hom}_R(k, -)$  gives

$$0 \rightarrow \text{Ext}_R^{t-1}(k, M/x_1M) \rightarrow \text{Ext}_R^t(k, M) \xrightarrow{x_1} 0$$

where the first term is zero as  $\text{depth } M > t - 1$  and the last is zero as  $x_1 \in m$ . Thus by induction  $\text{Hom}_R(k, M/\underline{x}M) \cong \text{Ext}_R^{t-1}(k, M/x_1M) \cong \text{Ext}_R^t(k, M)$ .

Let  $x_1, \dots, x_t \in S$  be  $M$ -regular. Then

$$\begin{aligned} \text{Ext}_S^t(k, M) &\cong \text{Hom}_{S/(\underline{x})}(k, M/\underline{x}M) \text{ by the claim} \\ &\cong \text{Hom}_{R/(\underline{\bar{x}})R}(k, M/\underline{x}M) \text{ as the } R/(\underline{\bar{x}})R \text{ and } S/(\underline{x}) \text{ structures are compatible} \\ &\cong \text{Ext}_R^t(k, M) \text{ by the claim.} \end{aligned}$$

In particular, this says  $r_S(M) = r_R(M)$  and we may replace  $R$  with  $S$ . So without loss of generality assume  $R$  is Gorenstein and hence  $\omega_R \cong R$ .

*Claim.*  $\dim R/p = \dim M/pM$

*Proof.* Note that  $p \in \text{Supp } M$  implies  $\text{Ann } M \subseteq p$ . Recall  $\sqrt{\text{Ann } M \otimes N} = \sqrt{\text{Ann } M + \text{Ann } N}$ . Thus we have  $\sqrt{\text{Ann } M/pM} = \sqrt{\text{Ann } M + p} = \sqrt{p} = p$ . Thus  $\dim M/pM = \dim R/p$ .

The claim gives us that

$$\dim R_p - \dim M_p = (\dim R - \dim R/p) - (\dim M - \dim M/pM) = \dim R - \dim M = d - t.$$

Thus by Proposition 2.57 we have

$$r(M_p) = \mu(\text{Ext}_R^{d-t}(M_p, R_p)) = \mu(\text{Ext}_R^{d-t}(M, R)_p) \leq \mu(\text{Ext}_R^{d-t}(M, R)) = r(M). \quad \square$$

**Note.** Bruns & Herzog prove the corollary in the case that  $R$  is a Cohen Macaulay local ring. Their proof shows it is enough to reduce to the case where  $R \cong S/I$ . However they use several results we have not proved and thus we only consider the special case above.

**Proposition 2.59.** *Let  $(R, m)$  be Cohen Macaulay and  $C$  a finitely generated  $R$ -module. TFAE*

- (1)  $C$  is the canonical module of  $R$ .
- (2)  $C$  is a faithful maximal Cohen Macaulay  $R$ -module of type 1.

*Proof.* Note that (1)  $\Rightarrow$  (2) follows from Proposition 2.57 and the definition of canonical module. For the other direction, we first want to show it is enough to reduce to the case where  $R$  is complete. Notice the following.

- $C$  is the canonical module of  $R$  if and only if  $\dim_k \text{Ext}_R^i(k, C) = \delta_{id}$  for  $d = \dim R$ . Now

$$\text{Ext}_{\hat{R}}^i(k, \hat{C}) \cong \text{Ext}_R^i(k, C) \otimes \hat{R} \cong k^{r_i} \otimes \hat{R} \cong \text{Ext}_R^i(k, C)$$

for some  $r_i$  where the second isomorphism comes from the facts that  $k$  and  $C$  are both finitely generated  $R$ -modules and  $\text{Ext}$  is a  $k$ -module. Since  $\dim R = \dim \hat{R}$ , we see  $C$  is the canonical module of  $R$  if and only if  $\hat{C}$  is the canonical module of  $\hat{R}$ .

- Recall  $C$  is faithful if and only if  $\phi : R \rightarrow \text{Hom}_R(C, C)$  is injective. Since  $R \rightarrow \hat{R}$  is faithfully flat, the completion of  $\phi$  (namely  $\hat{\phi} : \hat{R} \rightarrow \text{Hom}_{\hat{R}}(\hat{C}, \hat{C})$ ) is injective if and only if  $\phi$  is.
- We know

$$\begin{aligned} C \text{ is maximal Cohen Macaulay} &\Leftrightarrow \min\{i \mid \text{Ext}_R^i(k, C) \neq 0\} \\ &\Leftrightarrow \min\{i \mid \text{Ext}_R^i(k, C) \otimes_R \hat{R} \neq 0\} \\ &\Leftrightarrow \min\{i \mid \text{Ext}_{\hat{R}}^i(k, \hat{C}) \neq 0\} \\ &\Leftrightarrow \hat{C} \text{ is maximal Cohen Macaulay.} \end{aligned}$$

(Notice this also shows  $\text{depth } C = \text{depth } \hat{C}$ .)

- Similarly,  $C$  is type 1 if and only if  $\dim_k \text{Ext}_R^t(k, \hat{C}) = \dim_k \text{Ext}_R^t(k, C) = 1$  which is if and only if  $\hat{C}$  is type 1.

Thus we may assume  $R$  is complete. By Corollary 2.54  $R$  admits a canonical module  $\omega_R$ . By Proposition 2.57(2),  $1 = r(C) = \mu(\text{Hom}_R(C, \omega_R))$ . Let  $\text{Hom}_R(C, \omega_R) \cong R/I$ . By Theorem 2.56,  $C \cong \text{Hom}_R(\text{Hom}_R(C, \omega_R), \omega_R) \cong \text{Hom}_R(R/I, \omega_R)$ . So  $I$  annihilated  $C$ . Since  $C$  is faithful,  $I = 0$  and  $C \cong \text{Hom}_R(R, \omega_R) = \omega_R$ .  $\square$

**Exercise.** (3.3.26 in Bruns & Herzog) Let  $(R, m, k)$  be a Gorenstein local ring of dimension  $d$  and  $M$  a finite module of finite projective dimension. Show  $\text{Tor}_i^R(k, M) \cong \text{Ext}_R^{d-i}(k, M)$  for all  $i$ .

*Proof.* We will induct on  $\text{pd}_R M$  to show  $\text{Tor}_i^R(k, M) \cong \text{Ext}_R^{d-i}(k, M)$  and if  $\phi : M \rightarrow N$  are finite with finite projective dimension then following square commutes.

$$\begin{array}{ccc} \text{Tor}_0^R(k, M) & \xrightarrow{1 \otimes \phi} & \text{Tor}_0^R(k, N) \\ \downarrow \cong & & \downarrow \cong \\ \text{Ext}_R^d(k, M) & \longrightarrow & \text{Ext}_R^d(k, N) \end{array}$$

If  $\text{pd}_R M = 0$ , then  $M \cong R^n$  for some  $n$ . To show  $\text{Tor}_i^R(k, R^n) \cong \text{Ext}_R^{d-i}(k, R^n)$ , it is enough to show for the  $n = 1$  case (as we can pull the  $n$  out of the Tor and Ext terms). Since  $R$  is Gorenstein, we have  $R \cong \omega_R$  and so  $\dim_k \text{Ext}_R^{d-i}(k, R) = 1$  for  $i = 0$  and 0 otherwise. For Tor, note that  $0 \rightarrow \overset{0}{R} \rightarrow 0$  is a projective resolution for  $R$ . Tensor to get  $0 \rightarrow k \rightarrow 0$  and hence  $\dim \text{Tor}_i^R(k, R) = 1$  if  $i = 0$  and 0 otherwise. Thus  $\text{Tor}_i^R(k, R) \cong \text{Ext}_R^{d-i}(k, R)$ . For

the commutative diagram, we have  $\phi : R^n \rightarrow R^m$  and

$$\begin{array}{ccc} \mathrm{Tor}_0^R(k, R^n) & \xrightarrow{1 \otimes \phi} & \mathrm{Tor}_0^R(k, R^m) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ext}_R^d(k, R^n) & \longrightarrow & \mathrm{Ext}_R^d(k, R^m) \end{array}$$

commutes as Ext is multiplicative and additive. Now suppose  $\mathrm{pd}_R M \geq 0$ . Say  $\mathrm{pd}_R M = s$ . Let  $0 \rightarrow F_s \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$  be a minimal free resolution. Then  $\mathrm{rank} F_0 = \mu(M)$ . Let  $L = \mathrm{syz}_R^1(M)$ . Then  $\mathrm{pd}_R L = s - 1$  and the sequence  $0 \rightarrow L \rightarrow F_0 \rightarrow M \rightarrow 0$  is exact. This induces a long exact sequence on Tor:

$$\cdots \rightarrow \underbrace{\mathrm{Tor}_i^R(k, F_0)}_{=0} \rightarrow \mathrm{Tor}_i^R(k, M) \rightarrow \mathrm{Tor}_{i-1}^R(k, L) \rightarrow \underbrace{\mathrm{Tor}_{i-1}^R(k, F_0)}_{=0} \rightarrow \cdots$$

Note that the  $\mathrm{Tor}_i^R(k, F_0) = 0$  for all  $i > 1$  as  $F_0$  are free and  $\mathrm{Tor}_1^R(k, F_0) = 0$  by the  $\mathrm{pd}_R M = 0$  case. Thus  $\mathrm{Tor}_{i-1}^R(k, L) \cong \mathrm{Tor}_i^R(k, M)$  for  $i > 1$ . For  $i = 1$ , we have

$$\underbrace{\mathrm{Tor}_1^R(k, F_0)}_{=0} \rightarrow \mathrm{Tor}_1^R(k, M) \rightarrow k \otimes L \xrightarrow{\theta} k \otimes F_0 \xrightarrow{\cong} k \otimes M \rightarrow 0$$

where  $k \otimes F_0 \cong k \otimes M$  as the map is onto and  $\mathrm{rank} F_0 = \mu(M)$ . Thus  $\theta = 0$  and we get (\*)  $\mathrm{Tor}_{i-1}^R(k, L) \cong \mathrm{Tor}_i^R(k, M)$  for  $i \geq 1$ .

We could also apply  $\mathrm{Hom}(k, -)$  to our short exact sequence to get the long exact sequence on Ext :

$$\cdots \rightarrow \underbrace{\mathrm{Ext}_R^i(k, F_0)}_{=0} \rightarrow \mathrm{Ext}_R^i(k, M) \rightarrow \mathrm{Ext}_R^{i+1}(k, L) \rightarrow \underbrace{\mathrm{Ext}_R^{i+1}(k, F_0)}_{=0} \rightarrow \cdots$$

Since  $R$  is Cohen Macaulay and has finite injective dimension, we see  $F_0$  is Cohen Macaulay and has finite injective dimension and hence  $\mathrm{Ext}_R^i(k, F_0) = 0$  for all  $i \neq d$ . So (\*\*)  $\mathrm{Ext}_R^i(k, M) \cong \mathrm{Ext}_R^{i+1}(k, L)$  for all  $i \neq d, d - 1$ .

By induction,  $\mathrm{Tor}_i^R(k, L) \cong \mathrm{Ext}_R^{d-i}(k, L)$  for all  $i$ . Then

$$\mathrm{Tor}_i^R(k, M) \cong \mathrm{Tor}_{i-1}^R(k, L) \cong \mathrm{Ext}_R^{d-(i-1)}(k, L) \cong \mathrm{Ext}_R^{d-i}(k, M)$$

for  $i > 1$  by induction and (\*), (\*\*). For  $i = 0, 1$  consider

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^R(k, M) & \longrightarrow & \mathrm{Tor}_0^R(k, L) & \longrightarrow & \mathrm{Tor}_0^R(k, F_0) & \longrightarrow & \mathrm{Tor}_0^R(k, M) & \longrightarrow & 0 \\ & & & & \downarrow \cong & & \downarrow \cong & & & & \\ 0 & \longrightarrow & \mathrm{Ext}_R^{d-1}(k, M) & \longrightarrow & \mathrm{Ext}_R^d(k, L) & \longrightarrow & \mathrm{Ext}_R^d(k, F_0) & \longrightarrow & \mathrm{Ext}_R^d(k, M) & \longrightarrow & 0 \end{array}$$

By diagram chasing and the Five Lemma, we get our desired isomorphisms.  $\square$

**Lemma 2.60.** *Let  $R$  be Cohen Macaulay with canonical module  $\omega_R$ . Suppose  $\mathrm{Hom}_R(\omega_R, R) \cong R$ . Then  $R$  is Gorenstein.*

*Proof.* Let  $\underline{x}$  be a maximal  $R$ -sequence (and hence  $\omega_R$ -sequence). Then  $\mathrm{Hom}_{R/(\underline{x})}(\omega_{R/(\underline{x})}, R/(\underline{x})) \cong R/(\underline{x})$ . Since  $R$  is Gorenstein if and only if  $R/(\underline{x})$  is Gorenstein, we may assume  $\dim R = 0$ . By Proposition 2.57,  $\mu(M) = r(M^v)$  and  $r(M) = \mu(M^v)$  where  $M^v = \mathrm{Hom}_R(M, E)$  for  $E = E_R(M) = \omega_R$ . So it is enough to show  $1 = r(R) = \mu(R^v) = \mu(E)$ . Let  $r = r(R)$ . Then

$$\begin{aligned} r = \dim_k \mathrm{Hom}_R(k, R) &= \dim_k \mathrm{Hom}_R(k, \mathrm{Hom}_R(E, R)) \text{ by assumption} \\ &= \dim_k \mathrm{Hom}_R(k \otimes_R E, R) \\ &= \dim_k \mathrm{Hom}_R(k^r, R) \text{ as } k \otimes_R E \cong E/mE = k^{\mu(E)} = k^r \\ &= \dim_k \mathrm{Hom}_R(k, R)^r \\ &= \dim_k k^{r^2} = r^2 \end{aligned}$$

Thus  $r = r^2$  and so  $r = 1$ . □

**Exercise.** (3.3.25 in Bruns & Herzog) Let  $(R, m)$  be Gorenstein,  $I \subseteq R$  a perfect ideal of grade  $g$  (so  $\text{pd } R/I = \text{grade } I = \text{ht } I$ ). Let  $0 \rightarrow F_g \xrightarrow{\partial_g} F_{g-1} \xrightarrow{\partial_{g-1}} \dots \xrightarrow{\partial_1} F_0 \rightarrow 0$  be a free resolution of  $S = R/I$ . Then

- (1) The dual complex  $0 \rightarrow F_0^* \xrightarrow{\partial_1^*} \dots \rightarrow F_g^* \rightarrow 0$  is acyclic with  $\text{coker } \partial_g^* \cong \omega_S$  (and hence is a minimal free resolution of  $\omega_S$ ).
- (2)  $\text{Hom}_S(\omega_S, S) \cong \ker(F_g \otimes_R S \rightarrow F_{g-1} \otimes_R S) \cong \text{Tor}_R^g(S, S)$  (where the second isomorphism is just by definition)
- (3)  $S$  is Gorenstein if and only if  $\text{Tor}_R^g(S, S) \cong S$
- (4) If  $g = 2$ , then  $\mu(I) = r(S) + 1$  and if  $R$  is regular and  $S$  Gorenstein, then  $S$  is a complete intersection.

*Proof.* (1) We have  $\dim R - \dim R/I = g = \text{ht}(I) = \text{pd } R/I = \text{depth } R - \text{depth } R/I$ . Since  $R$  is Cohen Macaulay,  $\text{depth } R/I = \dim R/I$  and so  $S$  is Cohen Macaulay. Since  $I$  is a perfect ideal,  $R/I$  is a perfect module. Hence by Corollary 3.3.9 (since they only use  $R$  regular to get  $R/I$  is perfect), we are done.

(2) We first prove the following claim.

*Claim.* If  $R \rightarrow S$  a ring homomorphism and  $F$  a finitely generated free  $R$ -module then  $\text{Hom}_R(\text{Hom}_R(F, R), S) \cong F \otimes_R S$  naturally.

*Proof.* Say  $F = R^n$ . Then  $R^n \otimes_R S \cong S^n$ . Also,  $\text{Hom}_R(\text{Hom}_R(R^n, R), S) \cong \text{Hom}_R(R^n, S) \cong \text{Hom}_R(R, S)^n \cong S^n$ .

By (1), we see  $F_{g-1}^* \rightarrow F_g^* \rightarrow \omega_S \rightarrow 0$  is exact. Apply  $\text{Hom}_R(-, S)$  to get the following diagram.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}_R(\omega_S, S) & \longrightarrow & \text{Hom}_R(F_g^*, S) & \longrightarrow & \text{Hom}_R(F_{g-1}^*, S) & \text{exact} \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & F_g \otimes_R S & \longrightarrow & F_{g-1} \otimes_R S & \text{exact}
 \end{array}$$

By the Five Lemma,  $K \cong \text{Hom}_R(\omega_S, S) \cong \text{Hom}_S(\omega_S, S)$ .

- (3) Suppose  $S$  is Gorenstein. Then  $\text{Tor}_R^g(S, S) = \text{Hom}_S(\omega_S, S) = \text{Hom}_S(\omega_S, \omega_S) = S$ . Suppose  $\text{Tor}_R^g(S, S) = S$ . Then  $\omega_S^* = \text{Hom}_S(\omega_S, S) = \text{Tor}_R^g(S, S) = S$ . By the Lemma, done.
- (4) Let  $g = 2$  and  $\ell = \mu(I)$ . Then  $0 \rightarrow R^t \rightarrow R^\ell \rightarrow R \rightarrow R/I \rightarrow 0$  is exact with  $t = r(S)$ . Since  $\text{ht } I = 2$ , there exists a prime  $p \not\supseteq I$ . Localize at  $p$  to get  $0 \rightarrow R_p^t \rightarrow R_p^\ell \rightarrow R_p \rightarrow 0$ . Since  $R_p$  is free, the sequence splits. Thus  $R_p^\ell = R_p^t \oplus R_p$ , which implies  $\ell = t + 1$ . Thus  $\mu(I) = r(S) + 1$ . Lastly, if  $S$  is Gorenstein, then  $r(S) = 1$  which implies  $\mu(I) = 2 = g = \text{ht}(I)$ . Thus  $S$  is a complete intersection. □

**Exercise.** (3.3.27 in Bruns & Herzog) Let  $(R, m)$  be a Cohen Macaulay local ring with canonical module  $\omega_R$ . Suppose for all finite  $R$ -modules  $M$  there exists  $n$  and an epimorphism  $\omega_R^n \rightarrow M$ . Prove  $R$  is Gorenstein.

*Proof.* In particular, we have  $0 \rightarrow K \rightarrow \omega_R^n \rightarrow R \rightarrow 0$  where  $K = \ker(\omega_R^n \rightarrow R)$ . As  $R$  is projective, this sequence splits and  $\omega_R^n \cong K \oplus R$ . As  $\omega_R$  has finite injective dimension, so does  $\omega_R^n$ . Thus there exists  $m$  such that  $0 = \text{Ext}_R^i(k, \omega_R^n) = \text{Ext}_R^i(k, K) \oplus \text{Ext}_R^i(k, R)$  for all  $i \geq m$ , which implies  $0 = \text{Ext}_R^i(k, R)$  for all  $i \geq m$ . Thus  $R$  has finite projective dimension and is thus Gorenstein. □

**Lemma 2.61.** Let  $(R, m)$  be Cohen Macaulay,  $x \in R$  both  $R$  and  $M$  regular with  $M$  finitely generated. Then  $M$  is free if and only if  $M/xM$  is free as an  $R/(x)$ -module.

*Proof.* The forward direction is clear. For the backward direction, assume  $M/xM \cong (R/(x))^n$ . By NAK, we have a short exact sequence  $0 \rightarrow K \rightarrow R^n \xrightarrow{f} M \rightarrow 0$  where  $f$  is induced by the isomorphism and  $K = \ker f$  (NAK gives us the surjectivity of  $f$ ). Thus it is enough to show  $f$  is injective. Apply  $- \otimes R/(x)$ . Then  $f \otimes_{R/(x)} 1$  is an isomorphism by assumption and so  $f$  is an isomorphism by Lemma 2.48. □

**Exercise.** (3.3.28 in Bruns & Herzog) Let  $(R, m)$  be Cohen Macaulay with canonical module  $\omega_R$ .

- (1) If  $M$  is maximal Cohen Macaulay with  $\text{id}_R M < \infty$ , then  $M \cong \omega_R^n$  for some  $n \in \mathbb{N}$ .

- (2) Let  $M$  be a finite  $R$ -module. Show  $\text{id}_R M < \infty$  if and only if there exists a finite  $\omega_R$ -resolution  $0 \rightarrow \omega_R^{r_p} \xrightarrow{\phi_p} \cdots \rightarrow \omega_R^{r_0} \rightarrow M \rightarrow 0$  where  $p = \text{depth } R - \text{depth } M$ . [Note if  $M$  has finite projective dimension, then  $p = \text{pd}_R M$ .]
- (3) The  $\omega_R$ -resolution is minimal if  $\text{im } \phi_i \subseteq m\omega_R^{r_i-1}$  for all  $i = 1, \dots, p$ . Show a finite  $R$ -module  $M$  with finite injective dimension has a minimal  $\omega_R$ -resolution and in this case  $r_i = \mu_{d-i}(m, M)$ .

*Proof.* (1) Induct on  $\text{id}_R M < \infty$ . If  $\text{id}_R M = 0$ , then  $\text{depth}_R M = 0$  and so  $\dim R = 0$  as  $M$  is maximal Cohen Macaulay. So  $\omega_R \cong E_R(R/m)$ . As  $M$  is injective, it is the sum of indecomposable injectives. Thus  $M \cong E^n$  for some  $n$ . So suppose  $\text{id}_R M > 0$ .

*Claim.*  $\text{Hom}_R(M, \omega_R)$  is free.

*Proof.* Let  $x \in R$  by  $R$ -regular (and hence  $M$  and  $\omega_R$  regular). Then

$$(*) \quad \text{Hom}_R(M, \omega_R)/x \text{Hom}_R(M, \omega_R) \cong \text{Hom}_R(M, \omega_R) \otimes R/(x) \cong \text{Hom}_{R/(x)}(M/xM, \omega_{R/(x)}).$$

Notice  $M/xM$  is a maximal Cohen Macaulay  $R/(x)$ -module and  $\text{id}_{R/(x)} M/xM < \text{id}_R M$ .

By induction,  $M/xM \cong (\omega_{R/(x)})^n$  for some  $n$ . Thus equation (\*) above becomes

$$(*) \cong \text{Hom}_{R/(x)}((\omega_{R/(x)})^n, \omega_{R/(x)}) \cong \text{Hom}_{R/(x)}(\omega_{R/(x)}, \omega_{R/(x)})^n = (R/(x))^n.$$

By the Lemma, done.

By Theorem 2.56,  $M \cong \text{Hom}_R(\text{Hom}_R(M, \omega_R), \omega_R) \cong \text{Hom}_R(R^n, \omega_R) = \omega_R^n$ .

- (2) For the backward direction, recall that if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence where any two of  $A, B$ , and  $C$  have finite injective dimension, then so does the third. It follows from  $0 \rightarrow \omega_R^{r_p} \rightarrow \cdots \rightarrow \omega_R^{r_0} \rightarrow M \rightarrow 0$  that  $M$  has finite injective dimension as  $\omega_R^{r_i}$  does for all  $i$  (break up the long exact sequence into short exact sequences and successively show that the kernels have finite injective dimension).

For the forward direction, let  $d = \dim R$ . We will induct on  $p = \text{depth } R - \text{depth } M$ . If  $p = 0$ , then  $M$  is maximal Cohen Macaulay and we are done by part (1) as  $0 \rightarrow \omega_R^{r_0} \rightarrow M \rightarrow 0$  is a minimal resolution. Assume  $p > 0$ . From hint in Bruns & Herzog, there exists  $R$ -modules  $X, Y$  where  $X$  is Maximal Cohen Macaulay and  $\text{id}_R Y < \infty$  such that  $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$  is exact (this is called a **maximal Cohen Macaulay approximation** of  $M$  [1]). Since  $Y$  and  $M$  have finite injective dimension, so does  $X$ . By part (1),  $X \cong \omega_R^{r_0}$  for some  $r_0$ . Since  $M$  is not maximal Cohen Macaulay, but  $X$  is, we see  $\text{depth } M < \text{depth } X$ . The Depth Lemma then says  $\text{depth } Y = \text{depth } M + 1$ . By induction, there exists a finite  $\omega_R$ -resolution for  $Y$  (of length  $p - 1 = \text{depth } R - \text{depth } Y$ ). Glue this resolution to  $X \rightarrow M \rightarrow 0$  to get a resolution for  $M$  of length  $p$ .

- (3) Let  $M$  be an  $R$ -module. Consider an  $R$ -linear map  $\phi : M^r \rightarrow M^r$ . For  $1 \leq i \leq s$  and  $1 \leq j \leq r$ , let  $\phi_{ij}$  be the composition  $M \xrightarrow{\text{incl}_j} M^r \xrightarrow{\phi} M^s \xrightarrow{\pi_i} M$ . Then

$$\phi(\underline{x}) = \begin{pmatrix} \sum_{j=1}^r \phi_{1j}(x_j) \\ \vdots \\ \sum_{j=1}^r \phi_{sj}(x_j) \end{pmatrix} = [\phi_{ij}](\underline{x}), \text{ for } \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in M^r.$$

In this way, we get an isomorphism  $\text{Hom}_R(M^r, M^s) \cong M_{s \times r}(\text{End}_R(M))$ , the module of  $s \times r$  matrices with entries in  $\text{End}_R(M)$ . If  $R$  is Cohen Macaulay with canonical module  $\omega_R$ , then  $\text{End}_R(\omega_R) \cong R$  by Corollary 2.57. Thus  $\text{Hom}_R(\omega_R^r, \omega_R^s) \cong M_{s \times r}(R)$ .

*Claim.* Let  $(R, m)$  be Cohen Macaulay with canonical module  $\omega_R$ . Let  $M$  be a finitely generated  $R$ -module such that  $\text{id}_R M < \infty$ . By part (2) of the exercise, there exists  $s$  such that  $\omega_R^s \rightarrow M$ . Choose the least such  $s$ . Let  $K = \ker \phi$ . As  $\text{id}_R K < \infty$ , there exists  $r$  such that  $\omega_R^r \rightarrow K$ . This gives a short exact sequence  $\omega_R^r \xrightarrow{A} \omega_R^s \rightarrow M \rightarrow 0$  where  $A = (a_{ij}) \in M_{s \times r}(R)$ . Then  $a_{ij} \in m$  for all  $i, j$ .

*Proof.* Suppose not. Then some  $a_{ij}$  is a unit. So there exist invertible matrices  $U$  and  $V$  such that

$$UAV = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{bmatrix} = B$$

(by elementary row and column operations). This gives us the following commutative diagram

$$\begin{array}{ccccccc} \omega_R^r & \xrightarrow{B} & \omega_R^s & \longrightarrow & \text{coker } B & \longrightarrow & 0 \\ \downarrow V & & \downarrow U^{-1} & & & & \\ \omega_R^r & \xrightarrow{A} & \omega_R^s & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

By diagram chasing and the Five Lemma, we see  $\text{coker } B \cong M$ . Since  $B$  surjectively maps the first  $\omega_R$  in  $\omega_R^r$  onto the first  $\omega_R$  in  $\omega_R^s$ , we get  $\omega_R^{s-1} \rightarrow M$ , a contradiction to the choice of  $s$ .

Thus for all finitely generated modules of finite injective dimension there exists an  $\omega_R$ -presentation  $\omega_R^r \xrightarrow{(a_{ij})} \omega_R^s \rightarrow M \rightarrow 0$  such that  $a_{ij} \in m$  for all  $i, j$ . Inductively, we get a minimal  $\omega_R$ -resolution  $0 \rightarrow \omega_R^{r_p} \xrightarrow{\phi_p} \cdots \rightarrow \omega_R^{r_0} \rightarrow M \rightarrow 0$  where  $\text{im } \phi_i \subseteq m\omega_R^{r_{i-1}}$  for all  $i$ .

It is still left to show that  $r_i = \mu_{d-i}(m, M)$  for all  $i$  in our above minimal  $\omega_R$ -resolution for  $M$ . Recall  $\mu_{d-i}(m, M) = \dim_k \text{Ext}_R^{d-i}(k, M)$ . Let  $r_i(M)$  be the  $r_i$ 's in the resolution above and let  $p(M) = \text{depth } R - \text{depth } M$ . We will induct on  $p(M)$ . If  $p(M) = 0$ , then  $M$  is maximal Cohen Macaulay and  $M \cong \omega_R^n$ . Since  $\omega_R^{r_1} \xrightarrow{(a_{ij})} \omega_R^{r_0} \rightarrow M \rightarrow 0$  with  $a_{ij} \in m$ , applying  $-\otimes R/m$  gives  $(\omega_R/m\omega_R)^{r_0} \cong M/mM$ . Thus  $n = r_0$ . Now we have

$$\begin{array}{ccccc} \omega_R^r & \xrightarrow{(a_{ij})} & \omega_R^n & \xrightarrow{\phi} & \longrightarrow 0 \\ & \searrow & \nearrow & & \\ & & K = \ker \phi & & \end{array}$$

where  $K$  is maximal Cohen Macaulay and  $\text{id}_R K < \infty$ . Thus  $K \cong \omega_R^s$ . We have  $0 \rightarrow \omega_R^s \xrightarrow{(b_{ij})} \omega_R^n \rightarrow M \rightarrow 0$  where  $(b_{ij}) \in m$ . Apply  $\text{Hom}_R(k, -)$ . Since all the terms are maximal Cohen Macaulay, we see the Ext modules will be zero for all  $i \neq d$ . Thus we get the short exact sequence

$$0 \rightarrow \underbrace{\text{Ext}_R^d(k, \omega_R^s)}_{\cong k} \xrightarrow{0} \text{Ext}_R^d(k, \omega_R^n) \rightarrow \text{Ext}_R^d(k, M) \rightarrow 0$$

where the map is zero as  $b_{ij} \in m$  annihilates  $k$ . Thus  $s = 0$  and  $0 \rightarrow \omega_R^n \rightarrow M \rightarrow 0$  is our minimal resolution (since  $r_1$  is chosen minimally and  $s = 0$ , we must have  $r_1 = 0$  and similarly  $r_i = 0$  for all  $i > 1$ ). Now consider the syzygy  $0 \rightarrow L \rightarrow \omega_R^{r_0} \rightarrow M \rightarrow 0$  where  $p(L) < p(M)$  and use induction.  $\square$

**Proposition 2.62.** *Let  $(R, m)$  be a 0-dimensional Cohen Macaulay ring with canonical module  $\omega_R$ . Suppose  $r(R) = 2$  and  $\lambda(R) = 3$ . Then  $\omega_R$  is not reflexive.*

*Proof.* Recall that  $\omega_R = E_R(R/m) =: E$ . We first prove a claim.

*Claim.* For all finitely generated  $R$ -modules  $M$   $\lambda(M^*) \geq \mu(M) \cdot r(R)$ .

*Proof.* Let  $\alpha = \mu(M)$ . Then  $M/mM \cong k^\alpha$  and thus  $M \rightarrow k^\alpha \rightarrow 0$  is exact. Apply  $\text{Hom}_R(M, R) = M^*$ . Then additivity of length gives us

$$\lambda(M^*) \geq \lambda(\text{Hom}_R(k^\alpha, R)) = \lambda(\text{Hom}_R(k, R)^\alpha) = \lambda(\text{soc}(R)^\alpha) = \alpha \lambda(\text{soc}(R)) = \alpha \dim_k \text{soc}(R) = \alpha r(R).$$

In particular, the claim says  $\lambda(E^*) \geq 2\mu(E) = 2r(R) = 4$ . Suppose  $E_R^{**} = E_R$ . Then  $3 = \lambda(R) = \lambda(E) = \lambda(E^{**}) \geq 2\mu(E^*)$  by the claim. Thus  $\mu(E^*) = 1$ . So  $R \rightarrow E^* \rightarrow 0$  is exact. But then  $2 = \lambda(R) \geq \lambda(E^*) = 4$ , a contradiction.  $\square$

**Example.**  $R = k[x, y]/(x^2, xy, y^2) = k \oplus kx \oplus ky$  is a ring satisfying the hypotheses of the proposition.

**Conjecture.** *If  $R$  is not Gorenstein, then  $\omega_R$  is not reflexive.*

### 3. ALMOST COMPLETE INTERSECTIONS ARE NOT GORENSTEIN RINGS- KUNZ

Assume  $R$  is a complete Noetherian local ring. Then  $R = S/I$  where  $S$  is a regular local ring. Let  $d(R) = \mu(I) - \underbrace{(\dim S - \dim R)}_{=\text{ht } I = \text{grade } I}$ . Note that  $d(R)$  is independent of the choice for  $S$  and  $I$ . If  $R$  is a complete intersection, then  $d(R) = 0$ . Say  $R$  is an **almost complete intersection** if  $d(R) = 1$ . In this case,  $I = (x_1, \dots, x_n)$  where  $n-1 = \text{grade } I$  and  $x_1, \dots, x_{n-1}$  is a regular sequence.

**Example.**  $k[x, y]/(x^2, xy)$  is an almost complete intersection (that is not Cohen Macaulay).

To prove that almost complete intersections are not Gorenstein, we want to show  $H_1(\underline{x}) \cong \omega_R$  and then use a result of Gulliksen's that says  $H_1(\underline{x})$  is free (i.e.,  $\cong S/(\underline{x})$ ) if and only if  $(\underline{x})$  is a regular sequence.

**Theorem 3.1** (1.6.16 in Bruns & Herzog). *Let  $R$  be a ring,  $\underline{x} = x_1, \dots, x_n \in R$ . If  $I = (\underline{x})$  contains a weak  $R$ -sequence  $\underline{y}$  (i.e.,  $R/(\underline{y})$  may be 0), then*

- (1)  $H_{n+1-i}(\underline{x}, R) = 0$  for  $i = 1, \dots, m$ .
- (2)  $H_{n-m}(\underline{x}, R) \cong \text{Hom}_R(R/I, R/(\underline{y})) \cong \text{Ext}_R^m(R/I, R)$ .

*Proof.* We will induct on  $m$ . For  $m = 0$ , we have proved (in 915) that  $H_n(\underline{x}) = \text{Ann}_R(\underline{x}) \cong \text{Hom}_R(R/(\underline{x}), R)$ . For  $m > 0$ , set  $\bar{R} = R/y_1R$ . Then  $0 \rightarrow R \xrightarrow{y_1} R \rightarrow \bar{R} \rightarrow 0$  induces the long exact sequence

$$\dots \rightarrow H_i(\underline{x}, R) \xrightarrow{y_1} H_i(\underline{x}, R) \rightarrow H_i(\underline{x}, \bar{R}) \rightarrow H_{i-1}(\underline{x}, R) \xrightarrow{y_1} H_{i-1}(\underline{x}, R) \rightarrow \dots$$

*Claim.*  $I = (\underline{x})$  annihilates  $H_i(\underline{x})$  for all  $i$  and thus multiplication by  $y_1$  is the zero map.

*Proof.* Let  $x_1, \dots, x_n \in R, S = \mathbb{Z}[t_1, \dots, t_n]$  for  $t_i$  variables. Define  $\phi : S \rightarrow R$  by  $t_i \mapsto x_i$ . Then  $R$  is an  $S$ -algebra. Consider  $K(\underline{t}, S) : 0 \rightarrow \overset{n}{S} \rightarrow \dots \rightarrow \overset{1}{S} \rightarrow \overset{0}{S} \rightarrow 0$ . This is a free  $S$ -resolution of  $S/(t_1, \dots, t_n)$ . Apply  $-\otimes_S M$  for an  $R$ -module  $M$  to get  $K(\underline{t}) \otimes_S M : 0 \rightarrow \overset{n}{M} \rightarrow \dots \rightarrow \overset{1}{M} \rightarrow \overset{0}{M} \rightarrow 0$  where all the maps change from matrices in  $t_i$ 's to matrices in  $x_i$ 's. So  $K(\underline{t}) \otimes_S M = K(\underline{x}, M)$ . Then  $H_i(\underline{x}, M) = \text{Tor}_i^S(S/(t_1, \dots, t_n), M)$ . Since  $M$  was arbitrary, we see  $H_i(\underline{x}, -) = \text{Tor}_i^S(S/(t_1, \dots, t_n), -)$  is a functor. We know  $\text{Ann}_S \text{Tor}_i^S(S/(t_1, \dots, t_n), -) \supseteq (t_1, \dots, t_n)S$ . Thus  $\text{Ann}_R H_i(\underline{x}, M) \supseteq (x_1, \dots, x_n)R$ .

So we have short exact sequences

$$0 \rightarrow H_i(\underline{x}, R) \rightarrow H_i(\underline{x}, \bar{R}) \rightarrow H_{i-1}(\underline{x}, R) \rightarrow 0$$

for all  $i$ . Let  $i = (n+1) - (m-1)$ . Then we have

$$0 \rightarrow \underbrace{H_{(n+1)-(m-1)}(\underline{x}, R)}_{=0} \rightarrow \underbrace{H_{(n+1)-(m-1)}(\underline{x}, \bar{R})}_{=0} \rightarrow H_{n-(m-1)}(\underline{x}, R) \rightarrow 0.$$

So by induction  $H_{n-(m-1)}(\underline{x}, R) = 0$ . Let  $i = n - (m-1)$ . Then

$$0 \rightarrow \underbrace{H_{n-(m-1)}(\underline{x}, R)}_{=0} \rightarrow H_{n-(m-1)}(\underline{x}, \bar{R}) \rightarrow H_{n-m}(\underline{x}, R) \rightarrow 0.$$

Thus  $H_{n-m}(\underline{x}, R) \cong H_{n-(m-1)}(\underline{x}, \bar{R}) \cong \text{Hom}_R(R/I, \bar{R}/(y_2, \dots, y_m)\bar{R}) \cong \text{Hom}_R(R/I, R/(\underline{y}))$  by induction.  $\square$

Assume  $R$  is Gorenstein. Then since  $R = S/(\underline{x})$ , Theorem 2.53 says  $\omega_R \cong \text{Ext}_S^t(R, \omega_S)$  for  $t = \dim S - \dim R = \text{ht } I = \text{grade } I = n - 1$  and  $S \cong \omega_S$ . By the Theorem, we see  $\omega_R \cong \text{Ext}_S^{n-1}(S/(\underline{x}), S) = H_{n-(n-1)}(\underline{x}, S) = H_1(\underline{x}, S)$ . By Gulliksen's result,  $(\underline{x})$  is a regular sequence, a contradiction since  $\text{grade } I = n - 1$ , not  $n$ .

**3.1. Exterior Algebra and Gulliksen's Result.** Let  $M$  be an  $R$ -module,  $r \geq 0$ . Let  $T^r(M) = \underbrace{M \otimes \cdots \otimes M}_{r \text{ times}}$ .

Then  $T^r(M)$  has the following universal property:

- Given any  $R$ -multilinear map  $f : M \times \cdots \times M \rightarrow N$ , there exists a unique  $R$ -linear map  $\phi : T^r(M) \rightarrow N$  such that the following diagram commutes.

$$\begin{array}{ccc} M \times \cdots \times M & \xrightarrow{f} & N \\ \downarrow & \nearrow & \\ T^r(M) & & \end{array}$$

This universal property makes  $T^r(-) : \ll R\text{-mod} \gg \rightarrow \ll R\text{-mod} \gg$  a functor where  $h : M \rightarrow N$  goes to  $T^r(h) := h \otimes \cdots \otimes h$ . Given  $r, s$ , there exists an  $R$ -bilinear map  $T^r(M) \times T^s(M) \rightarrow T^{r+s}(M)$  defined by  $(m_1 \otimes \cdots \otimes m_r, x_1 \otimes \cdots \otimes x_s) \mapsto m_1 \otimes \cdots \otimes m_r \otimes x_1 \otimes \cdots \otimes x_s$ . The **tensor algebra** of  $M$  is defined to be  $T(M) = \bigoplus_{r=0}^{\infty} T^r(M)$  where  $T^0(M) := R$ . The product above makes  $T(M)$  into a ring. Note  $R \rightarrow T(M)$  defined by  $r \mapsto r \in T^0(M)$  is a ring homomorphism and  $T^0(M) \subseteq Z(T(M))$  as the tensor product is  $R$ -multilinear. Thus  $T(M)$  is an  $R$ -algebra. In general,  $m \otimes n \neq n \otimes m$  and so  $T(M)$  is noncommutative. Further  $T(M)$  is a graded  $R$ -algebra where  $T^r(M)$  is the homogenous component of degree  $r$ . In fact,  $T(M)$  is generated as a ring by elements in degree 1. If  $M$  is finitely generated, say  $M = Rx_1 + \cdots + Rx_n$ , then  $T(M) = R[x_1, \dots, x_n]$  where  $x_i \in T^1(M)$  (and multiplication is given by the tensor product). We want to think of elements of  $T^r(M)$  as  $\sum m_1 \otimes \cdots \otimes m_r$ .

Let  $J$  be the two-sided ideal of  $T(M)$  generated by  $\{x \otimes x | x \in M\}$ . Then  $J$  is a homogenous ideal of degree 2. Define  $\Lambda(M) := T(M)/J$  as the **Exterior algebra** of  $M$ . Since  $J$  is homogenous,  $\Lambda(M)$  is graded where  $\Lambda^r(M)$  is the  $r^{\text{th}}$  homogeneous component of  $M$ . So  $\Lambda(M) = \bigoplus_0^{\infty} \Lambda^r(M)$  where  $\Lambda^r(M) = T^r(M)/J \cap T^r(M)$ . The homogeneous component  $\Lambda^r(M)$  (as an  $R$ -module) is called the  $r^{\text{th}}$  **Exterior power** of  $M$ . Let  $x_1 \wedge \cdots \wedge x_r$  denote the image of  $x_1 \otimes \cdots \otimes x_r$  in  $\Lambda^r(M)$ .

- $\Lambda^0(M) = R$  and  $\Lambda^1(M) = M$ .
- $x_1 \wedge \cdots \wedge (rx_i + x'_i) \wedge \cdots \wedge x_r = r(x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_r) + x'_1 \wedge \cdots \wedge x'_i \wedge \cdots \wedge x_r$ .
- $x \wedge x = 0$  for all  $x \in M$  and  $x_1 \wedge \cdots \wedge x_r = 0$  if  $x_i = x_{i+1}$  for some  $i$  (since  $J$  is two-sided).
- For all  $x, y \in M$  we see  $0 = (x+y) \wedge (x+y) = x \wedge x + x \wedge y + y \wedge x + y \wedge y = x \wedge y + y \wedge x$ . Thus  $x \wedge y = -y \wedge x$ . So the exterior algebra is skew commutative (or anticommutative) in degree 1. This is not the case in general. Take  $\alpha = x \wedge y$  and  $\beta = z$ . Then  $\alpha\beta = x \wedge y \wedge z = -x \wedge z \wedge y = z \wedge x \wedge y = \beta\alpha$ . For  $x \in M$  and  $\alpha \in \Lambda^r(M)$ , the above shows  $x\alpha = (-1)^r \alpha x$ .
- Let  $\sigma \in S_n$ . Then  $x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma_n} = (\text{sgn } \sigma) x_1 \wedge \cdots \wedge x_n$ .
- Combining bullet 3 and 5, we see  $x_1 \wedge \cdots \wedge x_n = 0$  if  $x_i = x_j$  for some  $i \neq j$ .
- Every element in  $\Lambda^r(M)$  is a sum of elements of the form  $x_1 \wedge \cdots \wedge x_r$  for  $x_i \in M$ .
- Suppose  $M = Rx_1 + \cdots + Rx_n$ . Every element of  $\Lambda^r(M)$  is an  $R$ -linear combination of elements of the form  $x_{i_1} \wedge \cdots \wedge x_{i_r}$  where  $1 \leq i_1 < \cdots < i_r \leq n$ . Thus  $\Lambda^r(M)$  is generated by  $\binom{n}{r}$  element for  $0 \leq r \leq n$ . Also  $\Lambda^r(M) = 0$  for  $r > n$  (since  $x_{i_j} = x_{i_k}$  for some  $j \neq k$ ).

**Definition.** An  $R$ -multilinear map  $f : \underbrace{M \times \cdots \times M}_{r \text{ times}} \rightarrow N$  on  $M$  is called **alternating** if  $f(x_1, \dots, x_n) = 0$  whenever  $x_i = x_{i+1}$  for some  $i$ .

**Example.** Let  $M = R^n$  with basis  $\{e_1, \dots, e_n\}$ . Then  $\det : M \times \cdots \times M \rightarrow R$  is the unique alternating multilinear map such that  $\det(e_1, \dots, e_n) = 1$ .

We could also define  $\Lambda^r(M)$  by the following universal property.

- Given any multilinear alternating map  $f : M \times \cdots \times M \rightarrow N$

$$\begin{array}{ccc} M \times \cdots \times M & \xrightarrow{f} & N \\ \downarrow & \nearrow & \\ \Lambda^r(M) & & \end{array}$$

there exists a unique  $R$ -homomorphism  $\Lambda^r(M) \rightarrow N$ .

Suppose  $f : M \rightarrow N$  is  $R$ -linear. Define  $f \times \cdots \times f : M \times \cdots \times M \rightarrow N \times \cdots \times N$  and compose to get a map  $M \times \cdots \times M \rightarrow N \times \cdots \times N \rightarrow \Lambda^r(N)$ . Then there exists a map  $\Lambda^r(M) \rightarrow \Lambda^r(N)$ , called it  $f \wedge \cdots \wedge f$ .

$$\begin{array}{ccc} M \times \cdots \times M & \xrightarrow{f \times \cdots \times f} & N \times \cdots \times N \\ \downarrow & & \downarrow \\ \Lambda^r(M) & \xrightarrow{f \wedge \cdots \wedge f} & \Lambda^r(N) \end{array}$$

Thus  $\Lambda^r(-) : \langle\langle R\text{-mod} \rangle\rangle \rightarrow \langle\langle R\text{-mod} \rangle\rangle$  is an additive right exact covariant functor [right exact since we are just tensoring and modding where modding is a tensor product itself].

**Theorem 3.2.** Let  $M = R^n$  with basis  $\{e_1, \dots, e_n\}$ . Then for  $1 \leq r \leq n$   $\Lambda^r(M)$  is free of rank  $\binom{n}{r}$  with basis  $\{e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq n\}$ .

*Proof.* For  $r = n$ , we know  $\Lambda^n(M)$  is generated by  $e_1 \wedge \cdots \wedge e_n$ . Now  $\det : M^n \rightarrow R$  is a multilinear alternating function and  $\det(e_1, \dots, e_n) = 1$ . By the universal property, there exists an  $R$ -module homomorphism  $\phi : \Lambda^n(M) \rightarrow R$  such that  $\phi(e_1 \wedge \cdots \wedge e_n) = 1$ . Thus  $\text{Ann}_R(e_1 \wedge \cdots \wedge e_n) = 0$  and in particular  $R \cong \Lambda^n(M)$  via  $\phi$ .

For  $1 \leq r < n$ , let  $\Lambda_r = \{\{i_1, \dots, i_r\} \mid 1 \leq i_1 < \cdots < i_r \leq n\}$ . Then  $\Lambda^r(M)$  is generated by  $\{e_J\}_{J \in \Lambda_r}$  where  $e_J = e_{i_1} \wedge \cdots \wedge e_{i_r}$  if  $J = \{i_1, \dots, i_r\}$ . Suppose  $\sum_{J \in \Lambda_r} r_J e_J = 0$  for  $r_J \in R$ . Choose  $J_0 \in \Lambda_r$  and let  $J' \in \Lambda_{n-r}$  such that  $J_0 \cup J' = \{1, \dots, n\}$  and  $J_0 \cap J' = \emptyset$ . Then  $0 = (\sum_J r_J e_J) \wedge e_{J'} = r_{J_0} e_{J_0} \wedge e_{J'}$  since if  $J \neq J_0$ , then  $J \cap J' \neq \emptyset$  which implies  $e_J \wedge e_{J'} = 0$ . So  $0 = r_{J_0} e_{J_0} \wedge e_{J'} = \pm r_{J_0} (e_1 \wedge \cdots \wedge e_n)$ . As  $e_1 \wedge \cdots \wedge e_n$  is a basis for  $\Lambda^n(M)$ ,  $r_{J_0} = 0$ .  $\square$

If  $\phi : R^n \rightarrow R^n$  is  $R$ -linear, then  $\Lambda^n(\phi) : \Lambda^n(R^n) \rightarrow \Lambda^n(R^n)$  is defined by  $e_1 \wedge \cdots \wedge e_n \mapsto (\det \phi) e_1 \wedge \cdots \wedge e_n$ . We can derive all of the properties of determinants from the functorial properties of  $\Lambda^n(-)$ . For examples,  $\Lambda^n(\phi\psi) = \Lambda^n(\phi)\Lambda^n(\psi)$  and so  $\det \phi\psi = (\det \phi)(\det \psi)$ .

We can also apply exterior algebra to the Koszul complex. In fact, one could define the Koszul complex in this way. Let  $M = R^n$  and  $x = (x_1, \dots, x_n) \in M$ . Consider the cochain complex

$$0 \rightarrow \underbrace{\Lambda^0(M)}_{=R} \xrightarrow{d^0=x} \underbrace{\Lambda^1(M)}_{=M} \xrightarrow{d^1=x} \cdots \xrightarrow{x} \Lambda^{n-1}(M) \xrightarrow{d^{n-1}=x} \underbrace{\Lambda^n(M)}_{=R} \xrightarrow{d^n} 0$$

where the maps are given by left multiplication by  $x$  (that is,  $m \mapsto x \wedge m$ ). This is a complex as  $x \wedge x = 0$ . We call this the **Koszul cochain complex** of  $K(x)$ .

**Example.** Let  $n = 3$  and  $x = (x_1, x_2, x_3)$ . The the Koszul cochain complex is given by

$$0 \rightarrow R \xrightarrow{d^0} R^3 \xrightarrow{d^1} R^3 \xrightarrow{d^2} R \rightarrow 0.$$

Here  $d^0(1) = x_1 e_1 + x_2 e_2 + x_3 e_3$ . Let  $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$  be a basis for  $R^3$ . Then  $d^1(e_1) = -x_2(e_1 \wedge e_2) - x_3(e_1 \wedge e_3)$ ,  $d^1(e_2) = x_1(e_1 \wedge e_2) - x_3(e_2 \wedge e_3)$ , and  $d^1(e_3) = x_1(e_1 \wedge e_3) + x_2(e_2 \wedge e_3)$ . Then

$$d^0 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } d^1 = \begin{pmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{pmatrix}$$

So we have an  $R$ -linear degree 1 map  $d : \Lambda(M) \rightarrow \Lambda(M)$  defined by  $d \mapsto x\alpha$  such that  $d^2 = 0$ .

To get the Koszul complex, apply  $\text{Hom}_R(-, R)$  to  $K(\underline{x})$  to get

$$K(\underline{x}) : 0 \rightarrow \Lambda^n(M) \xrightarrow{d} \Lambda^{n-1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^0(M) \rightarrow 0$$

where  $d : \Lambda^r(M) \rightarrow \Lambda^{r-1}(M)$  is defined by  $d(e_{i_1} \wedge \dots \wedge e_{i_r}) = \sum_{j=1}^r (-1)^{j-1} x_{i_j} e_{i_1} \wedge \dots \wedge \widehat{e_{i_j}} \wedge \dots \wedge e_{i_r}$ .

**Notation.** Let  $M = R^n$  and  $e_1, \dots, e_n$  be a basis. Let  $A = \{(j_1, \dots, j_r) | 1 \leq j_1 < j_2 < \dots < j_r \leq n\}$ . For  $I = (j_1, \dots, j_r) \in A$ , let  $e_I = e_{j_1} \wedge \dots \wedge e_{j_r}$ . For  $I \in A, i \in I$ , define  $\text{sgn}(i, I) = |\{j \in I | j < i\}|$ . Then the **Koszul Complex**  $K(x_1, \dots, x_n)$  on  $x_1, \dots, x_n$  is give by

$$0 \rightarrow \Lambda^n(M) \xrightarrow{d} \Lambda^{n-1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^0(M) \rightarrow 0$$

where  $d : \Lambda^r(M) \rightarrow \Lambda^{r-1}(M)$  is defined by  $e_I \mapsto \sum_{j \in I} (-1)^{\text{sgn}(j, I)} x_j e_{I \setminus \{j\}}$ . On can check that  $d^2 = 0$  and also that the **Leibniz Rule** holds, that is, for all  $a \in \Lambda^r(M), b \in \Lambda^\ell(M)$  we have  $d(ab) = d(a)b + (-1)^r ad(b)$ .

*Proof.* We will induct on  $r$  to show the Leibniz Rule holds. Let  $a \in \Lambda^1(M)$  and  $b \in \Lambda^\ell(M)$ . By linearity, it is enough to show in the case  $a = e_j$  and  $b = e_I$  where  $|I| = \ell$ . If  $j \in I$ , then  $d(e_j e_I) = 0$  and

$$d(e_j) e_I - e_j d(e_I) = x_j e_I - e_j \sum_{k \in I} (-1)^{\text{sgn}(k, I)} x_k e_{I \setminus \{k\}} = x_j e_I - e_j (-1)^{\text{sgn}(j, I)} x_j e_{I \setminus \{j\}} = x_j e_I - x_j e_I = 0.$$

Similarly if  $j \notin I$ . So suppose for all  $a \in \Lambda^r(M)$  and  $b \in \Lambda^\ell(M)$  the formula holds. Let  $a \in \Lambda^{r+1}(M), b \in \Lambda^\ell(M)$ . Then  $a = \sum r_i e_i a'_i$  for  $a'_i \in \Lambda^r(M)$ . By linearity, it is enough to show when  $a = xc$  where  $x \in \Lambda^1(M)$  and  $c \in \Lambda^r(M)$ . Then

$$\begin{aligned} d(ab) = d(xcb) &= d(x)cb - xd(cb) \\ &= d(x)cb - x(d(c)b + (-1)^r cd(b)) \\ &= [d(x)c - xd(c)]b + (-1)^{r+1} xcd(b) \\ &= d(xc)b + (-1)^{r+1} xcd(b) \\ &= d(a)b + (-1)^{r+1} ad(b) \end{aligned}$$

□

**Definition.** Let  $R = \bigoplus_{i=0}^{\infty} R_i$  be a graded ring (hence an  $R_0$ -algebra). Then  $R$  is called a **DG algebra** or differential graded algebra if

- (1) For  $a \in R_i, b \in R_j$   $ab = (-1)^{ij} ba$  and  $a^2 = 0$  if  $i$  is odd, that is,  $R$  is **graded commutative**.
- (2) There exists an  $R_0$ -linear map  $d : R \rightarrow R$  of degree  $-1$  such that for all homogenous  $a, b$  we have  $d(ab) = d(a)b + (-1)^{|a|} ad(b)$ .

**Example.** The Koszul Complex is “the” canonical DG-algebra.

**Theorem 3.3** (Gulliksen). Let  $(R, m)$  be local,  $I = (x_1, \dots, x_n), \mu(I) = n$  with  $x_1, \dots, x_{n-1}$  a regular sequence. Assume  $H_1(\underline{x}, R) := H_1(K(\underline{x})) \cong R/I$ . Then  $\text{pd}_R R/I = \infty$ .

*Proof.* Let  $K. = K(x_1, \dots, x_n) : 0 \rightarrow K_n \xrightarrow{d} K_{n-1} \xrightarrow{d} \dots \xrightarrow{d} K_0 \rightarrow 0$ . Then  $H_0(K.) = R/I$  and  $H_1(K.) = R/I\bar{z}$  where  $z \in Z_1(K.)$  is a cycle. Since  $x_1, \dots, x_{n-1}$  is a regular sequence, Lori showed  $H_i(K.) = 0$  for all  $i > 1$ . Recall if  $C.$  is a complex, then  $C.[p]$  is the complex which in degree  $n$  is  $C.[p]_n = C_{n+p}$  with differential  $(-1)^p d$ . So shift the Koszul Complex by  $-1$ :

$$\begin{array}{ccccccccccc} K.[-1] : & 0 & \longrightarrow & K_n^{n+1} & \xrightarrow{-d} & K_{n-1}^n & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K_1^2 & \xrightarrow{-d} & K_0^1 & \longrightarrow & 0 \\ & & & \downarrow \psi & & \downarrow z & & & & \downarrow z & & \downarrow z & & \\ K. : & & & 0 & \xrightarrow{d} & K_n & \xrightarrow{d} & \dots & \xrightarrow{d} & K_2 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_0 & \longrightarrow & 0 \end{array}$$

Define  $\psi_i : K_{i-1} \rightarrow K_i$  by  $a \mapsto za = z \wedge a$ . Since  $zK_{i-1} \subseteq K_i$ , this is well-defined. We see  $\psi$  is a chain map as for  $a \in K_i$ , we have  $d(za) = d(z)a - zd(a) = -zd(a) = z(-d)(a)$ . So we may consider the mapping cone

$\text{cone}(\psi) = (k[-1])_{n-1} \oplus K_n = K_{n-2} \oplus K_n$  with differential  $\text{cone}(f)_n \rightarrow \text{cone}(f)_{n-1}$  defined by  $(a, b) \mapsto (d(a), d(b) - \psi(a)) = (d(a), d(b) - za)$ . Thus we get a short exact sequence of complexes  $0 \rightarrow K \xrightarrow{\delta} C = \text{cone}(\psi) \xrightarrow{\epsilon} K[-2] \rightarrow 0$  defined by  $\delta(c) = (0, c)$  and  $\epsilon(b, c) = b$  which gives rise to a long exact sequence on homology

$$\cdots \rightarrow H_n(C) \xrightarrow{\epsilon_*} H_{n-2}(K) \xrightarrow{\psi_*} H_{n-1}(K) \xrightarrow{\delta_*} H_{n-1}(C) \rightarrow \cdots$$

In particular, after plugging in the zeros we get the exact sequence

$$0 \rightarrow H_3(C) \xrightarrow{\cong} H_1(K) \rightarrow 0 \rightarrow H_2(C) \rightarrow H_0(K) \rightarrow H_1(K) \rightarrow H_1(C) \rightarrow 0 \rightarrow H_0(K) \xrightarrow{\cong} H_0(C) \rightarrow 0$$

where  $H_i(C) = 0$  for all  $i \geq 4$ . Also, we see  $H_0(C) \cong H_0(K) = R/I$  and  $H_3(C) \cong H_1(K) = R/I\bar{z}$ . So  $H_3(C) = R/I\overline{(0, z)}$ . Of course,  $H_0(K) = R/I \cong H_1(K) = R/I\bar{z}$  and so  $H_2(C) = H_1(C) = 0$ . Notice

- (1)  $C_i$  is free of rank  $\binom{n}{i-2} + \binom{n}{i}$  and  $C$  has length  $n+2$ .
- (2)  $d(C_i) \subseteq mC_{i-1}$  (that is, it is a minimal free resolution)

*Proof.* Let  $(a, b) \in C_n$ . Then  $d(a, b) = (d(a), d(b) - za)$ . We know  $d(a) \in mK_{n-3}$  and  $d(b) \in mK_{n-1}$ . The point is to show  $z \in mK_1$ . Notice  $z \in Z_1(K)$  implies  $z = \sum r_i e_i$  where  $d(z) = \sum r_i x_i = 0$ . If  $r_i \notin m$ , then  $x_i \in (x_1, \dots, \hat{x}_i, \dots, x_n)$ , a contradiction as  $\mu(I) = n$ . Thus  $r_i \in m$  and so  $z \in mK_1$ .  $\square$

We have a free complex  $C$  such that  $d(C_i) \subseteq C_{i-1}$  for all  $i$ ,  $H_0(C) \cong R/I$ ,  $H_3(C) \cong R/I\bar{z}$ , and  $H_i(C) = 0$  for all  $i \neq 0, 3$ . Thus  $\text{pd}_R R/I \geq 3$ . Now let  $D = \text{cone}(\phi)$  where  $\phi : K[-3] \xrightarrow{z} K[-2] \rightarrow C$ . Then we get  $H_0(D) = R/I$ ,  $H_5(D) = R/Iz$  and  $H_i(D) = 0$  for all  $i \neq 0, 5$ . Again,  $D$  is a minimal free resolution and so  $\text{pd}_R R/I \geq 5$ . Inductively, we see  $\text{pd}_R R/I \geq 2n+1$  for all  $n$ .  $\square$

**Corollary 3.4.** (Kunz) *If  $R$  is a regular local ring and  $I$  is an almost complete intersection (that is,  $\mu(I) = \text{ht } I + 1$ ), then  $R/I$  is not Gorenstein.*

#### 4. REDUCED NORMAL CONES ARE DOMAINS [2]

**Definition.** Say  $(R, m)$  is **quasi-unmixed** if  $\dim \hat{R}/q = \dim \hat{R}$  for every minimal prime  $q$  of  $\hat{R}$ .

Recall the associated graded ring of  $I$  is defined to be  $gr_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ . Also,  $gr_I(R) = R[It]/IR[It]$  where  $R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$  is the Rees ring.

**Goal.** Let  $(R, m)$  be a quasi-unmixed local ring and suppose  $p \in \text{Spec } R$  with  $\text{pd}_R p < \infty$ . If  $gr_p(R)$  is reduced, then it is a domain.

To prove our goal, we will prove a stronger statement giving a bijective correspondence between primes of  $gr_I R$  and primes minimal over  $I$  where  $I$  is an ideal with  $\text{pd}_R I < \infty$ . To do so, however, we need a few preliminary results.

**Lemma 4.1** (A&M Theorem 11.22). *Let  $(R, m, k)$  be a regular local ring. Then  $gr_m(R) \cong k[t_1, \dots, t_d]$  where  $t_i$  are indeterminates and  $d = \dim R$ . In particular,  $gr_m(R)$  is a domain.*

*Proof.* Recall (from last semester) that  $\dim R = \dim gr_m(R)$ . As  $R$  is regular,  $m = (x_1, \dots, x_d)$  for a regular sequence  $x_1, \dots, x_d$ . Then  $m/m^2 = (\bar{x}_1, \dots, \bar{x}_d)$  and  $gr_m(R)$  is generated by  $\bar{x}_1, \dots, \bar{x}_d$  over  $k$ . Let  $t_i$  be indeterminates for  $i \leq d$ . Define  $\phi : k[t_1, \dots, t_d] \rightarrow gr_m(R)$  by  $t_i \mapsto \bar{x}_i$ . Clearly  $\phi$  is onto and since  $\dim k[t_1, \dots, t_d] = d = \dim gr_m(R)$ , it is an isomorphism.  $\square$

**Fact.** Suppose  $V$  is a finite dimensional  $F$ -vector space with  $|F| = \infty$ . Then  $V$  is not the union of finitely many proper subspaces.

*Proof.* Suppose  $V = W_1 \cup \dots \cup W_s$  and fix a basis  $e_1, \dots, e_n$  for  $V$ . Assume  $\dim W_i = n - 1 = \dim V - 1$  (if not, enlarge the  $W_i$  so that this holds). Fix  $W$  and let  $u_1, \dots, u_{n-1}$  be a basis for  $W$ . Write  $u_i = \sum_j r_{ji} e_j$ . Then define

$$A = \begin{bmatrix} r_{1,1} & \cdots & r_{1,n-1} & T_1 \\ \vdots & \ddots & \vdots & \vdots \\ r_{n,1} & \cdots & r_{n,n-1} & T_n \end{bmatrix}$$

for indeterminates  $T_i$ . So  $f_W(T_1, \dots, T_n) := \det A \in F[T_1, \dots, T_n]$ . Note  $f_W(c_1, \dots, c_n) = 0$  if and only if  $\sum c_i e_i \in W$ . Since  $f_W(T_1, \dots, T_n) \neq 0$  and  $F$  is infinite,  $f_W$  has infinitely many nonzero non-roots. Consider  $g = f_{W_1} \cdots f_{W_s} \neq 0$ . Then there exists a non-root of this polynomial, but it is in none of the  $W_i$ , a contradiction.  $\square$

**Lemma 4.2.** *Let  $(R, m)$  be local and consider the short exact sequence  $0 \rightarrow C \rightarrow F \rightarrow T \rightarrow 0$  where  $F$  is a finitely generated free  $R$ -module. Suppose  $0 \rightarrow C/mC \rightarrow F/mF \rightarrow T/mT \rightarrow 0$  is exact. Then  $T$  is free (and hence  $C$  is free).*

*Proof.* Apply  $-\otimes R/m$  to  $0 \rightarrow C \rightarrow F \rightarrow T \rightarrow 0$  to get a long exact sequence on  $\text{Tor}$ .

$$\cdots \rightarrow \text{Tor}_1^R(T, R/m) \rightarrow C/mC \xrightarrow{\phi} F/mF \rightarrow T/mT \rightarrow 0.$$

By assumption,  $\phi$  is injective and so  $\text{Tor}_1^R(T, R/m) = 0$ . Thus  $\beta_1(T) = 0$  and so  $T$  is free.  $\square$

**Proposition 4.3.** *Let  $(R, m)$  be local. Let  $\phi : F \rightarrow G$  be a map of finitely generated free modules. If  $\phi \otimes \bar{1} : F/mF \rightarrow G/mG$  is injective, then  $\phi$  splits.*

*Proof.* Let  $T = \text{coker } \phi$  and  $C = \text{im } \phi$ . Then we have

$$\begin{array}{ccccc} F & \xrightarrow{\phi} & G & \longrightarrow & T & \longrightarrow & 0 \\ & \searrow \phi & \nearrow & & & & \\ & & C & & & & \\ & \nearrow & \searrow & & & & \\ 0 & & & & 0 & & \end{array}$$

Apply  $-\otimes R/m$  to get the following.

$$\begin{array}{ccccc} F/mF & \xrightarrow{\phi \otimes \bar{1}} & G/mG & \longrightarrow & T/mT & \longrightarrow & 0 \\ & \searrow \phi \otimes \bar{1} & \nearrow & & & & \\ & & C/mC & & & & \\ & & & & & & \\ & & & & & & 0 \end{array}$$

As  $\phi \otimes \bar{1}$  is injective and  $C = \text{im } \phi$ , we see  $C/mC \rightarrow G/mG$  is injective. By Lemma 4.2, we get that  $T$  and  $C$  are free. Consider  $0 \rightarrow K \rightarrow F \rightarrow C \rightarrow 0$  where  $K = \ker(F \rightarrow C)$ . Since  $C$  is free, the sequence splits. Thus  $F = C \oplus K$  and  $F/mF = C/mC \oplus K/mK$ . As  $\phi \otimes \bar{1}$  is injective,  $F/mF \cong C/mC$  and hence  $K/mK = 0$ . By Nakayama's Lemma,  $K = 0$ . Note that  $0 = \ker(F \rightarrow C) = \ker \phi$  and hence  $0 \rightarrow F \xrightarrow{\phi} G \rightarrow T \rightarrow 0$  is exact. Since  $T$  is free, the sequence splits and hence  $\phi$  splits.  $\square$

We now prove the main result.

**Theorem 4.4.** *Let  $R$  be a quasi-unmixed local ring and  $I < R$  with  $\text{pd}_R I < \infty$ . Assume  $\text{gr}_I(R)$  is reduced. Then there exists a bijective correspondence*

$$\{\text{minimal primes } \bar{Q} \text{ of } \text{gr}_I(R)\} \leftrightarrow \{\text{primes } q \text{ minimal over } I\}$$

*defined by  $\bar{Q} \cap R/I = q/I$ . In particular, if  $I$  is prime, then  $\text{gr}_I(R)$  is a domain.*

*Proof.* Assuming the correspondence, if  $I$  is prime then there is exactly one minimal prime over  $I$  (namely  $I$  itself). Hence there exists only one minimal prime of  $\text{gr}_I(R)$ , call it  $q$ . As  $\text{gr}_I(R)$  is reduced,  $0 = \cap \text{Min } \text{gr}_I(R) = q$ . Thus  $\text{gr}_I(R)$  is a domain.

To prove the bijective correspondence, first suppose we are given a prime  $q$  of  $R$  minimal over  $I$ . Let  $\bar{q} = q/I$ . Note  $R/I \subseteq \text{gr}_I(R)$  is reduced and hence  $(R/I)_{\bar{q}} \cong R_q/I_q$  is a field. Thus  $I_q = q_q$ . Since  $\text{pd}_R I < \infty$ , we see  $\text{pd}_{R_q} I_q < \infty$  (as localizing is faithfully flat) and so  $\text{pd}_{R_q} R_q/q_q = \text{pd}_{R_q} q_q + 1 < \infty$  (for the equality, consider the short exact sequence  $0 \rightarrow q_q \rightarrow R_q \rightarrow R_q/q_q \rightarrow 0$ ). Thus  $R_q$  is a regular local ring and hence  $\text{gr}_{I_q}(R_q)$  is a domain by Lemma 4.1.

*Claim.*  $K = \ker(\text{gr}_I(R) \rightarrow \text{gr}_{I_q}(R_q))$  is a minimal prime of  $\text{gr}_I(R)$  lying over  $\bar{q} = q/I$ .

*Proof.* Since  $\text{gr}_{I_q}(R_q) \cong \text{gr}_I(R)/K$  is a domain,  $K$  is prime. To show it is minimal, suppose  $K \supseteq Q$  for a prime  $Q$  in  $\text{gr}_I(R)$ . Notice  $K_0 = \bar{q}$  [if  $r \in R/I$  with  $r \mapsto 0$ , then  $\frac{r}{q} \in I_q = q_q$  implies there exists  $i \in I, s \in R \setminus q$  such that  $\frac{r}{1} = \frac{i}{s}$ , which implies there exists  $t \in R \setminus Q$  such that  $t(rs - i) = 0$ , that is,  $trs \in q$  and hence  $r \in q$ ]. So  $K$  and  $Q$  do not intersect  $(R/I) \setminus \bar{q}$ . By the bijective correspondence of primes when localizing, we see  $K_{\bar{q}} \supseteq Q_{\bar{q}}$  and  $K_{\bar{q}} = Q_{\bar{q}}$  if and only if  $K = Q$ . Of course,  $K_{\bar{q}} = 0$  as it is the kernel of the localizing map and so  $Q_{\bar{q}} = 0$ . So  $K = Q$ .

In particular, this gives us that  $K \cap R/I = q/I$ . We wish to show all minimal primes have this form.

Given a minimal prime of  $\text{gr}_I(R)$ , in order to show it corresponds to some minimal prime over  $I$ , we will replace  $R$  with  $R(x) = R[x]_{m[x]}$ . Then we may assume the residue field is infinite as  $R[x]_{m[x]}/m[x]_{m[x]} \cong (R[x]/mR[x])_W = (k[x])_W = k(x)$  where  $W$  is the set of all nonzero elements.

*Claim.* The assumptions of the theorem hold for  $R(x)$  and if the conclusions of the theorem hold for  $R(x)$ , then they hold for  $R$ .

*Proof.* The main point of the proof is the fact that  $R \rightarrow R[x] \rightarrow R[x]_{mR[x]} = R(x)$  is a faithfully flat extension. First we show the assumptions hold for  $R(x)$ .

- $R(x)$  is quasi-unmixed: Ratliff showed that  $R(x)$  is a quasi-unmixed local ring in the case that  $R$  is (Corollary 2.9 of [4]).
- $\text{pd}_{R(x)} IR(x) < \infty$ : Since  $R(x)$  is faithfully flat,  $- \otimes_R R(x)$  is exact. Take a finite free resolution for  $R/I$ :

$$0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow R/I \rightarrow 0$$

and apply  $- \otimes_R R(x)$ :

$$0 \rightarrow F_n \otimes_R R(x) \rightarrow F_{n-1} \otimes_R R(x) \rightarrow \cdots \rightarrow F_0 \otimes_R R(x) \rightarrow R(x)/IR(x) \rightarrow 0.$$

Since  $F_i \otimes_R R(x) \cong R^{n_i} \otimes_R R(x) \cong R(x)^{n_i}$  are free, we see  $\text{pd}_{R(x)} R(x)/IR(x) < \infty$  and so  $\text{pd}_{R(x)} IR(x) < \infty$ .

- $\text{gr}_{IR(x)} R(x)$  is reduced: First consider the faithfully flat extension  $R \rightarrow S := R[x]$ . Given a short exact sequence  $0 \rightarrow J \rightarrow I \rightarrow I/J \rightarrow 0$ , applying  $- \otimes_R S$  gives us  $0 \rightarrow J \otimes_R S \rightarrow I \otimes_R S \rightarrow I/J \otimes_R S \rightarrow 0$ . By the Five Lemma, since  $J \otimes_R S \cong JS$  and  $I \otimes_R S \cong IS$ , we see  $I/J \otimes_R S \cong IS/JS$ . In particular, this says  $\text{gr}_{IS}(S) \cong \text{gr}_I(R)[x]$ . Thus if  $\text{gr}_I(R)$  is reduced, then  $\text{gr}_{IS}(S)$  is reduced and stays reduced when we localize. Thus  $\text{gr}_{IR(x)}(R(x))$  is reduced.

We still need to show we can descend the conclusions of the theorem from  $R(x)$  to  $R$ . In general for a ring  $T$ , we know minimal primes of  $T$  are in 1-1 correspondence with minimal primes of  $T[x]$ . So

minimal primes of  $\text{gr}_I(R)$  are in 1-1 correspondence with minimal primes of  $\text{gr}_I(R)[x] = \text{gr}_{IR[x]}(R[x])$ . Now given a prime  $Q[x]$  in  $\text{gr}_I(R)[x]$  corresponding to  $Q \in \text{Min}(\text{gr}_I(R))$ , we see  $Q[x]_W$  is minimal in  $\text{gr}_{IR[x]}(R[x])_W = \text{gr}_{IR(x)}(R(x))$  if  $Q \cap W = \emptyset$  for  $Q = (R/I) \setminus (m/I)$ . Thus a minimal prime  $Q$  of  $\text{gr}_I(R)$  corresponds to a minimal prime  $\overline{Q}_W$  in  $\text{gr}_{IR(x)}(R(x))$  which as we will show corresponds to a prime  $q$  of  $R(x)$  minimal over  $IR(x)$ . Again,  $q$  corresponds to a prime  $q'$  minimal over  $IR[x]$  and minimal primes of  $R[x]$  over  $IR[x]$  are minimal over  $R[x]/IR[x] = R/I[x]$  which are in 1-1 correspondence with primes over  $I$ .

So replace  $R$  with  $R(x)$  and assume  $R$  has an infinite residue field. Let  $\overline{Q}$  be a minimal prime of  $\text{gr}_I(R)$  and  $Q = \phi^{-1}(\overline{Q})$  for  $\phi: S = R[It] \rightarrow S/IS = \text{gr}_I(R)$ . Then  $Q$  is minimal over  $IS$ . Let  $q = Q \cap R$ . It suffices to show  $q$  is minimal over  $I$ . Ratliff showed that  $R_q$  is still quasi-unmixed (see Lemma 2.5 of [3]) and so we may replace  $R$  with  $R_q$  and assume  $Q \cap R = m$ . By Theorem 3.8 of [4], we have that  $\dim S/Q = \dim R =: d$ .

Since  $Q \cap R = m$ , we have  $IS \subseteq mS \subseteq Q$ . Since  $(\text{gr}_I(R))_Q = (S/IS)_Q = S_Q/(IS)_Q$  is a field, we see  $(IS)_Q = Q_Q$  and in particular  $(IS)_Q = (mS)_Q$ . Now  $mS = (x_1, \dots, x_k)$  for some homogenous  $x_i \in S$ . So for all  $i$  there exists  $f_i \in S$  and  $n_i \in \mathbb{N}$  such that  $f_i t^{n_i} x_i \in IS$  (in general we know there exists some  $s \in S \setminus Q$  such that  $s x_i \in IS$ , but  $IS$  is homogenous and so each homogenous component must also be in  $IS$ ). Define  $f := f_1 \cdots f_k$  and  $n := n_1 + \cdots + n_k$ . Then  $f t^n x_i \in IS$  for all  $i$  and thus  $f t^n (mS) \subseteq IS$ . Translating this back to  $R$ , we see by comparing homogenous parts that

$$f \in I^n, mf \subseteq I^{n+1}, \text{ and } f t^n \notin Q \text{ implies } f \notin mI^n (*).$$

Now  $S/Q$  is a graded  $k$ -algebra generated in degree 1 (since  $S = R[It]$ ) and  $k$  is infinite.

*Claim.* We may choose  $a_1 t, \dots, a_d t \in S$  such that their images in  $S/Q$  form a homogenous system of parameters.

*Proof.* Let  $A := S/Q = k \oplus A_1 \oplus \cdots$  where  $A_i = (A_1)^i$  as  $A$  is generated in degree 1. Let  $P_i \in \text{Ass } A$ . We wish to show  $A_1 \not\subseteq (P_1 \cap A_1) \cup \cdots \cup (P_S \cap A_1)$ . So suppose not. Then by the Fact we must have  $A_1 = P_1 \cap A_1$  and so  $A_1 \subseteq P_1$ . Thus  $A \subseteq P_1$  since  $A_1$  generated  $A$ , which implies  $A$  is 0-dimensional. This contradicts the fact that  $\dim S/Q = d$ . Thus  $A_1 \not\subseteq (P_1 \cap A_1) \cup \cdots \cup (P_S \cap A_1) \subseteq \cup P_i$ . So there exists a homogenous degree 1 element in  $A \setminus \cup P_i$ , call it  $a_1 t$ . This element is  $A$ -regular. Given  $a_1 t, \dots, a_i t$  for  $i < d$ , repeat the argument on  $M = S/(Q, a_1 t, \dots, a_i t)$ , which has dimension  $d - i$  (since  $a_1 t, \dots, a_i t$  form an  $A$ -regular sequence) to get  $a_{i+1} t$ . Stop at  $i = d$ .

Note that  $a_1, \dots, a_d \in I$ . We claim the images of  $a_1, \dots, a_d$  are linear independent in  $m/m^s$ . Suppose not. Then there exists  $r_i \in R$  such that  $\sum_{i=1}^d r_i a_i \in m^2$  with some  $r_i$ , say  $r_j$ , not in  $m$ . Let  $a'_j = \sum_{i=1}^d r_i a_i \in m^2$  and note  $(a_1 t, \dots, a_d t) = (a_1 t, \dots, a_{j-1} t, a'_j t, a_{j+1} t, \dots, a_d t)$ . Hence  $a_1 t, \dots, a'_j t, \dots, a_d t$  form a system of parameters for  $S/Q$ . Since  $\dim S/Q = d$ , we see  $a'_j \notin Q$ . Now

$$(a'_j)^n f = (a'_j)^{n-1} a'_j f \in I^{n-1}(m^2 f) \subseteq mI^{n-1}(mf) \subseteq mI^{2n}$$

by (\*). So  $(a'_j t)^n f t^n \in mI^{2n} t^{2n} \subseteq mS \subseteq Q$ . Since  $Q$  is prime and neither  $a'_j t$  nor  $f t^n$  are in  $Q$ , we have a contradiction. Thus  $a_1, \dots, a_d \in I$  are linearly independent modulo  $m^2$ .

Since  $I$  is reduced,  $I = \sqrt{I}$  and thus  $I = \sqrt{I} = \cap \{Q_i | Q_i \text{ is a minimal prime over } I\}$ . If  $m$  is not minimal over  $I$ , then  $m \neq Q_i$  for all  $i$ . By the prime avoidance lemma, there exists  $x \in m \setminus \cup Q_i$ . Then  $\bar{x}$  is a regular sequence in  $R/I$  and so  $\text{depth } R/I \geq 1$ . By Auslander Buchsbaum,  $\text{pd}_R(R/I) + \text{depth } R/I = \text{depth } R \leq \dim R = d$  and so  $\text{pd}_R(R/I) < d$ . Let  $F = 0 \rightarrow F_k \rightarrow \cdots \rightarrow F_1 \rightarrow R \rightarrow R/I \rightarrow 0$  be a minimal finite free resolution of  $R/I$  and  $K$  be the Koszul Complex for  $a_1, \dots, a_d$ . Since  $R/(a_1, \dots, a_d) \rightarrow R/I$ , the comparison theorem for projective resolutions

gives us the following

$$\begin{array}{ccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & F_k & \longrightarrow & \cdots & \longrightarrow & F_1 & \longrightarrow & R \\
& & & & \uparrow \phi_k & & & & \uparrow \phi_1 & & \parallel \phi_0 \\
\phi_d \uparrow & & & & & & & & & & \\
0 & \longrightarrow & K_d & \xrightarrow{\alpha_d} & \cdots & \longrightarrow & K_k & \xrightarrow{\alpha_k} & \cdots & \longrightarrow & K_1 & \xrightarrow{\alpha_1} & R
\end{array}$$

We will prove inductively that  $\phi_i$  split, that is, there exists  $\theta_i : F_i \rightarrow K_i$  such that  $\theta_i \phi_i = 1_{K_i}$ . Since  $\phi_d$  certainly does not split, we will reach a contradiction to the fact that  $m$  is not minimal over  $I$ .

*Claim.* Suppose  $z \in K_i, z \notin mK_i$ . Then  $\alpha_i z \notin m^2 K_{i-1}$ .

*Proof.* Recall  $K_i$  is free with basis  $\{e_{j_1} \wedge \cdots \wedge e_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq d\}$ . Write  $z = \sum_{|I|=i} r_I e_I$  where  $I = (j_1, \dots, j_i)$  for  $j_1 < \cdots < j_i$  and  $e_I = e_{j_1} \wedge \cdots \wedge e_{j_i}$ . Let  $e_{I-\{k\}} = e_{j_1} \wedge \cdots \wedge e_{j_k} \wedge \cdots \wedge e_{j_i}$ . Then by definition we have  $\alpha_i(e_I) = \sum_{k=1}^i (-1)^{k-1} a_{j_k} e_{I-\{k\}}$ . Let  $\text{sgn}(k, I) = |\{\ell \in I \mid \ell < k\}|$ . So

$$\alpha_i(z) = \sum_{|I|=i} \sum_{k=1}^i (-1)^{\text{sgn}\{k, I\}} r_I a_{j_k} e_{I-\{k\}}.$$

If  $|J| = i - 1$ , then the coefficient of  $e_J$  in  $\alpha_i(z)$  is

$$\sum_{\ell \notin J, 1 \leq \ell \leq d} (-1)^{\text{sgn}\{\ell, J \cup \{\ell\}\}} r_{J \cup \{\ell\}} a_\ell.$$

If  $\alpha_i(z) \in m^2 K_{i-1}$ , then the coefficient of  $e_J$  would be in  $m^2$  since the  $e_J$ 's form a basis. Since  $\{a_i\}$  are linearly independent in  $m/m^2$ , we must in fact have  $r_I \in m$ . Then  $z \in mK_i$ , a contradiction.

Obviously,  $\phi_0$  splits. So suppose  $\phi_i$  splits for  $i < d$ . Consider the diagram

$$\begin{array}{ccc}
F_{i+1} & \xrightarrow{\beta_{i+1}} & F_i \\
\phi_{i+1} \uparrow & & \uparrow \phi_i \\
K_{i+1} & \xrightarrow{\alpha_i} & K_i
\end{array}$$

Suppose  $z \in K_{i+1}$  with  $z \notin mK_{i+1}$ . We claim  $\phi_{i+1}z \notin mF_{i+1}$ . If  $\phi_{i+1}z \in mF_{i+1}$ , then  $\phi(\alpha_{i+1}z) = \beta_{i+1}(\phi_{i+1}z) \in \beta_{i+1}(mF_{i+1}) \subseteq m^2 F_i$  since  $F$  is a minimal free resolution (so all entries are in  $m$ ). As  $\phi_i$  splits,  $\alpha_i z \in m^2 K_i$ , a contradiction to the claim. Thus  $\phi_{i+1}z \notin mF_{i+1}$ . Thus  $\bar{\phi}_{i+1} : K_{i+1}/mK_{i+1} \rightarrow F_{i+1}/mF_{i+1}$  is injective. By Proposition 4.3,  $\phi_{i+1}$  splits.  $\square$

**Remark.** We can apply the theorem to a regular local ring  $R$  with prime  $p$ . In the proof there exists  $f \in p^n \setminus mp^n$  such that  $mf \subseteq p^{n+1}$ . In particular,  $\bar{f} \in p^n/p^{n+1}$  is a socle element which is a minimal generator. Thus a copy of  $k = R/m$  must split out of  $p^n/p^{n+1}$ . **Conjecture:** This is impossible unless  $p = m$ .

**Corollary 4.5.** *Let  $R$  be a quasi-unmixed local ring and  $I$  a reduced ideal having finite projective dimension. TFAE*

- (1)  $\text{gr}_I(R)$  is reduced
- (2)  $\text{gr}_I(R)$  is torsion free over  $R/I$  (that is, every non-zero-divisor of  $R/I$  is a non-zero-divisor on  $\text{gr}_I(R)$ )

*Proof.* For (1)  $\Rightarrow$  (2), the theorem says if  $Q$  is a minimal prime of  $\text{gr}_I(R)$ , then  $Q \cap R/I = q/I$  where  $q$  is minimal over  $I$ . Since  $\text{gr}_I(R)$  is reduced,  $\text{Ass}(\text{gr}_I(R)) = \text{Min}(\text{gr}_I(R))$ . Hence a non-zero-divisor of  $R/I$  is not in any associated prime of  $\text{gr}_I(R)$  (as otherwise, it would be in  $q/I \in \text{Min } R/I \subseteq \text{Ass } R/I$ ). Thus a non-zero-divisor of  $R/I$  is a non-zero-divisor on  $\text{gr}_I(R)$ .

For (2)  $\Rightarrow$  (1), let  $Q \in \text{Ass } \text{gr}_I(R)$  and set  $Q \cap R/I = q/I$ . Since  $\text{gr}_I(R)$  is torsion free, the prime avoidance lemma gives us that  $q \subseteq q' \in \text{Ass } R/I$ . Also  $I$  reduced implies  $I = \sqrt{I}$  and so  $R/I$  is reduced. Then  $q'$  is minimal over  $I$  (since  $\text{Min } R/I = \text{Ass } R/I$ ) and so  $q = q'$ . As in the proof of the theorem,  $q$  minimal implies  $I_q = q_q$ . Thus  $\text{pd}_{R_q} q_q < \infty$ , which implies  $R_q$  is a regular local ring and hence  $\text{gr}_{q_q}(R_q)$  is a domain. So  $(\text{gr}_I(R))_Q$  is a domain.

Since  $Q \in \text{Ass gr}_I(R)$ ,  $Q_Q$  is minimal and thus  $(\text{gr}_I(R))_Q$  is a zero-dimensional domain, that is, a field. Since  $Q$  was arbitrary,  $\text{gr}_I(R)$  is reduced [since a ring  $S$  is reduced if and only if  $S_p$  is a field for all  $p \in \text{Ass } S$ ].  $\square$

## 5. EVERY ALGEBRAIC SET IN $n$ -SPACE IS THE INTERSECTION OF $n$ HYPERSURFACES

This paper was written by Eisenbud and Evans in 1973. We split the result into two cases: the affine and projective case. We first prove the result in the affine case.

**Theorem 5.1.** *Let  $R$  be a Noetherian ring of the form  $R = S[x]$  for a ring  $S$  with  $\dim R = n$ . Let  $I \subseteq R$  be an ideal. Then there exists  $g_1, \dots, g_n \in I$  such that  $\sqrt{I} = \sqrt{(g_1, \dots, g_n)}$ .*

*Proof.* We induct on  $n$ . If  $n = 1$ , then  $\dim S = 0$  and so  $S$  is Artinian. Let  $N$  be the nilpotent ideal of  $S$ . Let  $\bar{R} = R/NR = (S/N)[x]$  and  $\bar{I} = (I + NR)/NR$ . Since  $S$  is Artinian,  $N = \cap m_i$  for  $m_i$  maximal and so by the Chinese Remainder Theorem we see  $S/N$  is the product of finitely many fields.

*Claim.*  $\bar{R}$  is a principal ideal ring.

*Proof.* Say  $S/N = F_1 \times \dots \times F_k$  for  $F_i$  fields. Notice  $(F_1 \times \dots \times F_k)[x] \cong F_1[x] \times \dots \times F_k[x]$ . Let  $0 \neq I \subseteq \bar{R}$  be an ideal. Choose  $f \in I$  such that  $f = (f_1, \dots, f_k)$  where  $f_i$  is such that  $(0, \dots, f_i, \dots, 0) \in I$  has least degree of its form. Then for any  $g \in I$ , say  $g = (g_1, \dots, g_k)$ , the division algorithm on  $F_i[x]$  gives us  $g_i = f_i h_i + r_i$  where  $r_i = 0$  and  $\deg r_i < \deg f_i$ . Since  $f_i$  has least degree,  $r_i = 0$ . Thus  $g = f(h_1, \dots, h_k)$ .

Since  $\bar{R}$  is a PIR, there exists  $\bar{g} \in \bar{I}$  such that  $(\bar{g}) = \bar{I}$ . Now since  $NR$  is nilpotent, it is contained in every prime. So, since  $(g + NR)/NR = (I + NR)/NR$ , we see  $\sqrt{(g)} = \sqrt{(I)}$ .

Suppose  $n > 1$ . Let  $P_1, \dots, P_k$  be the minimal primes of  $S$  and let  $U = S \setminus \cup_1^k P_i$ . Then  $S_U$  has dimension 0 and  $R_U = S_U[x]$  has dimension 1. By the  $n = 1$  case, there exists  $g_1 \in I_U$  such that  $\sqrt{(g_1)}_U = \sqrt{I}_U$ . Multiplying by an element of  $U$  if necessary, we may assume  $g_1 \in I$ . Now since  $I$  is finitely generated, we can find  $u \in U$  such that  $uI \subseteq \sqrt{(g_1)}$ . Since  $u \notin \cup_1^k P_i$ , we have  $\dim S/(u) \leq n - 2$  and  $\dim R/(u) \leq n - 1$ . By induction, there exists  $\bar{g}_2, \dots, \bar{g}_n \in (I + (u))/(u)$  such that  $\sqrt{(\bar{g}_2, \dots, \bar{g}_n)} = \sqrt{(I + (u))/(u)}$ . Let  $g_2, \dots, g_n \in I$  such that they reduce to  $\bar{g}_2, \dots, \bar{g}_n$  in  $(I + (u))/(u)$ .

We claim  $\sqrt{I} = \sqrt{(g_1, \dots, g_n)}$ . We know  $(g_1, \dots, g_n) \subseteq I$  and so  $\sqrt{(g_1, \dots, g_n)} \subseteq \sqrt{I}$ . We want to show for a prime  $p$  with  $p \supseteq (g_1, \dots, g_n)$  that  $p \supseteq I$ . Suppose  $p \supseteq (g_1)$ . Then  $p \supseteq (u)$  or  $p \supseteq I$ . If  $p \supseteq (u)$ , then  $\bar{p} = p/(u) \subseteq \bar{R}$  is prime and so  $\bar{p} \supseteq \bar{I}$ . Thus  $p = p + (u) \supseteq I + (u) \supseteq I$ . If  $p \supseteq I$ , done.  $\square$

**Corollary 5.2.** *Every algebraic set in affine  $n$ -space is the intersection of  $n$  hypersurfaces.*

*Proof.* Let  $V$  be an algebraic set. Then  $V = V(I) = V(\sqrt{I}) = V(\sqrt{(g_1, \dots, g_n)}) = V(g_1, \dots, g_n) = \cap_1^n V(g_i)$  for some ideal  $I$  and  $g_1, \dots, g_n \in I$ .  $\square$

We now show the result in the projective case. Let  $S = \oplus_{i \geq 0} S_i$  be a graded ring,  $S_+ = \oplus_{i > 0} S_i$ . If  $R = S[x]$ , then  $R$  is graded by  $R_i = \oplus_{j+k=i} S_j x^k$ .

**Definition.** *A relevant prime of  $S$  is a homogenous prime ideal  $p$  such that  $S_+ \not\subseteq p$ . If  $S$  is Noetherian, the projective dimension of  $S$  is the length of a maximal chain of relevant primes.*

**Example.** Let  $S = k[x_1, \dots, x_n]$ . Then  $S_+ = (x_1, \dots, x_n)$  does not correspond to a point in projective space. So the relevant primes are precisely those that correspond to a point in projective space.

**Note.**  $\text{projdim } S = -1$  if and only if  $S_+$  is nilpotent. If  $\text{projdim } S = n$ , then  $\text{projdim } S[x] \geq n + 1$ .

**Lemma 5.3.** *Let  $S$  be a ring and  $f, g \in S[x]$  polynomials of degrees  $d, e$  respectively with  $d \leq e$ . If  $u \in S$  is the leading coefficient of  $f$ , then for all  $n > e - d$  there exists  $h, r \in S[x]$  such that  $u^n g = fh + r$  where  $\deg r < d$ . Moreover, if  $S$  is graded and  $f, g$  are homogenous, then  $h$  and  $r$  can be chosen to be homogenous as well.*

**Theorem 5.4.** *Let  $R$  be a graded Noetherian ring of the form  $R = S[x]$  for some graded Noetherian ring  $S$  of projective dimension  $n - 1$ . Suppose  $I \subseteq S_+R$  is a homogenous ideal. Then there exists  $n$  homogenous elements  $g_1, \dots, g_n$  such that  $\sqrt{I} = \sqrt{(g_1, \dots, g_n)}$ .*

*Proof.* Induct on  $n$ . If  $n = 0$ , then  $\text{projdim } S = -1$  and  $\sqrt{I} = \sqrt{(0)} = \sqrt{\emptyset}$ . Suppose  $n > 0$ . Let  $p_1, \dots, p_k$  be the minimal relevant primes of  $S$ . Want  $U \notin \cup_1^k p_i$  and a homogenous element  $g_1 \in I$  such that  $uI \subseteq \sqrt{(g_1)}$ . The result will then follow as in the affine case. For  $i = 1, \dots, k$  let

- $I_i = (I + p_i R)/p_i R \subseteq R/p_i R = S/p_i[x]$ .
- If  $I \subseteq p_i$ , define  $h_i = 0$ . Otherwise, choose  $h_i \in I \setminus \{0\}$  which reduces modulo  $p_i R$  to an element  $\overline{h_i}$  of  $I_i$  of least degree in  $x$ .
- If  $\overline{h_i} = 0$ , choose  $u_i$  to be any homogenous element not in  $p_i$ . Otherwise, choose  $u_i$  to be an element reducing modulo  $p_i R$  to the leading coefficient of  $\overline{h_i}$ .
- Choose  $s_i \in S$  homogenous such that  $s_i \in \cap_{j \neq i} p_j - p_i$  (since  $p_j$  are minimal,  $p_j - p_i \neq 0$  for all  $j$  so this is well-defined)
- Choose  $s \in S_1 \setminus \cup_1^k p_i$  (this exists as otherwise  $p_i \subseteq S_+$  for some  $i$ , a contradiction)

Multiplying each  $s_i, u_i, h_i$  be a power of  $s$  if necessary, we may assume  $\deg u_i = \deg u_j, \deg s_i = \deg s_j$ , and  $\deg h_i = \deg h_j$  for all  $i, j$ . Let  $g_1 = \sum_1^k s_i h_i$  (homogenous by previous sentence) and  $u = (s(\sum_1^k s_i u_i))^N$ , where  $N$  is yet to be determined. Note  $u \notin \cup_1^k p_i$  since otherwise  $s \in p_i$  or  $\sum s_i u_i \in p_i$  for some  $i$  but  $s \notin p_i$  and if  $\sum s_j u_j \in p_i$  then  $s_i u_i \in p_i$  as  $s_j p_j \in p_i$  for all  $j \neq i$ , a contradiction.

Fix  $i$  and notice  $\overline{g_1} = \overline{s_i h_i} \in R/p_i R$ . So  $\overline{g_1}$  has least degree in  $x$  in  $I_i$ . Also  $\overline{u} = (\overline{s s_i u_i})^N$  is a multiply of a power of the leading coefficient of  $\overline{g_1}$ . Let  $I = (f_1, \dots, f_r)$  so that  $I_i = (\overline{f_1}, \dots, \overline{f_r})$ . Applying the lemma, we see  $\frac{\overline{u}}{\overline{s}} \overline{f_j} = \overline{g_1} \overline{q_j} + \overline{r_j}$  where  $\deg_x \overline{r_j} < \deg_x \overline{g_1}$  and we now impose  $N > \max\{\deg_x f_i - \deg_x g_1\}$ . Since  $\frac{\overline{u}}{\overline{s}} \overline{f_j}, \overline{g_1} \overline{q_j} \in I_i$ , we see  $\overline{r_j} \in I_i$ . Since  $\overline{g_1}$  has least degree, we see  $\overline{r_j} = 0$ . Thus  $\overline{u} I_i \subseteq (\overline{g_1})$  and thus  $uI \subseteq S_+R \cap ((g_1) + p_i R)$  for all  $i$ . Since every prime of  $R$  contains either  $S_+R$  or some  $p_i R$ , it follows that  $uI \subseteq \sqrt{(g_1)}$ .  $\square$

**Corollary 5.5.** *Every algebraic set in projective  $n$ -space is the intersection of  $n$  hypersurfaces.*

**Example.** Let  $R = k[x_0, \dots, x_n]$  and  $V \subseteq \mathbb{P}^n$ . Then  $V = V(I)$  for  $I \subseteq R$  homogenous. Let  $S = K[x_0, \dots, x_{n-1}]$  so that then  $S_+ = (x_0, \dots, x_{n-1})$ . For every homogenous ideal in  $S_+R$ , the theorem says  $\sqrt{I} = \sqrt{(g_1, \dots, g_n)}$ .

Suppose  $k$  is algebraically closed. If  $V(I) \neq \emptyset$ , then we may do a change of coordinates so that  $I \subseteq S_+k[x_0, \dots, x_n]$  and thus the theorem applies. In general, we do not seem to have this luxury.

**Question.** Given *any* homogenous ideal  $I \subseteq k[x_0, \dots, x_n]$ , is  $\sqrt{I} = \sqrt{(g_1, \dots, g_n)}$  for  $g_i \in I$  homogenous?

- Yes when  $k = \overline{k}$  (as proven by Eisenbud and Evans above) and when  $\text{char } k = p$  (as proven by Ganadi).
- It is still unknown when  $\text{char } k = 0$ , even in the case of  $\mathbb{P}^2$ . In particular, if  $R = \mathbb{Q}[x_0, x_1, x_2]$  and  $p$  is a homogenous prime of height 2, we do not know if there exists homogenous  $f, g \in R$  such that  $p = \sqrt{(f, g)}$ .

#### REFERENCES

- [1] Auslander, Maurice; Buchweitz, Ragnar-Olaf *The homological theory of maximal Cohen-Macaulay approximations*. Mémoires de la Société Mathématique de France Sér. 2, 38 (1989), p. 5-37 [[http://www.numdam.org/item?id=MSMF\\_1989\\_2\\_38\\_5\\_0](http://www.numdam.org/item?id=MSMF_1989_2_38_5_0)]
- [2] Huneke, C., Simis, A., and Vasconcelos, W., *Reduced Normal Cones are Domains*, Invariant Theory, Proceedings of an AMS Special Session held Oct 31- Nov 1, 1986.
- [3] Ratliff, L. J., *On Quasi-Unmixed Semi-Local Rings and the Altitude Formula*, American Journal of Math 87 (1965), 178-284. [<http://www.jstor.org/stable/237005>]
- [4] Ratliff, L. J., *On Quasi-Unmixed Local Domains, the Altitude Formula, and the Chain Condition for Prime Ideals II*, American Journal of Math 92 (1970), 99-144. [<http://www.jstor.org/stable/2373501>]