1. Graded Rings

Definition. *Let G be an abelian group (written additively) and R a commutative ring. A $G-$**grading** for R is a family $\{R_g\}_{g\in G}$ of abelian groups of $(R,+)$ such that $R=\oplus_{g\in G}R_g$ (as abelian groups) and $R_gR_h\subseteq R_{g+h}$ for all $g,h\in G$. The elements of R_g are called the **homogeneous elements** of R of degree g. If $r\in R_g$, we write the **degree** of r as $\deg r=g$ or $|r|=g$.*

Note. Any ring is a graded ring by letting $R_0=R$ and $R_i=0$ for all $i\neq 0$.

Examples.

(1) Let $R=S[x]$ and G be any abelian group. Set $|x|=g$ for some $g\in G$. For $h\in G$, we see $R_h=\oplus_{ig=h}Sx^i$. Note that $S\subseteq R_0$.

- If $G=\mathbb{Z}$ and $|x|=1$, then for $n\in\mathbb{Z}$, $R_h=\begin{cases}Sx^h & \text{if } n\geq 0,\\ 0 & \text{if } n<0.\end{cases}$. This is an $\mathbb{N}-$grading.
- If $G=\mathbb{Z}/n\mathbb{Z}$ and $|x|=\overline{1}$, then for $\overline{m}\in\mathbb{Z}_n$ with $0\leq m\leq n-1$, we see $R_{\overline{m}}=\oplus_{\overline{i}=\overline{m}}Sx^i=\oplus_{k=0}^{\infty}Sx^{m+kn}$. So $R_{\overline{0}}=\oplus_{k=0}^{\infty}Sx^{kn}=S[x^n]\subseteq R$.

(2) Let $R=S[x_1,...,x_d]$ and G be any abelian group. Set $|x_i|=g_i$. For $h\in G$, we have $R_h=\displaystyle\bigoplus_{\alpha_1g_1+\cdots+\alpha_dg_d=h}Sx_1^{\alpha_1}\cdots x_d^{\alpha_d}$.

- If $R=S[x,y], G=\mathbb{Z}$, and $|x|=|y|=1$, then for $n\in\mathbb{Z}$, $R_h=\oplus_{i+j=n,i,j\geq 0}Sx^iy^j$.
- If $R=S[x,y], G=\mathbb{Z}$, and $|x|=2,|y|=3$, then $R_m=\oplus_{2i+3j=m}Sx^iy^j$.
- If $R=S[x,y], G=\mathbb{Z}$, and $|x|=1,|y|=-1$, then $R_n=\oplus_{i-j=n}Sx^iy^j$. In particular, $R_0=\oplus_{i\geq 0}Sx^iy^i=S[xy]\supsetneq S$.
- If $R=S[x,y], G=\mathbb{Z}\oplus\mathbb{Z}$, and $|x|=(1,0),|y|=(0,1)$, then for $g=(m,n)\in G$, we see $R_g=Sx^my^n$ (if $m,n\geq 0$).

Proposition 1.1. *Let $R=\oplus_{g\in G}R_g$ be a graded ring by an abelian group G. Then R_0 is a ring with identity. In fact, $1_R\in R_0$ and each R_g is an R_0-module.*

Proof. We will prove $1_R\in R_0$. As $R=\oplus R_g$, we can write $1_R=e_{g_1}+\cdots+e_{g_n}$ where $e_{g_i}\in R_{g_i}\setminus\{0\}$. Now, multiplying by e_{g_i}, we see $e_{g_i}=e_{g_i}e_{g_1}+\cdots+e_{g_i}e_{g_n}$. By properties of the direct sum, we must have $e_{g_i}=e_{g_i}e_{g_j}$ for some j and $e_{g_i}e_{g_k}=0$ for all other k. Note that $g_i+g_j=g_i$, that is, $g_j=0$. WLOG, let $j=1$. Thus, we have $1=e_0+e_{g_2}+\cdots+e_{g_n}$. Notice that

$$1=1^2=(e_0+e_{g_2}+\cdots+e_{g_n})^2=\sum_{i,j}e_{g_i}e_{g_j}=e_0^2=e_0\in R_0.$$ □

Lemma 1.2. *Let M be an R_0-submodule of R_i for some fixed i. Let RM be the ideal of R generated by M. Then $RM\cap R_i=M$.*

Proof. It is clear that $M\subseteq RM\cap R_i$. So let $r\in RM\cap R_i$. Then $r=\sum_{j=1}^k f_jm_j$ where $f_j\in R, m_j\in M$. Write $f_j=\sum_{finite}f_{j,\ell}$ where $f_{j,\ell}\in R_\ell$. Then $R=\sum_{j,\ell}\underbrace{f_{j,\ell}m_j}_{\in R_{i+\ell}}$. As $r\in R_i$, we have $r=\sum_j f_{j,0}m_j\in R_0M\subseteq M$. □

Proposition 1.3. *Let $R=\oplus_{g\in G}R_g$ and suppose R is Noetherian (resp. Artinian). Then each R_i is a Noetherian (resp. Artinian) R_0-module. In particular, R_0 is a Noetherian (resp. Artinian) ring.*

Proof. Let $M_0\subseteq M_1\subseteq\cdots$ be an ascending chain of R_0-modules in R_i. Then $RM_0\subseteq RM_1\subseteq\cdots$ is an ascending chain of ideals in R. As R is Noetherian, there exists k such that $RM_{k+1}=RM_{k+2}=\cdots$. Intersecting with R_i gives us $M_{k+1}=M_{k+2}=\cdots$ by the Lemma. □

Corollary 1.4. *If R has finite length, R_i does for all i.*

Proof. Recall that $\lambda(R)<\infty$ if and only if R is Noetherian and Artinian. □

Corollary 1.5. *If $R = \oplus_{g\in G} R_g$ is Noetherian, then R_0 is Noetherian and each R_i is a finitely generated R_0-module.*

By the Hilbert Basis Theorem, if R_0 is Noetherian and R is a finitely generated R_0-algebra, then R is Noetherian.

Definition. *Let $R = \oplus_{g\in G} R_g$ and M an $R-$module. We say M is a **graded** $R-$**module** if there exists abelian subgroups M_g of M for all $g \in G$ such that $M = \oplus_{g\in G} M_g$ and $R_h M_g \subseteq M_{h+g}$ for all $g, h \in G$.*

Note. As $R_0 M_g \subseteq M_g$, we see each M_g is an R_0-module. Also note that R is a graded $R-$module.

Example. If $\{M_i\}_{i\in I}$ is a family of graded $R-$modules, then $\oplus_{i\in I} M_i$ and $\prod_{i\in I} M_i$ are graded in the obvious way. Thus $R \oplus R$ is a graded $R-$module and R^n is a graded $R-$module.

Definition. *Let M be a $G-$graded $R-$module and $g \in G$. Define M **shifted by** $\mathbf{g}$ (or **twisted by** $\mathbf{g}$) by $M(g) = M$ as $R-$modules, but for $h \in G$, $M(g)_h = M_{g+h}$. Then $M(g) = \oplus_{h\in G} M(g)_h$.*

Example. Let $R = K[x], G = \mathbb{Z}$, and $|x| = 1$. Then $k[x] = k \oplus kx \oplus kx^2 \oplus \cdots$ where $k[x]_0 = k$ but $k[x](2)_0 = kx^2$.

Proposition 1.6. *Let $R = \oplus_{g\in G} R_g, M = \oplus_{g\in G} M_g$. Let $N \subseteq M$ be an $R-$submodule. TFAE*

(1) *$N = \oplus_{g\in G} N_g$ where $N_g = N \cap M_g$.*
(2) *Given $f \in M, f \in N$ if and only if each homogenous component of f is in N.*
(3) *N has a homogenous generating set.*

*If (1) – (3) are satisfied, N is called a **graded submodule** or **homogeneous submodule** of M.*

Proof. (1)⇒(2): Let $f \in M$. Write $f = f_{g_1} + \cdots + f_{g_n}, f_{g_i} \in M_{g_i}$. If $f \in N$, then $f = \sum_{g\in G} f'_g$ where $f'_g \in N \cap M_g$. By uniqueness of the decomposition of f, $f'_g = f_g$ for all $g \in G$. Thus $f_g \in N$ for all $g \in G$. The other direction is clear.

(2)⇒(1): Similar.

(2)⇒(3): Let $S = \{f \in N | f \text{ is homogeneous}\}$. By (2), S generates N. [Note that when N is finitely generated, one can show there exists a finite homogeneous generating set].

(3)⇒(2): Let S be a homogeneous generating set for N. Let $f \in M$. The backward direction is clear. So suppose $f = \sum_{g\in g} f_g \in N$. Now, $f = \sum_{i=1}^n r_i s_i$ where $s_i \in S$ are homogeneous and $r_i \in R$. Let $|s_i| = g_i$ and $r_i = \sum_{g\in G} r_{i,g}$. So $f = \sum_{i,g} r_{i,g} s_i$. For $h \in G$, $f_h = \sum_i r_{i,h-g_i} s_i \in N$ as $s_i \in N$. □

Examples.

(1) Let $R = k[x, y, z], G = \mathbb{Z}$, and $|x| = |y| = |z| = 1$. Then $I_1 = (x^2yz + z^4 - x^2z^2, xy + y^2)$ is a homogeneous ideal (by (3) of the proposition). Also, $I_2 = (x^2 + y, y^2, x^2 + x^2y)$ is homogeneous because $I_2 = (x^2, y)$. However, $I_3 = (x^2 + y)$ is not homogeneous as $y \notin I$ but y is a homogeneous component of $x^2 + y \in I_3$.
(2) Let $R = k[x, y, z], G = \mathbb{Z}^3$, and $|x| = (1, 0, 0), |y| = (0, 1, 0), |z| = (0, 0, 1)$. Note that I_1 is not homogeneous as it does not contain xy, a homogeneous component of $xy + y^2 \in I_1$. I_2 is again homogeneous, but I_3 is again not.

Remarks. Let R, M be graded.

(1) Let $\{N_\lambda\}_{\lambda\in I}$ be a family of graded $R-$submodules of M. Then $\cap N_\lambda$ and $\sum N_\lambda$ are graded.
(2) If I is a graded ideal and N a graded submodule, then IN is a graded submodule of N.

Proof. As I is a graded ideal, $I = (r_1, r_2, ...)$ where the r_i are homogeneous. Similarly, as N is a graded submodule, it has a homogeneous generating set $S = \{f_i\}$. Then $\{r_i f_i\}$ is a homogeneous generating set for IN as for $rf \in IN$, we see $rf = (\sum_{i\in I} a_i r_i)(\sum_{j\in J} b_j f_j) = \sum_{i,j} a_i b_j r_i f_j$ (and similarly for $\sum s_i g_i \in IN$). Thus IN is graded. □

In particular, if I and J are homogeneous ideals, then $IJ, I \cap J, I + J$ are all homogeneous.

(3) Let $N_1, N_2 \subseteq M$ be graded submodules. Then $(N_1 :_R N_2) = \{r \in R | rN_2 \subseteq N_1\}$ is a homogenous ideal of R.

Proof. Let $f \in (N_1 :_R N_2)$ and write $f = \sum_{g\in G} f_g$. Let T be a homogeneous generating set for N_2. Note $f \in (N_1 :_R N_2)$ if and only if $ft \in N_1$ for all $t \in T$. Thus $(\sum f_g)t \in N_1$. But $(\sum f_g)t = \sum f_g t$ is a homogeneous decomposition of ft. As N_1 is homogeneous, $f_g t \in N_1$ for all $g \in G, t \in T$. Thus $f_g \in (N_1 :_R N_2)$ for all g. □

Special Cases:

- If M is graded, then $Ann_R M = (0 :_R M)$ is a homogeneous ideal.
- If $x \in M$ is homogeneous, then $(0 :_R x) = (0 :_R Rx)$ is a homogeneous ideal.
- If I is a homogeneous ideal and x a homogeneous element, then $(I :_R x)$ is a homogeneous ideal.

(4) If $N \subseteq M$ is a graded submodule and $I \subseteq R$ is homogeneous, then $(N :_M I) = \{m \in M | Im \subseteq N\}$ is a graded submodule of M.

Proof. Let $m \in (N :_M I)$. As M is graded, $m = \sum m_i$ where m_i are homogeneous. Let T be a homogeneous generating set for I. Then $tm \in N$ for all $t \in T$. Note $tm = \sum tm_i$ is a homogeneous decomposition of an element in N. Thus $tm_i \in N$ for all i as N is graded. Of course, this is true for all $t \in T$, which generates I. So $Im_i \subseteq N$, that is, $m_i \in (N :_M I)$ for all i. Thus $(N :_M I)$ is graded as it contains the homogeneous components of any element. □

(5) If $I \subseteq R$ is a homogeneous ideal, then R/I is a graded ring (as $(R/I)_g = \{r + I | r \in R_g\}$).

Definition. *Let $R = \oplus_{g\in G} R_g$ be a graded ring. Let $S \subseteq R$ be a subring. Then S is a* ***graded subring*** *of R if $S = \oplus S_g$ where $S_g = S \cap R_g$.*

Exercises.

(1) Let $R = \oplus_{g\in g} R_g, S \subseteq R$ a subring. Let $S_0 = S \cap R_0$, a ring. Then S is a graded subring if and only if S is generated over S_0 by homogeneous elements of R. In particular, if $f_1, ..., f_n$ are homogeneous elements of R, then $R_0[f_1, ..., f_t]$ is a graded subring of R.

Proof. For the forward direction, assume $S = \oplus_{g\in G} S_g$ where $S_g = S \cap R_g$. As $1_R \subset S$ and $1_R \subset R_0$, we know $1_R \subset S_0$. Then it is clear to see $S = \sum_{s\in S_g, g\in G} S_0 s$. For the backward direction, let T be the generating set and set $S_g = \sum_{s\in T\cap R_g} S_0 s$. The clearly $S_g = S \cap R_g$ and $S = \oplus S_g$. □

(2) If $R = \oplus_{i\in\mathbb{Z}} R_i$ is a field, then $R_i = 0$ for all $i \neq 0$.

Proof. Consider the element $r_g + r_h \in R$ where $r_g \in R_g \setminus \{0\}$ and $r_h \in R_h$. Suppose $g < h$ and note $r_g + r_h \neq 0$. Then there exists $s \in R$ such that $s(r_g + r_h) = 1 \in R_0$. Write $s = s_1 + ... + s_t$ where $s_i \in R_{m_i}$ and $m_1 < \cdots < m_t$. Then, $1 = s_1 r_g +$ (higher terms). Certainly $s_1 r_g \neq 0$ as $s_1, r_g \neq 0$. Thus $s_1 r_g = 1$. This says $sr_h = 0$, which implies $r_h = 0$. Thus every element of R is homogeneous. Furthermore, as $1 + r$ is a unit for all $r \in R \setminus \{-1\}$ and $1 \in R_0$, we must have $r \in R_0$ for all $r \in R$. □

(3) Suppose R is $\mathbb{Z}-$graded and I a homogeneous ideal. Then $\sqrt{I}$ is homogeneous.

Proof. Let $r \in \sqrt{I}$ and write $r = \sum r_i$ where $r_i \in R_{m_i}$ and $m_1 < \cdots < m_t$. Then $r^k \in I$ for some I. Of course, $r^k = r_1^k +$ (higher terms) and as I is homogenous, we must have that the homogenous component $r_1^k \in I$. Thus $r_1 \in \sqrt{I}$, which implies $r - r_1 \in \sqrt{I}$. Now, induct on the number of homogeneous components to see that $r_i \in \sqrt{I}$ for all $1 \leq i \leq t$. Thus $\sqrt{I}$ is homogeneous. □

Note. In the above two exercises, we could have generalized $\mathbb{Z}$ to the following:

Definition. *A **totally ordered abelian group** $(G,+)$ is an abelian group with a partial order $<$ such that G is totally ordered with respect to $<$ and if $g_1 < g_2$ and $h \in G$, then $g_1 + h < g_2 + h$.*

Example. Let $R = k[t], G = \mathbb{Z}$, and $|t| = 1$. Then $S = k[t^2, t^3]$ is a graded subring of R.

Definition. *Let R be a commutative ring and I an ideal of R. Consider $R[t]$, where t is a variable, as a graded ring over $\mathbb{Z}$ with $|t| = 1$. Define $R[It] := R[\{it|i \in I\}] \subseteq R[t]$, a graded ring, to be the **Rees ring** of I.*

If $I = (a_1, ..., a_n)$, then $R[It] = R[a_1t, ..., a_nt]$. Therefore, if R is Noetherian, $R[It]$ is a finitely generated algebra over a Noetherian ring and hence Noetherian. Often, the Rees ring is written without the t, that is, $\mathcal{R}(I) = R \oplus I \oplus I^2 \oplus \cdots$, but we will write it with the t, in order to keep a better grasp on the grading.

Let M be an $R-$module. Consider the graded $R[t]-$module $M[t] = \{m_0 + m_1t + \ldots + m_kt^l|m_i \in M\} = M \oplus Mt \oplus Mt^2 \oplus \cdots$. Note $M[t]$ is generated over $R[t]$ by M (degree 0). So, if M is finitely generated as an $R-$module, then $M[t]$ is finitely generated as an $R[t]-$module. We define the **Rees module of** I **and** M to be $\mathcal{R}(I, M) = M[IMt] = M \oplus IMt \oplus I^2Mt^2 \oplus \cdots = R[It]Mt^0 \subseteq M[t]$. This is a graded $R[It]-$submodule of $M[t]$. Further, $M[IMt]$ is the $R[It]-$submodule of $M[t]$ generated by M (degree 0). Again, if M is finitely generated over R, then $M[IMt]$ is a finitely generated $R[It]-$module.

Remark. If R is Noetherian and M a finitely generated $R-$module, then $M[IMt]$ is a Noetherian (so finitely generated) $R[It]-$module.

Theorem 1.7 (**Artin-Rees Theorem**). *Let R be Noetherian, $I \subseteq R$ an ideal, M a finitely generated $R-$module and $N \subseteq M$ a submodule. Then there exists k such that for all $n \geq k$ we have $I^nM \cap N = I^{n-k}(I^kM \cap N)$.*

Proof. Note that $\supseteq$ is always true. So we need only to prove $\subseteq$. As $N[t] \subseteq M[t]$ is a graded $R[It]-$module, we can consider $N[t] \cap M[IMt]$ to be a graded $R[It]-$submodule of $M[IMt]$. Note

$$N[t] \cap M[IMt] = N \oplus (IM \cap N)t \oplus (I^2M \cap N)t^2 \oplus \cdots \oplus (I^nM \cap N)t^n \oplus \cdots.$$

As R is Noetherian, so is $R[It]$ as a ring. Further, as $M[IMt]$ is a finitely generated $R[It]-$module, $M[IMt]$ is a Noetherian $R[It]-$module.. Hence $N[t] \cap M[IMt]$ is a finitely generated $R[It]-$module, which implies it has a finite homogeneous generating set $\{x_1t^{m_1}, ..., x_st^{m_s}\}$. Thus $x_i \in I^{m_i}M \cap N$. Let $k = \max\{m_1, ..., m_s\}, n \geq k$ and $f \in I^nM \cap N$. Then $ft^n \in (N[t] \cap M[IMt])_n$, which implies $ft^n = \sum_{i=1}^{s}(r_it^{n-m_i})(x_it^{m_i}) = (\sum_1^s r_ix_i)t^n$, where $r_i \in I^{n-m_i}$. So $r_ix_i \in I^{n-m_i}(I^{m_i}M \cap N) \subseteq I^{n-m_i-1}(I^{m_i+1}M \cap N) \subseteq \cdots \subseteq I^{n-k}(I^kM \cap N)$. Therefore, $f = \sum r_ix_i \in I^{n-k}(I^kM \cap N)$. $\square$

Note. The integer k above depends on I, M and N.

Theorem 1.8 (**Krull's Intersection Theorem**). *Let R be Noetherian, I an ideal of R, and M a finitely generated $R-$module. Then there exists $s \in I$ such that $(1 - s) \cap_{n=1}^{\infty} I^nM = 0$.*

Proof. Let $N = \cap_{n=1}^{\infty} I^nM \subseteq M$. By the Artin-Rees Lemma, there exists k such that for all $n \geq k$, $I^nM \cap N \subseteq I^{n-k}N$. Let $n = k+1$. Then $N = I^{k+1}M \cap N \subseteq IN$. Note N is finitely generated and so by NAK, $I + Ann_RN = R$ (a 902 exercise). Thus $1 = s + t$ for $s \in I, t \in Ann_RN$. Then $t = 1 - s \in Ann_RN$. So $(1 - s) \cap I^nM = 0$. $\square$

Special Cases.

(1) If $I \subseteq J(R)$, then $\cap_1^{\infty} I^nM = 0$ (as if $s \in J(R)$, then $1 - s$ is a unit).
(2) IF R is a domain and $I \neq R$, then $\cap_1^{\infty} I^n = 0$ (as $1 - s \neq 0$, you can cancel).

Corollary 1.9. *Let R be Noetherian, $I \subseteq J(R), N \subseteq M$ finitely generated. Then $\cap_{n=1}^{\infty}(N + I^nM) = N$.*

Proof. Note that $I^n(M/N) = (N + I^nM)/N \subseteq M/N$, which is finitely generated. As $I \subseteq J(R)$, Krull's Intersection Theorem says $\cap_0^{\infty} I^n(M/N) = 0$. Thus $0 = \cap_0^{\infty}(N + I^nM)/N = (\cap_0^{\infty}(N + I^nM))/N$, and so $\cap_0^{\infty}(N + I^nM) = N$. $\square$

As $R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$, we see $IR[It] = I \oplus I^2t \oplus I^3t^2 \oplus \cdots$, a homogenous ideal. This yields the graded ring $R[It]/IR[It] = R/I \oplus (I/I^2)t \oplus (I^2/I^3)t^2 \oplus \cdots$, the **associated graded ring** of I. We denote $R[It]/IR[It]$ by $gr_I(R)$ or $gr(I, R)$. As $R[It]$ is Noetherian, $gr_I(R)$ is Noetherian. Further, if $I = (a_1, ..., a_n)$, then $gr_I(R) = R/I[\overline{a_1}t, ..., \overline{a_n}t]$, where $\overline{a_i} \in I/I^2$. Suppose $\overline{x} \in I/I^2$ is a non-zerodivisor on $gr_I(R)$. Let $y \in I^n$ and $\overline{y} = y + I^{n+1} \in I^n/I^{n+1} = gr_I(R)_n$. Suppose $xy \in I^{n+2}$. Then $\overline{x} \cdot \overline{y} = \overline{xy} \in I^{n+1}/I^{n+2}$. Thus $\overline{xy} = 0$ which implies $\overline{y} = 0$ as x is a non-zerodivisor. Thus $y \in I^{n+1}$. This gives us $(I^{n+2} : x) \cap I^n = I^{n+1}$ for all n.

Exercise. Let (R, m) be local, $I \subseteq m$, and $x \in I$ a non-zerodivisor on R. Suppose $(I^n : x) \cap I^{n-2} = I^{n-1}$ for all $n \geq 2$. Prove $(I^n : x) = I^{n-1}$ for all $n >> 0$.

Proof. Note that we can show inductively that $I^{m-1} = \cap_2^m (I^k : x)$. Of course, $(I^m : x) \subseteq (I^{m-1} : x) \subseteq \cdots$. So $I^{m-1} = \cap_2^m (I^k : x) = (I^m : x)$. □

Thus, in the situation above, if x is a non-zerodivisor, then $(I^n : x) = I^{n-1}$ for all n. Conversely, if $x \in I \setminus I^2$ and $(I^n : x) = I^{n-1}$ for all n, then $\overline{x} \in I/I^2$ is a non-zerodivisor on $gr_I(R)$. This gives us the following criteria:

Proposition 1.10. *Let $x \in I \setminus I^2$. Then $\overline{x} \in I/I^2$ is a non-zerodivisor on $gr_I(R)$ if and only if $(I^n : x) = I^{n-1}$ for all n.*

Exercise. Let (R, m) be local, $I \subseteq m$, $x \in I$ a non-zerodivisor on R. Then there exists k such that $(I^n : x) \subseteq I^{n-k}$ for all k.

Proof. First note that $x(I^n : x) = (x) \cap I^n$. By Artin-Rees (let $M = R, N = (x)$), we see $x(I^n : x) = (x) \cap I^n \subseteq I^{n-k}(x)$. Thus $x(I^n : x) \subseteq xI^{n-k}$. As x is a non-zerodivisor, we can cancel to get $(I^n : x) \subseteq I^{n-k}$. □

Now, suppose we know $(I^{n+2} : x) \cap I^n = I^{n+1}$ for all $n \geq k$. Then, one can show inductively that $(I^{k+n} : x) \cap I^k = I^{k+n-1}$. Using the exercise, we get $(I^\ell : x) = I^{\ell-1}$ for all $\ell >> 0$. This requirement says x is a **superficial element.**

With the above assumption, Krull's Intersection Theorem tells us $\cap I^{n-k} = 0$ and so $\cap(I^n : x) = 0$. If we choose $I = m$, then this exercise shows x a non-zerodivisor implies $\cap(m^n : x) = 0$. Of course, if x is a zerodivisor, then $\alpha x = 0$, which implies $\alpha \in \cap(m^n : x)$. So we now have a criteria for non-zerodivisors in a local ring, namely

$$x \text{ is a non-zerodivisor if and only if } \cap (m^n : x) = 0.$$

Exercise. If (R, m) is a local ring, then we can define $\rho(x, y) = \begin{cases} 2^{-n} & \text{if } x - y \in m^n \setminus m^{n+1} \\ 0 & \text{if } x = y \end{cases}$. This definition makes sense by Krull's Intersection Theorem. Prove that ρ is a metric.

Proof. Clearly ρ is symmetric and $\rho(x, y) = 0$ if and only if $x = y$. So we need only show the triangle inequality holds. If $x = z, z = y$, or $x = y$, we are done. So suppose $\rho(x, y) = 2^{-n_1}, \rho(x, z) = 2^{-n_2}$, and $\rho(y, z) = 2^{-n_3}$. WLOG, assume $n_2 \leq n_3$. Then $x - y = (x - z) + (z - y) \in m^{n_2}$, that is, $n_1 \geq n_2$. So $\rho(x, y) \leq 2^{-n_2} = \rho(x, z) \leq \rho(x, z) + \rho(z, y)$. □

The resulting topology on R is called the $m-$adic topology. Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in R. Define $\{x_n\} \sim \{y_n\}$ if and only if $\lim_{n \to \infty} x_n - y_n = 0$. Let $\hat{R}$ denote the equivalence classes of Cauchy Sequences, that is, $\hat{R} = \{[\{x_n\}] | \{x_n\} \text{ is a Cauchy sequence}\}$. Define $+, \cdot$ on $\hat{R}$ by $[\{x_n\}][\{y_n\}] = [\{x_n y_n\}]$ and $[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$. These are well-defined

- Assume $[\{x_n\}] = [\{r_n\}]$ and $[\{y_n\}] = [\{s_n\}]$. Then, we wish to show $[\{x_n y_n\}] = [\{r_n s_n\}]$, that is $\lim x_n y_n - r_n s_n = 0$. So it suffices to show $\rho(x_n y_n - r_n s_n, 0)$ is arbitrarily small, that is, for all k there exists N such that $x_n y_n - s_n t_n \in m^k$ for all $n \geq N$. Let k be given. Then, there exists N_1 such that $x_n - r_n \in m^k$ for all $n \geq N_1$ and there exists N_2 such that $y_n - s_n \in m_k$ for all $n \geq N_2$. So, for all $n > \max\{N_1, N_2\}$ we have

$$x_n y_n - r_n s_n = x_n y_n - r_n y_n + r_n y_n - r_n s_n = \underbrace{(x_n - r_n)}_{\in m^k} y_n + r_n \underbrace{(y_n - s_n)}_{\in m^k} \in m^k.$$

Thus $[\{x_n y_n\}] = [\{r_n s_n\}]$.

- Assume $[\{x_n\}] = [\{r_n\}]$ and $[\{y_n\}] = [\{s_n\}]$ and define N_1, N_2 as above. We want to show $[\{x_n + y_n\}] = [\{r_n + s_n\}]$, that is, for all k there exists N such that $(x_n + y_n) - (r_n + s_n) \in m^k$. Of course, for all $n > \max\{N_1, N_2\}$, we see $(x_n + y_n) - (r_n + s_n) = (x_n - r_n) + (y_n - s_n) \in m^k$.

and they make $\hat{R}$ into a commutative ring with identity (where $1 = [\{1\}]$). Note that there is a natural ring homomorphism $\phi : R \to \hat{R}$ defined by $r \mapsto \hat{r} = [\{r, r, ...\}]$.

- ϕ is injective.

  Proof. Suppose $\hat{r} = \hat{s}$. Then $\lim \hat{r} - \hat{s} = 0$, which implies $\rho(r - s, 0) \to 0$. Of course, $\rho(r - s, 0)$ is constant and so we must indeed have $\rho(r - s, 0) = 0$, that is, $r = s$. □

Further, $\hat{R}$ is faithfully flat as an $R-$module and is a Noetherian local ring with $\hat{m} = m\hat{R}$.

Definition. *A ring R is complete if it is complete in the $m-$adic topology, that is, $\phi : R \hookrightarrow \hat{R}$ defined by $r \mapsto \hat{r}$ is an isomorphism.*

In particular, the definition says $\hat{\hat{R}} \cong \hat{R}$.

Examples.

(1) Any local ring R with $\dim R = 0$ is complete: Note $\sqrt{(0)} = m$ and so $m^n = 0$ for some n. Thus $\{x_n\}$ is a Cauchy sequence if and only if there exists k such that $x_n = x$ for all $n \geq k$. So $\{x_n\} \sim \{x\}$, the constant sequence. So $R \hookrightarrow \hat{R}$ is an isomorphism.

(2) Let $R = k[x]_{(x)}$. Then $\{1, 1 + x, 1 + x + x^2, ...\}$ is a Cauchy sequence. So $\hat{R} = k[[x]]$ [Recall $\sum r_i x^i$ is a unit if and only if r_o is a unit. So $\hat{R}$ is local with maximal ideal $m = (x)$ as the only non-units are those with zero constant term]. In general, for $R = k[x_1, ..., x_n]_{(x_1, ..., x_n)}$, we see $\hat{R} = k[[x_1, ..., x_n]]$.

Theorem 1.11 (**Cohen Structure Theorem**). *Let (R, m) be a complete local ring. Assume $R \supseteq$ a field. Then $R \cong k[[x_1, ..., x_n]]/I$ for some field k.*

Definition. *Let M, N be graded $R-$modules where $R = \oplus_{g \in G} R_g$. An $R-$homomorphism $f : M \to N$ is called* ***graded*** *or* ***homogeneous*** *of degree $g \in G$ if $f(M_h) \subseteq N_{h+g}$ for all $h \in G$.*

Exercise. If $f : M \to N$ is graded, then $\ker f$ is a graded submodule of M and $\operatorname{im} f$ is a graded submodule of N.

Proof. Let $m \in \ker f$. As $m \in M$ is graded, we know $m = \sum_{i=1}^{t} m_{g_i}$ where $m_{g_i} \in M_{g_i}$. Note that $0 = f(m) = f(\sum m_{g_i}) = \sum f(m_{g_i})$. Thus $\sum f(m_{g_i})$ is a decomposition of 0, which is homogenous. By the uniqueness of decompositions, we must have $f(m_{g_i}) = 0$, that is, $m_{g_i} \in \ker f$. Thus $m \in \ker f$ if and only if its homogenous components are, that is, $\ker f$ is graded.

Let $n \in \operatorname{im} f$. Say $n = \sum_{i=1}^{t} n_{g_i}$ where $n_{g_i} \in N_{g_i}$. As $n \in \operatorname{im} f$, there exists $m \in M$ such that $f(m) = n$. Say $m = \sum_{j=1}^{t} n_{g_i}$ where $m_{h_j} \in M_{h_j}$. Then, $n = f(\sum m_{h_j}) = \sum f(m_{h_j})$ is a decomposition. By uniqueness, we must have $s = t$ and (after reordering) $f(m_i) = n_i$ for all i. Thus $n_i \in \operatorname{im} f$ for all i, that is $\operatorname{im} f$ is graded. □

Remarks.

(1) Suppose $f : M \to N$ is graded of degree g. Define $f[-g] : M(-g) \to N$ by $m \mapsto f(m)$. So $f[-g](M(-g)_h) = f(M_{h-g}) \subseteq N_{h-g+g} = N_h$. Note $f[-g]$ is degree 0.

(2) If $0 \to A \to B \to C \to 0$ is a short exact sequence of graded $R-$modules with degree 0 maps, then $0 \to A_g \to B_g \to C_g \to 0$ is a short exact sequence of R_0-modules for all $g \in G$. In particular, if R_0 is a field, then we have infinitely many short exact sequences of vector spaces.

(3) If $C. : \cdots C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} C_{i-2} \to \cdots$ is a complex of graded $R-$modules and degree 0 maps, then H_i are graded $R-$modules for all i by the exercise.

2. Primary Decomposition

Definition. *Let R be a commutative ring with identity and M a finitely generated $R-$module. An $R-$submodule $N \subseteq M$ is called* ***primary*** *if for all $f \in R$ the map $f_{M/N} : M/N \to M/N$ defined by $\overline{m} \mapsto \overline{fm}$ is either injective or nilpotent, that is, for all $f \in R, m \in M$, if $fm \in N$, then either $m \in N$ or $f \in \sqrt{\operatorname{Ann}_R M/N}$.*

Special Case. An ideal $I \subseteq R$ is primary if for all $a, b \in R$ if $ab \in I$ then $b \in I$ or $a \in \sqrt{\operatorname{Ann}_R R/I} = \sqrt{I}$.

Remarks.

(1) Prime ideals are primary.
(2) If $N \subseteq M$ is primary, then $\sqrt{\operatorname{Ann}_R M/N}$ is prime.

Proof. Suppose $ab \in \sqrt{\operatorname{Ann}_R M/N}$, but $a \notin \sqrt{\operatorname{Ann}_R M/N}$. Say $(ab)^\ell M \subseteq N$, that is, $a^\ell(b^\ell M) \subseteq N$. Then $a^\ell \notin \sqrt{\operatorname{Ann}_R M/N}$ implies $b^\ell M \subseteq N$. Thus $b^\ell \in \sqrt{\operatorname{Ann}_R M/N}$. □

Definition. *If $N \subseteq M$ is primary, we say N is $p-$**primary** in M if $p = \sqrt{\operatorname{Ann}_R M/N}$.*

Proposition 2.1. *If $\sqrt{\operatorname{Ann}_R M/N} = m$, where m is maximal, then N is primary in M.*

Proof. Let $f \in R, u \in M$. Suppose $fu \in N$. Assume $f \notin \sqrt{\operatorname{Ann}_R M/N} = m$. Then $m + (f) = R$, that is, $1 = rf + a$ for some $r \in R, a \in m$. Say $a^\ell M \subseteq N$. Then $1 = 1^\ell = (rf + a)^\ell = r'f + a^\ell$ for $r1 \in R$. Then $u = r'fu + a^\ell u \in N$. □

Corollary 2.2. *If $I \subseteq R$ and $\sqrt{I}$ is maximal, then I is primary.*

Example. Let $R = k[x, y], I = (x^2, y^3)$. Then $\sqrt{I} = (x, y)$ maximal implies I is primary.

Exercise. Find an example where $\sqrt{I}$ is prime, but I is not primary.

Proof. Consider $R = k[x, y, z]/(xy - z^2)$, graded in the usual way. Let $p = (\overline{x}, \overline{z})$. We will show $\sqrt{p^2}$ is prime, but p^2 is not primary. First note that $\sqrt{p^2} = p$ (In general, $\sqrt{q^n} = q$ for a prime q as if $q' \supseteq q^n$, then $q' \supseteq q$. Thus $\sqrt{q^n} = \cap q' = q$.). So $\sqrt{p^2}$ is prime. Now, p^2 is not primary as $\overline{xy} = \overline{z}^2 \in p^2$, but $\overline{x} \notin p^2$ (as p^2 is generated by elements of degree 2 while $\overline{x}$ has degree 1) and $\overline{y} \notin p$. □

Example. Let $R = \mathbb{Z}_p[x]/(x^p - 1)$ for a prime p. Let R be $\mathbb{Z}_p-$graded with $\deg x = \overline{1}$. Then $\deg x^p = p \cdot \overline{1} = 0$. So $(x^p - 1)$ is homogenous. As $char R = p$, we see $(x^p - 1) = (x - 1)^p$. So $x - 1$ is nilpotent in R, but $1, x$ are not nilpotent. Hence $\sqrt{(0)}$ is not a homogenous ideal.

From now on, we will assume a graded ring will be graded by a totally ordered abelian group. Then, by the earlier exercise, I homogenous will imply $\sqrt{I}$ is homogenous.

Proposition 2.3. *Let R be graded, $N \subseteq M$ graded $R-$modules with M finitely generated. Then N is primary if and only if for all $f \in R$ homogeneous and $m \in M$ homogenous with $fm \in N$, then $f \in \sqrt{\operatorname{Ann}_R M/N}$ or $m \in N$.*

Proof. The forward direction is clear. For the backward direction, we will induct on the number of nonzero homogenous components of f. So first suppose f is homogenous and $m = \sum_{i=1}^t m_{g_i}$ where $g_1 < g_2 < \cdots$ with $fm \in N$. If $m \in N$, done. Note also that if $m_{g_i} \in N$, then $f(m - m_{g_i}) \in N$. So, WLOG assume $m_{g_i} \notin N$ for all i. Now $\sum fm_{g_i} = fm \in N$, which implies $fm_{g_i} \in N$ for all g_i as N is graded. Thus $f \in \sqrt{\operatorname{Ann}_R M/N}$ by assumption. Now assume $f = \sum_{j=1}^s f_{h_j}$, where $h_1 < h_2 < \cdots$. Again suppose $m \notin N$ and thus $m_{g_i} \notin N$ for all i. So $\sum_{i,j} f_{h_j} m_{g_i} = fm \in N$ implies $f_{h_1} m_{g_1} \in N$, as it is a homogeneous component. Thus $f_{h_1} \in \sqrt{\operatorname{Ann}_R M/N}$ by assumption. As $m \notin N$, we can find a smallest t such that $f_{h_1}^t m \in N$. Let $m' = f_{h_1}^{t-1} m \notin N$ and $f' = f - f_{h_1}$. Then $f'm' \in N$ and f' has one less homogenous component. By induction, $f' \in \sqrt{\operatorname{Ann}_R M/N}$. Thus $f = f' + f_{h_1} \in \sqrt{\operatorname{Ann}_R M/N}$. □

Corollary 2.4. *Let R be graded and p a homogenous ideal. Then p is prime if and only if for all $f, g \in R$ homogenous with $fg \in p$, we have $f \in p$ or $g \in p$.*

Proof. Since $\sqrt{p}$ is homogenous, the condition implies $p = \sqrt{p}$: take a homogeneous element f in $\sqrt{p}$ and consider $f^t \in p$. Then either $f^{t-1} \in p$ or $f \in p$ and either way we see inductively that $f \in p$. This condition also gives that p is primary. Of course, we've shown that the radical of a primary ideal is prime. This $p = \sqrt{p}$ is prime. □

Note. As a graded ring is a domain if and only if 0 is prime, the corollary shows a graded ring is a domain if and only if there are no homogenous zero-divisors.

Definition. *Let $R = \oplus R_g$ and M a graded $R-$module. M is called* ****Noetherian*** *if one of the equivalent conditions hold:*

(1) *M satisfies ACC on graded submodules.*
(2) *Every homogenous submodule of M is finitely generated.*
(3) *Every nonzero set of graded submodules of M has a maximal element.*

Similarly, one can define ****Artinian****.*

Definition. *A graded $R-$module M is called* ****simple*** *if $M \neq 0$ and M has no nontrivial graded submodules.*

Definition. *Let $N \subseteq M$ be graded $R-$modules. N is* ****primary*** *if for all $f \in R$ homogeneous, $f_{M/N}$ is either nilpotent or injective.*

Note. If G is totally ordered, $N \subseteq M$, then *primary is equivalent to primary.

Definition. *Let $N \subseteq M$ be graded $R-$modules. Say N is* ****irreducible*** *if whenever $N = N_1 \cap N_2$ where N_1, N_2 are graded $R-$submodules of M, then $N = N_1$ or $N = N_2$.*

Q: Is *irreducible equivalent to irreducible?

Proposition 2.5. *Let $N \subseteq M$ be graded $R-$modules with M *Noetherian. Suppose N is *irreducible. Then N is *primary.*

Proof. Let $f \in R$ homogeneous and suppose $f_{M/N}$ is not nilpotent. Then $f \in \sqrt{\operatorname{Ann}_R M/N}$. We will show $(N :_M f) = N$, that is, $f_{M/N}$ is injective. Note that $(N :_M f)$ is homogeneous. Consider $(N :_M f) \subseteq (N :_M f^2) \subseteq \cdots \subseteq (N :_M f^n) \subseteq \cdots$, an ascending chain of graded submodules. As M is *Noetherian, we must have $(N :_M f^n) = (N :_M f^{n+1}) = \cdots$. Let $g = f^n$. Note g is homogeneous, $g \notin \sqrt{\operatorname{Ann}_R M/N}$, and $(N :_M g) = (N :_M g^2)$. If $(N :_M g) = N$, then $(N :_M f) = N$.

Claim. $N = (N :_M g) \cap (gM + N)$.

Proof. Note that $\subseteq$ is clear. So let gm_n be a homogeneous element of $(N :_M g) \cap (gM + N)$, that is, m and n are homogeneous. Then $g^2m + gn \in N$, which implies $g^2m \in N$. As $(N :_M g) = (N :_M g^2)$, this says $gm \in N$. Thus $gm + n \in N$.

As N is *irreducible, either $N = (N :_M g)$ or $N = gM + N$. In the latter case, we have $gM \subseteq N$ which says $g \in \operatorname{Ann}_R M/N$, a contradiction. Thus $N = (N :_M g)$. □

Theorem 2.6. *Let M be a *Noetherian graded $R-$module. Then every graded submodule of M is a finite intersection of *irreducible submodules and thus a finite intersection of *primary submodules.*

Proof. Suppose not. Let $\Gamma = \{N \subseteq M | N \text{ is graded and not the intersection of *irreducibles}\} \neq \emptyset$. As M is *Noetherian, there exists $N \in \Gamma$ maximal. In particular, N is not *irreducible and so $N = N_1 \cap N_2$ where $N_1, N_2 \supsetneq N$ are graded. By maximality, we must have N_1 and N_2 are the intersections of finitely many *irreducibles and therefore N is, a contradiction. □

Exercises.

(1) Give an example of a *field (i.e., a graded ring in which every nonzero homogeneous element is a unit) which is not a field. [Thus *simple does not imply simple.]

Proof. Consider $R = \mathbb{Q}[x, \frac{1}{x}]$ where $\deg x = 1, \deg \frac{1}{x} = -1$. Then R is a *field, but not a field as $1 + x$ is not a unit. $\square$

(2) Show that if a $\mathbb{Z}-$graded ring is *Noetherian then it is Noetherian.

Proof. By the Hilbert Basis Theorem, it is enough to show R_0 is a Noetherian ring and R is a finitely generated R_0-algebra.

Claim. R_i is a Noetherian R_0-module for all i.

Proof. We will show R_i satisfies ACC on submodules. If $M_1 \subseteq M_2 \subseteq \cdots$ is an ascending chain of R_0-submodules of R_i, then $RM_1 \subseteq RM_2 \subseteq \cdots$ is an ascending chain of homogeneous $R-$modules. Thus, there exists n such that $RM_n = RM_{n+1} = \cdots$. We've shown $RM_n \cap R_i = M_n$ and thus $M_n = M_{n+1} = \cdots$.

Thus, by the claim R_0 is Noetherian. Note that this also shows R_i is finitely generated as an R_0-module for all i. To show R is a finitely generated R_0-algebra, let $R_- = \oplus_{n<0} R_n$ and $R_+ = \oplus_{n>0} R_n$. Then $R_- R$ is a homogenous ideal of R and thus finitely generated. Say $R_- R = (y_1, ..., y_d)R$ where y_i are homogeneous. Let $-k = \min\{\deg y_1, ..., \deg y_d\}$ and $N = R_{-k} \oplus R_{-k+1} \oplus \cdots \oplus R_{-1}$. As each R_i is finitely generated, we see N is. Say $N = R_0 x_1 + ... + R_0 x_t$ where x_i are homogenous.

Claim. $(x_1, ..., x_t)R = R_- R$.

Proof. It is enough to show $(x_1, ..., x_t) = (y_1, ..., y_d)$. Of course, $y_i \in R_n$ for some $n \geq k$ and thus $y_i \in N$. So $y_i = r_1 x_1 + ... + r_t x_t$. Similarly, $x_i \in N$ implies $x_i \in R_n$ for $n < 0$ and so $x_i \in I$.

We now claim $R_0[R_-] = R_0[x_1, ..., x_t]$. Let $S = R_0[R_-]$ and $T = R_0[x_1, ..., x_t]$. We'll show inductively that $S_{-n} = T_{-n}$ for all $n \geq 0$. Clearly, $S_0 = R_0 = T_0$. So suppose $n > 0$. Let $r \in S_n$. If $n \leq k$, then $r \in N = R_0 x_1 + ... + R_0 x_t \subseteq T$. If $n > k$, then $r \in R_- R = (x_1, ..., x_t)R$. So there exists homogenous elements $u_1, ..., u_t \in R$ such that $r = \sum_1^t u_i x_i$. Note that $\deg u_i + \deg x_i = \deg r = -n$. Thus $0 > \deg u_i > -n$ for all i, which implies $u_i \in T$ by induction. Thus $r \in T$ and $S_n = T_n$. Thus $R_0[R_-] = R_0[x_1, ..., x_t]$.

We will show R is finitely generated over S. Let $R_+ R = (z_1, ..., z_m)R$. As above, we can show $R = S[z_1, ..., z_m]$ (as S is Noetherian by the Hilbert Basis Theorem). Now, R finitely generated over S and S finitely generated over R_0 implies R is in fact finitely generated over R_0. $\square$

The above exercise can in fact be generalized to a ring graded by any finitely generated group. It is not true in general, however, as the following example shows:

Example. Let $R = k[x_1, x_2, ...]$ where $G = \oplus_{i=1}^{\infty} \mathbb{Z}$ and $\deg x_i = (0, ..., 0, 1, 0, ..., 0)$ (where the 1 is in the i^{th} spot). Let S be the set of nonzero homogeneous elements. Then R_S is graded ($\deg \frac{r}{s} = \deg r - \deg s$). Further, R_S is a *field (every homogeneous element is a unit), but not Noetherian.

Since most of the groups we will consider are finitely generated, we will from now on replace *Noetherian with Noetherian.

Let R be a Noetherian graded ring, M a finitely generated $R-$module, $N \subseteq M$ graded. Then there exists a primary decomposition $N = Q_1 \cap \cdots \cap Q_\ell$ where Q_i are graded primary submodules of M. If $\sqrt{\operatorname{Ann}_R M/Q_i} = P_i$, a prime, then we say P_i is **associated** to Q_i and Q_i is P_i-**primary**.

Remark. If Q_1, Q_2 are (homogeneous) primary submodules of M with the same associated prime P, then $Q_1 \cap Q_2$ is a (homogenous) primary ideal with associated prime P.

Proof. It is an elementary exercise to show $\sqrt{\operatorname{Ann}_R M/(Q_1 \cap Q_2)} = \sqrt{\operatorname{Ann}_R M/Q_1 \cap Ann_R M/Q_2} = \sqrt{\operatorname{Ann}_R M/Q_1} \cap \sqrt{\operatorname{Ann}_R M/Q_2} = P \cap P = P$. Now suppose $f \in R$ homogeneous,$m \in M$ homogeneous, and $fm \in Q_1 \cap Q_2$. Suppose $f \notin \sqrt{\operatorname{Ann}_R M/(Q_1 \cap Q_2)} = P$. Then $fm \in Q_i$ and $f \notin P$ implies $m \in Q_i$. Thus $m \in Q_1 \cap Q_2$. $\square$

Definition. *Say $N = Q_1 \cap \cdots \cap Q_\ell$ is an* ***irredundant primary decomposition*** *(ipd) for N if the following hold*

(1) *It is a primary decomposition*
(2) $Q_i \not\supset Q_1 \cap \cdots \cap \hat{Q}_i \cap \cdots \cap Q_\ell$ *(that is, the intersection without* Q_i*).*
(3) $\sqrt{\operatorname{Ann} M/Q_i} \neq \sqrt{\operatorname{Ann}_R M/Q_j}$ *for all* $i \neq j$.

By the previous theorem and the remark, M Noetherian implies every submodule $\neq M$ has an ipd.

Suppose $N \subseteq Q$ are graded submodules of M where Q is $p-$primary. Then Q/N is a $p-$primary submodule of M/N. Thus, if $N = Q_1 \cap \cdots \cap Q_\ell$ is an ipd, then $(\overline{0}) = Q_1/N \cap \cdots \cap Q_\ell/N$ is an ipd of $(\overline{0})$ in M/N. Thus, we often just consider ipd's of $(\overline{0})$ in various modules.

Example. Let $R = k[x, y]$, k a field. Then $(x^2, xy) = (x) \cap (x^2, y)$ (to prove this, give R the standard $\mathbb{Z}_2-$grading where all the ideals are homogenous). Note (x) is prime and thus primary. Also $\sqrt{(x^2, y)} = (x, y)$, the maximal ideal. So (x^2, y) is primary. As the radicals are different, this forms an ipd. Notice, though, that $(x) \cap (x^2, xy, y^2)$ is also an ipd. Thus ipd's are not unique.

Remark. Suppose Q is a graded $p-$primary submodule of M. Then $\sqrt{(Q :_R f)} = \begin{cases} p & \text{if } f \notin Q \\ R & \text{if } f \in Q. \end{cases}$

Proof. If $f \in Q$, clear. So suppose $f \notin Q$. Let $r \in R$ be homogeneous such that $rf \in Q$. So $r \in \sqrt{\operatorname{Ann}_R M/Q} = p$. If $r \in p$, then $r^n M \subseteq Q$ which implies $r^n f \in Q$, that is, $r \in \sqrt{(Q :_R f)}$. □

Theorem 2.7 (**First Uniqueness Theorem**). *Let* R *be a graded Noetherian ring,* $N \subseteq M$ *finitely generated* $R-$*modules,* $N = Q_1 \cap \cdots \cap Q_\ell$ *be a graded ipd. Let* $P_i = \sqrt{\operatorname{Ann}_R M/Q_i}$ *for* $i = 1, ..., \ell$ *and let* $P \in \operatorname{Spec}(R)$. *Then* $P = P_i$ *for some* i *if and only if* $P = (N :_R f)$ *for some homogeneous* $f \in M$. *Thus* $\{P_1, ..., P_\ell\}$ *depend only on* N, *not the ipd. This set of primes is denoted* $\operatorname{Ass}_R M/N$, *the set of* ***associated primes of*** $\mathbf{M/N}$. *Note these primes are all homogeneous and the number of primary components in an ipd is uniquely determined by* N.

Proof. For the backward direction, suppose $P = (N :_R f)$ for some homogeneous $f \in M$. Then

$$P = (N :_R f) = (Q_1 \cap \cdots \cap Q_\ell :_R f) = \cap_1^\ell (Q_i :_R f).$$

This implies $P = (Q_i :_R f)$ (if $P = I \cap J$, then $P = I$ or $P = J$) and so $P = \sqrt{P} = \sqrt{(Q_i :_R f)} = P_i$ as $f \notin Q$ by the remark.

For the forward direction, choose f homogeneous such that $f \in Q_1 \cap \cdots \cap \hat{Q}_i \cap \cdots \cap Q_\ell \backslash Q_i$. Then $\sqrt{(Q_i :_R f)} = P_i$. Also, $(N :_R f) = \cap_1^\ell (Q_j :_R f) = (Q_i :_R f)$ by the remark as $f \in Q_j$ for all $j \neq i$. So $\sqrt{(N :_R f)} = P_i$. Consider the set $\Gamma = \{(N :_R yf) | y \in R \text{ homogeneous}, yf \notin N\}$. Choose $(N :_R yf)$ maximal in Γ.

Claim. $(N :_R yf) = P_i$.

Proof. Observe that $(N :_R f) \subseteq (N :_R yf)$ and $(N :_R yf) \subseteq P_i$ [We see $yf \in Q_j$ for all $j \neq i$ and $yf \notin Q_i$, else $yf \in N$. So suppose $r \in (N :_R yf)$ is homogeneous. Then $ryf \in N \subseteq Q_i$, which implies $r \in \sqrt{\operatorname{Ann} M/Q_i} = P_i$.] So, we see

$$P_i = \sqrt{(N :_R f)} = \sqrt{(N :_R yf)} = \sqrt{P_i} = P_i,$$

that is, $\sqrt{(N :_R yf)} = P_i$. Notice that if $r \in \sqrt{(N :_R yf)}$ is homogeneous, then $r^n \in (N :_R yf)$. If $ryf \in N$, then $r \in (N :_R yf)$ and if $ryf \notin N$, then $(N :_R yf) \subseteq (N :_R ryf)$ which implies $(N :_R yf) = (N :_R ryf)$ by maximality of y and thus $r^{n-1} \in (N :_R yf)$. Inductively, we see $r \in (N :_R yf)$. Thus $(N :_R yf) = \sqrt{(N :_R yf)} = P_i$. □

Note. For a homogeneous $f \in M$, we have $(0 :_R f)$ is an associated prime of M if and only if $(0 :_R f)$ is prime.

Remarks. Let R be Noetherian, $N \subseteq M$ be finitely generated, and everything graded.

(1) $M = N$ if and only if $\operatorname{Ass}_R M/N = \emptyset$ and so $M = 0$ if and only $\operatorname{Ass}_R M = \emptyset$.
(2) Q is a primary submodule of M if and only if $|\operatorname{Ass}_R M/Q| = 1$.

(3) $\sqrt{\operatorname{Ann}_R M} = \bigcap_{p \in \operatorname{Ass}_R M} p.$

Proof. Let $0 = Q_1 \cap \cdots \cap Q_\ell$ be an ipd. Then

$$\sqrt{\operatorname{Ann}_R M} = \sqrt{\operatorname{Ann}_R M/(Q_1 \cap \cdots \cap Q_\ell)} = \sqrt{\operatorname{Ann} M/Q_1 \cap \cdots \cap \operatorname{Ann} M/Q_\ell} = \cap_1^\ell \sqrt{\operatorname{Ann} M/Q_i} = \cap P_i. \quad \square$$

(4) The set of zerodivisors of M, $Z_R(M) = \bigcup_{p \in \operatorname{Ass}_R M} p.$

Proof. To show $\supseteq$, let $x \in \cup_{p \in Ann_R M} p$. Then $x \in p$ for some associated prime, which says $x \in (0 :_R f)$ for some homogeneous element. Then $xf = 0$ implies $x \in Z_R(M)$. For $\subseteq$, note that $Z_R(M) = \cup_{x \in M \setminus \{0\}}(0 :_R x)$ and so

$$Z_R(M) = \sqrt{Z_R(M)} = \sqrt{\cup_{x \in M \setminus \{0\}}(0 :_R x)} = \cup \sqrt{(0 :_R x)}.$$

Now, for $x \in M$, we see $(0 :_R x) = ((Q_1 \cap \cdots \cap Q_\ell :_R x) = \cap_1^\ell (Q_i :_R x)$. By the earlier remark (before the first uniqueness thm), we see $\sqrt{(0 :_R x)} = \cup_1^\ell \sqrt{(Q_i : x)} = \cap_{x \notin Q_i} P_i \subseteq P_j$ for some j. Thus $Z_R(M) = \cup_{x \in M \setminus \{0\}} \sqrt{(0 :_R x)} \subseteq \cup_{p \in \operatorname{Ass}_R M} p$. $\square$

Consequently, if $f \in Z_R(M)$, then all homogeneous components of f are in $Z_R(M)$ as each P is homogeneous. This also says that $(0 :_R x) \subseteq Z_R(M) = P_1 \cup \cdots \cup P_S$, which implies $(0 :_R x) \subset P$, for some associated prime. Thus, if $(0 :_R x)$ is not an associated prime, it is at least contained in one.

(5) $N = 0$ if and only if $N_p = 0$ for all $p \in \operatorname{Ass}_R M$. In particular, if $I \subseteq R$, then $I = 0$ if and only if $I_p = 0$ for all $p \in \operatorname{Ass}_R R$.

Proof. Suppose $N_p = 0$ for all associated primes and consider a homogeneous element $x \in N \setminus \{0\}$. Say $p \supseteq (0 :_R x)$ for $p \in \operatorname{Ass}_R M$. Then $N_p = 0$ implies $\frac{x}{1} \equiv \frac{0}{1}$, that is, there exists $a \in R \setminus p$ such that $ax = 0$. This of course says $a \in p$, a contradiction. Thus $N = 0$. $\square$

(6) R is reduced if and only if R_p is a field for every $p \in \operatorname{Ass}_R R$.

Proof. For the forward direction, let $p \in \operatorname{Ass}_R R$. Then $p = (0 :_R f)$ for some homogeneous f. Note $f \notin (0 :_R f)$ as R is reduced. Let $\frac{a}{s} \in R_p$. If $a \in (0 :_R f)$, then $fa = 0$ implies $\frac{a}{s} \cong \frac{0}{1}$. Otherwise, $\frac{s}{a} \in R_p$, which implies every nonzero element is a unit.

For the backward direction, suppose $r^n = 0$ for some $r \neq 0$. Say $p \supseteq (0 :_R r)$ for some $p \in \operatorname{Ass}_R R$. Consider R_p. If $\frac{r}{s} \cong \frac{0}{1}$, then there exists $t \in R \setminus p$ such that $tr = 0$, that is, $t \in p$, a contradiction. Otherwise $\frac{r}{s}$ is a unit, but also a nilpotent element, a contradiction. $\square$

Note that this can not be weakened to the set of minimal primes. For example, consider $R = k[x, y]/(x^2, xy)$, where the only minimal prime is (x). Clearly, R is not reduced as $x^2 = 0$. However,

$$R_{(x)} = k[x, y]_{(x)}/(x^2, xy)_{(x)} \cong k[x, y]_{(x)}/(x)_{(x)} \cong (k[x, y]/(x))_{(x)} \cong k[y]_{(0)} = k(y),$$

a field. Recall for an $R-$module M, the set of **minimal primes** is

$$\operatorname{Min}_R M = \{p \in \operatorname{Spec} R | p \text{ is minimal over } \operatorname{Ann}_R M\}.$$

(7) $\operatorname{Min}_R M \subseteq \operatorname{Ass}_R M$. Thus $\operatorname{Min}_R M = \{p \in \operatorname{Ass}_R M | p \text{ is minimal}\}$.

Proof. By Remark 3, we see $P \supseteq \sqrt{\operatorname{Ann}_R M} = \cap_{Q \in \operatorname{Ass}_R M} Q$. Thus $P \supseteq Q \supseteq \operatorname{Ann}_R M$ for some Q and thus $P = Q$. $\square$

Hence, $\operatorname{Min}_R M$ is a finite set and each minimal prime is homogeneous.

Combining the previous two remarks, we see that in a reduced ring R, the minimal primes are exactly the associated primes.

(8) $P \in \operatorname{Ass}_R M$ if and only if there exists a degree 0 injective homomorphism $0 \to R/p(g) \to M$ for some $g \in G$.

Proof. If $p \in \operatorname{Ass}_R M$, then $p = (0 :_R f)$ where f is homogeneous. Let $\deg f = g$. Define $\phi : R/p(-g) \to M$ by $\overline{r} \mapsto rf$. This is injective. Similarly, if $R/p(d) \to M$ is defined by $1 \mapsto f$, then $p = (0 :_R f)$. ☐

(9) Let $0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N$ be an exact sequence of degree 0 homomorphisms of finitely generated $R-$modules. Then $\operatorname{Ass}_R L \subseteq \operatorname{Ass}_R M \subseteq \operatorname{Ass}_R L \cup \operatorname{Ass}_R N$.

Proof. For the first containment, let $p \in \operatorname{Ass}_R L$. Then $p = (0 :_R f)$ for some homogeneous $f \in L$. Note then that $p = (0 :_R \phi(f))$ as for $a \in p$, we have $af = 0$ which implies $0 = \phi(af) = a\phi(f)$ and for $a \in (0 :_R \phi(f))$, we have $0 = a\phi(f) = \phi(af)$ which implies $af = 0$ as ϕ is 1-1.

For the second containment, let $p \in \operatorname{Ass}_R M$. By the above remark, we see $R/p(g) \cong M' \subseteq M$ for some submodule M' of M and $g \in G$. Note that for all $x \in M'$, we have $p = (0 :_R x)$.

Case 1: $M' \cap \phi(L) \neq (0)$. Say $x \in M' \cap \phi(L)$ with $x \neq 0$. Say $\phi(y) = x$ for $y \in L$. Then $p \subseteq (0 :_R y)$ as for $a \in p$ we have $0 = ax = a\phi(y) = \phi(ay)$ and thus $ay = 0$ as ϕ is injective. In fact $p = (0 :_R x)$ as for $a \in (0 :_R y)$ we have $ay = 0$ implies $0 = \phi(ay) = a\phi(y) = ax$. Thus $p \in \operatorname{Ass}_R L$.

Case 2: $M' \cap \phi(L) = \emptyset$. Then $\psi|_{M'}$ is injective by exactness. So $R/p(g) \cong M' \cong \psi(M')$ and so $p \in \operatorname{Ass}_R N$ by the above remark. ☐

Proposition 2.8. *Let R be Noetherian, $M \neq 0$ finitely generated graded $R-$module. Then there exists a sequence of graded submodules of M $(0) = N_t \subsetneq N_{t-1} \subsetneq \cdots \subsetneq N_0 = M$ such that $N_i/N_{i+1} \cong (R/p_i)(g_i)$ for some homogeneous P_i for $i = 0, ..., t-1$ and $g_i \in G$.*

Proof. Let $\Gamma = \{N \subseteq M \text{ graded} | N \text{ has such a filtration}\}$. As $M \neq 0$, there exists $p \in \operatorname{Ass}_R M$ and thus an injection $\phi : R/p(f) \hookrightarrow M$. Let $N = \phi(R/p(g))$, a graded submodule of M, so that $N \cong R/p(g)$. Thus $N \in \Gamma$ and so $\Gamma \neq \emptyset$. Choose $N \in \Gamma$ maximal. Suppose $N \neq M$. Let $p \in \operatorname{Ass}_R M/N$. So there exists an degree 0 injection $\phi : R/p(g) \hookrightarrow M/N$. Let $N \subseteq N' \subseteq M$ be graded with $N'/N = \phi(R/p(g))$. Then $N' \supseteq N$ has a filtration, a contradiction to maximality. Thus $N = M$. ☐

Lemma 2.9 (Prime Avoidance Lemma). *Let $I, P_1, ..., P_k$ be ideals in a ring R. Suppose*

(1) *At least $k-2$ of the P_i's are prime*
(2) *$I \not\subset P_i$ for $i = 1, ..., k$.*

Then $I \not\subset \cup_{i=1}^{\ell} P_i$.

Application. Let R be Noetherian, I an ideal. Then I contains a non-zerodivisor if and only if $\operatorname{Ann}_R I = 0$ if and only if $\operatorname{Hom}_R(R/I, R) = 0$.

Proof. The second implication statement and the forward implication of the first are clear. So suppose $\operatorname{Ann}_R I = (0)$, but I consists of zerodivisors. Then $I \subset \cup_{p \in \operatorname{Ass} R} P$, which implies $I \subset P$ for some $p \in \operatorname{Ass} R$. Say $p = (0 :_R x)$. Then $x \in \operatorname{Ann}_R I$, a contradiction. ☐

This sets up the homological aspect of depth.

Example. Let $R = k[x, y], I = (x, y), P_1 = (x), P_2 = (y)$. Clearly, $I \not\subset (x)$ and $I \not\subset (y)$. By the PAL, $I \not\subset (x) \cup (y)$. (for example, $x + y \in I \setminus ((x) \cup (y))$.

If $\deg x = \deg y = 1$, we have a homogeneous element in $I \setminus ((x) \cup (y))$. If $\deg x = (1, 0), \deg y = (0, 1)$, then $x + y$ is no homogeneous. In fact, there does not exist any homogeneous element in $I \setminus ((x) \cup (y))$.

Proof. Suppose there exists $f \in I$ homogeneous. Say $\deg f = (r, s)$. Then $f = cx^r y^s \in (x) \cup (y)$. So every homogeneous element of I is in $(x) \cup (y)$. ☐

A similar proof shows no homogeneous elements exist in $I \setminus ((x) \cap (y))$ when $\deg x = 0, \deg y = 1$.

Thus (this version) of the Prime Avoidance Lemma can not be modified to a general graded ring so that we can find a homogeneous element in $I \setminus (\cup P_i)$. However, we can talk about a relaxed prime avoidance lemma in the graded case.

Exercise. Let R be local, Noetherian. Suppose I contains a non-zerodivisor. Prove I has a minimal generating set consisting of non-zerodivisors.

Proof. First, we show we can find a generating set consisting of non-zerodivisors. Recall, by NAK, $I = (x_1, ..., x_n)$ if and only if $I/mI = (\overline{x_1}, ..., \overline{x_n})$. So any $x \in I \setminus mI$ is part of a generating set of I. Now, as I contains a non-zerodivisor,$I \not\subset P = ZD(R)$ for $p \in \operatorname{Ass} R$ and $I \not\subset mI$. Thus by PAL, there exists $x_1 \in I \setminus (mI \cup (\cup_{p\in \operatorname{Ass} R} p))$. Then x_1 is a non-zerodivisor and part of a generating set for I. If $I = (x_1)$, done. Otherwise, suppose we have $\{x_1, ..., x_{n-1}\}$. If $I = (x_1, ..., x_{n-1})$, done. Otherwise, $I \not\subseteq (x_1, ..., x_{n-1})$ which implies, by PAL, that there exists $x_n \in I \setminus ((x_1, ..., x_{n-1}) \cup mI \cup (\cup_{p\in \operatorname{Ass} R} p))$. If $I = (x_1, ..., x_n)$, done. Otherwise, continue. By ACC, we must eventually stop. Say $I = (x_1, ..., x_k)$ where x_i are non-zerodivisor. Then $\overline{I} = (\overline{x_1}, ..., \overline{x_k})$. This is a generating set over a field and thus some subset must be a basis. Lift the subset back up to get a minimal generating set of non-zerodivisors for I. ☐

Lemma 2.10 (**Graded Verson of the Prime Avoidance Lemma**). *Let R be a $\mathbb{Z}-$graded ring and $I, P_1, ..., P_l$ homogeneous ideals such that*

(1) *P_i is prime for all i.*
(2) *I is generated by elements in strictly positive degree*
(3) *$I \not\subset P_i$, $i = 1, ..., k$.*

Then, there exists a homogeneous $f \in I \setminus \cup_1^k P_i$ with $\deg f > 0$.

Proof. Induct on k. If $k = 1$, clear. Assume true for $k - 1$ primes. By induction, we get homogeneous $f_i \in I \setminus (P_1 \cup \cdots \cup \hat{P_i} \cup \cdots \cup P_k)$ for $i = 1, ..., k$ with $\deg f_i = m_i$. If $f_i \notin P_i$ for some i, done. So assume $f_i \in P_i$ for $i = 1, ..., k$. Let $s = \prod m_i$ and so $\deg f_i^{s/m_i} = s$ for all i. Note $f_i^{s/m_i} \in I$ and $f_i^{s/m_i} \notin P_i$ as P_i is prime. By replacing f_i with f_i^{s/m_i}, we can assume f_i are homogeneous of the same degree. Consider $f = f_1^{k-1} + f_2 f_3 \cdots f_k$, a homogeneous element in I. If $f \in P_1$, then $f_2 \cdots f_k \in P_1$, which implies $f_j \in P_1$ for some $j > 1$, a contradiction to the definition of f_j. Similarly, if $f \in P_j$ for $j > 1$, then $f_1^{k-1} \in P_j$ and so $f_1 \in P_j$, a contradiction. Thus $f \in I \setminus \cup_1^k P_i$ is homogeneous. ☐

Q: Can we extend this proof so that P_1, P_2 are not primes?

Example. Let $R = k[x, y]/(xy), (\overline{o}) = (\overline{x}) \cap (\overline{y}), Z_R(R) = (\overline{x}) \cup (\overline{y})$. If R is graded in one of the standard ways, then R has non-zerodivisors, but no homogeneous ones.

Homogeneous Localization

Let R be a $G-$graded ring, S a mcs consisting of homogeneous elements. Then the localization R_S has a natural grading where, for $g \in G$,

$$(R_S)_g = \left\{ \frac{r}{s} | r, s \text{ are homogeneous }, \deg r - \deg s = g \right\}.$$

- Check that this is closed under addition, is an abelian group, every element can be written as a sum of $\frac{r}{s}$ (of course, $\frac{f}{s} = \frac{f_1}{s} + \cdots + \frac{f_t}{s}$).

Then $R_S = \oplus_{g\in G}(R_S)_g$ is a $G-$graded ring. If M is a graded $R-$module, then $M_S = \oplus_{g\in G}(M_S)_g$ where $(M_S)_g = \{\frac{m}{s} | m, s \text{ are homogeneous}, \deg m - \deg s = g\}$ is a graded R_S-module.

If $p \in \operatorname{Spec} R$, let $(p) = \{s \in R | s \text{ is homogeneous}, s \notin P\}$. Then $R_{(p)}$ is the homogeneous localization of R at p.

Q: If p is homogeneous, is this *local?

Example. Let $R = k[t]$, $\deg t = 1$. Let $S = \{t^n | n \geq 0\}$. Then R_S is a $\mathbb{Z}-$graded ring and $R_S \cong k[t, t^{-1}]$.

Exercise. Let R be graded, M a graded $R-$module, S a homogeneous mcs, and $Q \subseteq M$ $p-$primary and graded. Then

(1) $Q_S \subseteq M_S$ is p_S-primary if $S \cap p = \emptyset$.

Proof. Note that if $S \cap p = \emptyset$, then p_S is a prime in M_S. First, we show $p_S = \sqrt{\operatorname{Ann} M_S/Q_S}$. Then, we show Q_S is in fact primary. Recall that localizing commutes with annihilators and radicals. Thus

$$p_S = \left(\sqrt{\operatorname{Ann} M/Q}\,\right)_S = \sqrt{(\operatorname{Ann} M/Q)_S} = \sqrt{\operatorname{Ann}(M/Q)_S} = \sqrt{\operatorname{Ann} M_S/Q_S}.$$

To show Q_S is primary, suppose $\frac{f}{s} \cdot \frac{m}{s'} \in Q_S$. If $\frac{f}{s} \in p_S$, done. Otherwise, there exists $s'' \in S$ such that $s''fm \in Q$. Now $s'' \in S$ and $p \cap S = \emptyset$ implies $s'' \notin p$. Thus $fm \in Q$. Of course, $\frac{f}{s} \notin p_S$ implies $f \notin p$. Thus $m \in Q$ and so $\frac{m}{s'} \in Q_S$. ☐

(2) $Q_S = M_S$ if $S \cap p \neq \emptyset$.

Proof. If $S \cap p \neq \emptyset$, then $p_S = R_S$ which implies $R_S = \sqrt{\operatorname{Ann} M_S/Q_S}$. Then $1 \in \operatorname{Ann} M_S/Q_S$ which implies $Q_S = M_S$. ☐

Lemma 2.11. *With the same hypotheses as above, consider $\phi : M \to M_{(p)}$ defined by $m \mapsto \frac{m}{1}$. Then $\phi^{-1}(Q_{(p)}) = Q$.*

Proof. $\supseteq$ is clear. Let $q \in \phi^{-1}(Q_{(p)})$. Then $\frac{q}{1} = \frac{q'}{s}$, for $s \notin p$ homogeneous. So there exists $s' \in (p)$ such that $s'(sq - q') = 0$, i.e., $s'sq = s'q' \in Q$. Now $s's \notin P$ and so $q \in Q$ by definition of primary. ☐

Exercise. Let R be graded, Noetherian, M a finitely generated $R-$module, and S a homogeneous mcs. Then $\operatorname{Ass}_{R_S} M_S = \{p_S | p \in \operatorname{Ass} M, p \cap S = \emptyset\}$.

Proof. To show $\subseteq$, let $0 = Q_1 \cap \cdots \cap Q_t$ be an ipd. Then $0_S = (Q_1)_S \cap \cdots \cap (Q_t)_S$ is a primary decomposition. If $S \cap p_i \neq \emptyset$, then $(Q_i)_S = M_S$ and so we can remove it from the intersection. This leaves us with (after indexing) $0_S = (Q_1)_S \cap \cdots \cap (Q_n)_S$. To show this is irredundant, we need only show that we can not remove another one. We know there exists $x \in (Q_2 \cap \cdots \cap Q_n) \setminus Q_1$. If it were the case that $x \in (Q_1)_S$, then there would exist $s \in S$ such that $xs \in Q_1$. Now Q_1 is primary and $s \notin P_1$ (since $P_1 \cap S = \emptyset$). Thus $x \in Q_1$, a contradiction. So we have an ipd for 0_S. ☐

Theorem 2.12 (**Second Uniqueness Theorem**). *Let R be graded, Noetherian, M finitely generated and graded, and $N \subsetneq M$ graded. Let $N = Q_1 \cap \cdots \cap Q_t$ be an ipd, where Q_i are p_i-primary. Suppose $p_i \in \operatorname{Min}_R M/N$ (that is, P_i is minimal in $\operatorname{Ass}_R M/N$). Then $Q_i = N_{(p_i)} \cap M = \phi^{-1}(N_{(p_i)})$ where $\phi : M \to M_{(p_i)}$. In particular, the primary component of N corresponding to p_i is unique. Say Q_i is an* ***isolated primary component****.*

Proof. Let $S = (p_i)$. Then $N_S = (Q_1 \cap \cdots \cap Q_t)_S = (Q_1)_S \cap \cdots \cap (Q_t)_S$. Suppose $j \neq i$. Then $P_j \not\subset P_i$ as P_i is minimal. So there exists a homogeneous element in $P_j \setminus P_i$, that is, $P_j \cap S \neq \emptyset$. Thus $N_S = (Q_i)_S$. Therefore, $Q_i \phi^{-1}((Q_i)_S) = \phi^{-1}(N_S)$. ☐

This shows that if all the associated primes are minimal, then there is a unique ipd.

Exercise. Prove or Disprove. If Q is a uniquely determined primary component for $N \subseteq M$ (that is, Q is in every ipd for N), then Q is isolated.

2.1. **Symbolic Powers.** Let R be Noetherian, $p \in \operatorname{Spec} R$. Note $\operatorname{Min}_R R/p^n = \{p\}$ (as if $Q \supseteq p^n$ is prime, then $Q \supseteq p$). Thus the $p-$primary component of p^n is uniquely determined, denoted $p^{(n)}$. By the theorem, $p^{(n)} = \phi^{-1}(p^n R_p) = p^n R_p \cap R$. One easily sees the following:

- $p^{(n)} \supseteq p^{(n+1)} \supseteq \cdots$
- $p^n \subseteq p^{(n)}$ with equality if and only if p^n is primary.
- $p^{(n)} p^{(m)} \subseteq p^{(n+m)}$.

This allows us to talk about the **Symbolic Rees Algebra** $S(p) = R \oplus pt \oplus p^{(2)}t^2 \oplus \cdots$.

Let R be a regular local ring of dimension d, $p \in \operatorname{Spec} R$ such that $\dim R/p = 1$ (that is, $\operatorname{Spec} R/p = \{\overline{(0)}, \overline{m}\}$).

Open Question. Does there exists $a_1, ..., a_{d-1} \in p$ such that $p = \sqrt{(a_1, ..., a_{d-1})}$, that is, is p a set theoretic complete intersection?

R. Cowsik showed in 1978 that, with the above hypotheses, this is true if $S(p) = R \oplus pt \oplus p^{(2)}t^2 \oplus \cdots$ is Noetherian. Unfortunately, Paul Roberts gave an example in the 80s that showed the symbolic Rees algebra is not always Noetherian. Note however the being Noetherian implies there exists $k \geq 1$ such that $(p^{(k)})^n = p^{(kn)}$ for all $n \geq 1$.

Proposition 2.13. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be an $\mathbb{N}-$graded ring. Suppose R is Noetherian. Then there exists $k \geq 1$ such that $R_{kn} = (R_k)^n$ for all $n \geq 1$.*

Proof. Let $R_+ = R_1 \oplus R_2 \oplus \cdots$, an ideal of R. Suppose R_+ is generated in degrees $R_1, ..., R_\ell$. So $(R_1 \oplus \cdots \oplus R_\ell)R = R_+ (*)$.

Claim. For all $m \geq 1$, $R_m = \sum_{\substack{(j_1,\ldots,j_\ell) \\ \sum i j_i = m}} R_1^{j_1} \cdots R_\ell^{j_\ell}$.

Proof. Note that $R_{\ell+1} = R_1 R_\ell + R_2 R_{\ell-1} + ... + R_\ell R_1$ by $(*)$. Proceed by induction.

Claim. Let $k = \ell \cdot \ell!$. For all $m \geq k$, $R_m = R_{m-\ell!} R_{\ell!}$.

Proof. Note that $\supseteq$ is true by the grading. So we only prove $\subseteq$. Now $R_m = \sum_{(j)} R_1^{j_1} \cdots R_\ell^{j_\ell}$. Let $(j_1, ..., j_\ell) \in \mathbb{N}^\ell$ such that $\sum_1^k i j_i = m \geq k = \ell \cdot \ell!$. Then there exists $i \in \{1, ..., \ell\}$ such that $i j_i \geq \ell!$ by the pigeonhole principle. Thus $j_i \geq \frac{\ell!}{i}$. So

$$R_1^{j_1} \cdots R_\ell^{j_\ell} = R_1^{j_1} \cdots R_{i-1}^{j_{i-1}} (R_i^{\ell!/i} \cdot R_i^{j_i - \ell!/i}) R_{i+1}^{j_{i+1}} \cdots R_\ell^{j_\ell} \subseteq R_{m-\ell!} R_{\ell!}.$$

To finish the proof, we will induct on n to show $R_{kn} = (R_k)^n$. If $n = 1$, clear. For $n > 1$, we have

$$\begin{aligned} R_{kn} = R_{n \cdot \ell \cdot \ell!} &= R_{(n\ell-1)\ell!} R_{\ell!} \\ &= R_{(n\ell-2)\ell!} R_{\ell! \ell!} \\ &\vdots \\ &= R_{(n\ell-\ell)\ell!} R_{\ell \cdot \ell!} \\ &= R_{(n-1)k} R_k = (R_k)^{n-1} R_k = (R_k)^n. \end{aligned}$$

□

If $S(p)$ is Noetherian, then there exists k such that $p^{(kn)} = (p^{(k)})^n$ for all $n \geq 1$ (this is in fact an if and only if statement). Thus $\operatorname{Ass}_R R/(p^{(k)})^n = \{p\}$. Let $I = p^{(k)}$. Choose $x \in m \setminus p$. Then $x \notin \cup_{\operatorname{Ass}_R R/I^n} p$ which implies x is a non-zerodivisor on R/I^n for all $n \geq 1$. So $(I^n :_R x) = I^n$ for all $n \geq 1$.

Consider $gr_I(R) = R/I \oplus I/I^2 \oplus \cdots$ and $\overline{x} \in R/I$. Then $\overline{x}$ is a non-zerodivisor on $gr_I(R)$. (If not, then $\overline{x} \in Q \subseteq \operatorname{Ass}_R gr_I(R)$. So $Q = (0 :_{gr_I(R)} \overline{t})$ for some homogeneous element t with $\overline{t} \in I^n/I^{n+1}$. then $\overline{xt} = \overline{0}$ implies $xt \in I^{n+1}$ and so $t \in I^{n+1}$, a contradiction.) Hence $\overline{x}$ is not in any minimal prime of $G = gr_I(R)$. So $\dim G/xG \leq \dim G - 1 = \dim R - 1$. From here, one could prove Cowsik's result.

Proposition 2.14. *Let R be a $G-$graded Noetherian ring, M a finitely generated graded $R-$module. Let $p \in \operatorname{Ass}_{R_0} M_g$ for some $g \in G$. Then there exists $Q \in \operatorname{Ass}_R M$ such that $Q \cap R_0 = p$. Consequently*

$$\bigcup_{g \in G} \operatorname{Ass}_{R_0} M_g \subseteq \{Q \cap R_0 | Q \in \operatorname{Ass}_R M\},$$

which is a finite set.

Proof. Let $S = R_0 \setminus p$, a multiplicatively closed set of homogeneous elements. Then $p_S \in \operatorname{Ass}_{(R_0)_S}(M_g)_S$. If we show there exists $Q_S \in \operatorname{Ass}_{R_S} M_S$ such that $Q_S \cap (R_0)_S = p_S$, we will be done as $Q_S \cap (R_0)_S = p_S$ if and only if $Q \cap R_0 = p$. Thus it suffices to prove in the case where (R_0, m) is local and $p = m$. Now $m \in \operatorname{Ass}_{R_0} M_g$ implies m consists of zerodivisors on M_g and hence M. Thus $m \subset Q$ for some $Q \in \operatorname{Ass}_R M$. So $m \subseteq Q \cap R_0$ which implies $m = Q \cap R_0$. □

Theorem 2.15. *Let R be Noetherian, M a finitely generated $R-$module, I an ideal. Then $\cup_1^\infty \operatorname{Ass}_R M/I^n M$ is a finite set.*

Proof. Let $G = gr_I(R)$ and $\tilde{M} = M/IM \oplus IM/I^2M \oplus \cdots$. Then $\tilde{M}$ is a finitely generated graded $G-$module and G is Noetherian. By the proposition $\cup_0^\infty \operatorname{Ass}_{R/I} I^n M/I^{n+1}M$ is finite. Note that if M is an $R-$ and $R/I-$module, then $\operatorname{Ass}_{R_I} M = \{p/I | p \in \operatorname{Ass}_R M\}$.

Claim. $\cup \operatorname{Ass}_R M/I^n M \subseteq \operatorname{Ass}_{R/I} I^n M/I^{n+1}M = \cup_R I^n M/I^{n+1}M = A$.

Proof. We will induct on n to show $\cup \operatorname{Ass}_R M/I^n M \subseteq A$. If $n = 1$, then $\operatorname{Ass}_R M/IM = \operatorname{Ass}_R I^0 M/IM \subseteq A$. For $n > 1$ we have

$$0 \to I^n M/I^{n+1}M \to M/I^{n+1}M \to M/I^n M \to 0.$$

So $\operatorname{Ass}_R M/I^{n+1}M \subseteq \operatorname{Ass}_R I^n M/I^{n+1}M \cup \operatorname{Ass}_R M/I^n M \subseteq A$.

□

3. Dimension Theory

Theorem 3.1 (Krull's Principal Ideal Theorem). *Let R be Noetherian, $p \in \operatorname{Spec} R$. Suppose p is minimal over (x) for some $x \in R$. Then $\operatorname{ht} p \leq 1$.*

Proof. Note that localizing at p does not change the height of p. So WLOG (R, m) is local with m minimal over x, that is, $m = \sqrt{(x)}$ (since m is the only prime containing (x)). Let $q \in \operatorname{Spec} R$, $q \neq m$. We need to show $\operatorname{ht} q = 0$. This would show $\operatorname{ht} m \leq 1$.

Recall $q^{(n)} = \phi^{-1}(q^n R_q)$ where $\phi : R \to R_q$ is the canonical map. We have shown $q^{(n)}$ is the $q-$primary component of q^n. Note $q^{(n)} \supseteq q^{(n+1)} \supseteq \cdots$. Note $x \notin q$ as m is the only prime over (x). So consider the following chain of ideals in $R/(x)$:

$$\frac{(q,x)}{(x)} \supseteq \frac{(q^{(2)},x)}{(x)} \supseteq \frac{(q^{(3)},x)}{(x)} \supseteq \cdots .$$

As $R/(x)$ is Noetherian and $\dim R/(x) = 0$, we see $R/(x)$ is Artinian. So the chain stabilizes. Say $(q^{(n)}, x)/(x) = (q^{(n+1)}, x)/(x)$ for some n. Then $q^{(n)} \subseteq (q^{(n+1)}, x)$.

Claim. $q^{(n)} = xq^{(n)} + q^{(n+1)}$.

Proof. Note that $\supseteq$ is clear. So let $a \in q^{(n)}$. Then $a = b + rx$ where $b \in q^{(n+1)}$ and $r \in R$. So $rx = a - b \in q^{(n)}$. As $q^{(n)}$ is $q-$primary and $x \notin q$, we have $r \in q^{(n)}$.

Hence $q^{(n)} = q^{(n+1)} + mq^{(n)}$. By NAK, $q^{(n)} = q^{(n+1)}$. Note $q^{(n)} R_q = q^n R_q$ and $q^{(n+1)} R_q = q^{n+1} R_q$. Thus $q^n R_q = q^{n+1} R_q$ and again by NAK, $q^n R_q = 0$. So $\operatorname{ht} q R_q = 0$ which implies $\operatorname{ht} q = 0$. □

If $p \in \operatorname{Spec} R$ and $I \subset p$, then p/I is a prime in R/I. Is $q \subset p$ are primes, then $\operatorname{ht}(p/q)$ is defined as the length of the longest chain of primes $q = q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_s = p$. Thus $\operatorname{ht}(p/q) = 0$ if and only if $p = q$ and $\operatorname{ht}(p/q) = 1$ if and only if there are no primes between p and q.

Corollary 3.2. *Let $q \subset p$ be primes in a Noetherian ring R. Suppose $\operatorname{ht}(p/q) > 1$. Then there are infinitely many primes between p and q.*

Proof. By modding by q and localizing at p, it is enough to show in the case where (R, m) is a local domain, $q = 0$ and $p = m$. Then $\operatorname{ht} m > 1$. So suppose there are only finitely many primes between m and 0. Then $m \not\subset p_i$ for all i

and so $m \not\subset p_1 \cup \cdots \cup p_\ell$ by PAL. Let $x \in m \setminus (p_1 \cup \cdots \cup p_\ell)$. Then m is minimal over (x) by $\mathrm{ht}(m) > 1$, a contradiction to Krull's Principal Ideal Theorem. □

Theorem 3.3 (**Generalized Principal Ideal Theorem**). *Let R be Noetherian, $p \in \operatorname{Spec} R$. Suppose p is minimal over $(a_1, ..., a_\ell)$. Then $\operatorname{ht} p \leq \ell$.*

Proof. By localizing at p we may assume (R, m) is local and m is minimal over $(a_1, ..., a_\ell)$. We want to show $\operatorname{ht} m \leq \ell$. So suppose not. Then $\operatorname{ht} m > \ell \geq 2$ (the $\ell = 1$ is true by the above). Let $m = p_0 \supsetneq p_1 \supsetneq \cdots \supsetneq p_{\ell+1}$ be a chain of primes descending from m. Assume $\mathrm{ht}(p_i/p_{i+1}) = 1$ for $i = 0, ..., \ell$. We know $(a_1, ..., a_\ell) \not\subseteq p_1$, so WLOG $a_1 \notin p_1$. As $\operatorname{ht} m/p_1 = 1$ and $a \in m \setminus p_1$, we see $m = \sqrt{(p_1, a_1)}$. Then there exists t such that $a_i^t \in (p_1, a_1)$ for $i = 2, ..., \ell$. Say $a_i^t = b_i + r_i a_1$ for $b_i \in p_1, r_i \in R$ and $i = 2, ..., \ell$. Let $J = (b_2, ..., b_\ell) \subset p_1$. Note $\operatorname{ht} p_1 \geq \ell$ as $\operatorname{ht} m > \ell$ and $\operatorname{ht} m/p_1 = 1$. So p_1 is not minimal over J. Thus there exists $Q \in \operatorname{Spec} R$ with $P_1 \subsetneq Q \subseteq J = (b_2, ..., b_\ell)$. Note $a_i^t = b_i + r_i a_1 \in (J, a_1) \subseteq (Q, a_1)$. So $(a_1, ..., a_\ell) \subseteq \sqrt{(Q, a_1)}$ which implies $m = \sqrt{(a_1, ..., a_\ell)} = \sqrt{(Q, a_1)}$. Thus m is minimal over (Q, a_1). In R/Q, we see $\overline{m} = m/Q$ is minimal over $(\overline{a_1})$. However $m/Q \supsetneq p_1/Q \supsetneq Q/Q$ implies $\mathrm{ht}(\overline{m}) \geq 2$, a contradiction to Krull's principal ideal theorem. □

This says every prime has finite height (as it is minimal over a generating set and has a finite generating set by Noetherianness).

Lemma 3.4. *Let R be a Noetherian ring, p a prime and $I \subseteq p$ an ideal. Then p is minimal over I if $p^n \subseteq I$ for some n. If (R, p) is local, then the converse is true.*

Proof. First suppose $p^n \subseteq I$ for some n. Note that $p^n \subseteq I$ are still proper ideals when we localize at p. So, WLOG let R be local and p maximal. Note that p/I is maximal in R/I and $p^n \subseteq I$. Thus $(p/I)^n = 0$, which implies R/I is Artinian (Prop 8.6 A&M) and thus has dimension 0. So p is the only prime containing I and is therefore minimal over I.

For the converse, if p is minimal over I, then R/I is Artinian (the only prime is p/I) and so $p^n \subseteq I$ as the nilradical is nilpotent in a Artinian Ring. □

Lemma 3.5 (**"Converse" to the Principal Ideal Theorem**). *Let R be Noetherian, I an ideal of height n. Then there exists $x_1, ..., x_n \in I$ such that $\mathrm{ht}(x_1, .., x_i) = i$ for $i = 1, ..., n$.*

Proof. If $\mathrm{ht}(I) = 0$, let $x \in I$. Then $\mathrm{ht}(x) = 0$. So suppose $\mathrm{ht}(I) > 0$ and induct on $i \leq n$. Say we have found $x_1, ..., x_{i-1} \in I$ with $\mathrm{ht}(x_1, ..., x_{i-1}) = i - 1$. Note that $I \not\subseteq (x_1, ..., x_{i-1})$ as otherwise $\mathrm{ht}(I) = i - 1$. Let $P_1, ..., P_m$ be the height $i - 1$ primes over $(x_1, ..., x_{i-1})$. Then again we see $I \not\subseteq P_i$ for all i as otherwise $\mathrm{ht}(I) = i - 1$. By prime avoidance, there exists $x_i \in I \setminus (\cup_{i=1}^m P_i)$. Thus $\mathrm{ht}(x_1, ..., x_i) \geq i$. By the Principal Ideal Theorem, $\mathrm{ht}(x_1, ..., x_i) \leq i$ and so $\mathrm{ht}(x_1, ..., x_i) = i$. □

Exercises.

(1) If R is Noetherian, then $\dim R[x] = \dim R + 1$.

Proof. Let $Q \subset R[x]$ be a maximal ideal and $P = Q \cap R$. Then P is prime in R. Say $\mathrm{ht}(P) = n$. We want to show $\mathrm{ht}(Q) = n + 1$. Then, as we can choose Q lying over P with $\operatorname{ht} P = \dim A$, we will be done. Let $S = R \setminus P$. Then $\mathrm{ht}(P_S) = \mathrm{ht}(P)$ and $\mathrm{ht}(Q_S) = \mathrm{ht}(Q)$ by our choice of S. So we may assume R is local with maximal ideal P.

Consider $PR[x]$. By the converse to the principal ideal theorem, there exists $a_1, ..., a_n \in R$ such that P is minimal over $I = (a_1, ..., a_n)$. By the lemma, there exists $P^n \subset I$ for some n. Then $(PR[x])^n \subseteq IR[x]$, and so $PR[x]$ is minimal over $IR[x]$.

Now consider Q and mod out by $PR[x]$. Then R/P is a field and thus $R[x]/PR[x] = (R/P)[x]$ is a PID. So $\overline{Q} = \overline{(f)}$ for some $f \in R[x]$.

Let $Q' \subset Q$ be a minimal prime over $(I, f)R[x]$. Then $PR[x] \subseteq Q'$ as $(PR[x])^n \subseteq IR[x] \subseteq Q'$, a prime. But now $\overline{Q'}$ contains $(((I, f) + p)/p)R[x] = \overline{(f)}$. Thus $Q' = Q$ and Q is minimal over an ideal generated by $n + 1$ elements. This implies $\operatorname{ht}(Q) \leq n + 1$ by the Principal Ideal Theorem. So $\dim R[x] \leq \dim R + 1$.

For the other inequality, let m be maximal with $\operatorname{ht} m = \dim R$. Then $\operatorname{ht} mR[x] = \dim R$ as the chain of primes $m = p_0 \supsetneq p_1 \supsetneq \cdots \supsetneq p_m = 0$ yields the chain $mR[x] = p_0R[x] \supsetneq p_1R[x] \supsetneq \cdots \supsetneq p_mR[x] = 0$. Let Q be maximal with $Q \supset mR[x]$. If $Q \cap R \neq m$, then $1 \in Q \cap R$ and so $1 \in Q$. Thus $Q \cap R = m$. Further, Q/m is maximal in $(R/m)[x]$ and so $Q \supsetneq mR[x]$. So $\operatorname{ht} Q = \dim R + 1$ and thus $\dim R[x] = \dim R + 1$. □

(2) Let $R = k[x_1, ..., x_n]$, k a field. Prove $\operatorname{ht} m = n$ for all maximal ideals m.

Proof. By the above, $\dim R = n$. So $\operatorname{ht} m \leq n$ for all maximal ideals m. If $k = \overline{k}$ the Nullstellensatz says every maximal ideal has the form $(x_1 - a_1, ..., x_n - a_n)$. Clearly, this has height n. If $k \neq \overline{k}$, then we refer to the following:

Strong Nullstellensatz: *If A/k is an extension of fields and A is finitely generated over k as an algebra, then A/k is algebraic.*

Let $m \subseteq k[x_1, ..., x_n]$ be maximal. Then R/m is a field containing k, say $R/m = k[\overline{x_1}, ..., \overline{x_n}]$. By the Strong Nullstellensatz, $(R/m)/k$ is algebraic. So for each i there exists $f_i(t) \in k[t]$ such that $\overline{f_i(x_i)} = \overline{0}$, that is, $f_i(x_i) \in m$.

We will induct on n. If $n = 1$, we are in a PID and it is clear. If $n > 1$ consider the algebraic extension of fields

$$\begin{array}{c} k[x_1, ..., x_n]/m \\ | \\ k[x_1, ..., x_{n-1}]/m \cap k[x_1, ..., x_{n-1}] \\ | \\ k \end{array}$$

The middle ring is stuck between two fields and so must itself be a field. So $m \cap k[x_1, ..., x_{n-1}] =: \mathfrak{n}$ is maximal in $k[x_1, ..., x_{n-1}]$. By induction, $\mathfrak{n}$ is generated by $n - 1$ elements and has height $n - 1$. Consider $\overline{m}$, maximal in $k[x_1, ..., x_n]/\mathfrak{n}k[x_1, ..., x_n] = (k[x_1, ..., x_{n-1}]/\mathfrak{n})[x_n]$, a PID as $k[x_1, ..., x_{n-1}]/\mathfrak{n}$ is a field. So $\overline{m} = \overline{(f(x_n))}$ which implies $m = (\mathfrak{n}, f(x_n))$. So m is generated by n elements and has height n as $\mathfrak{n}k[x_1, ..., x_n] \subsetneq m$. □

Example. Let $R = \mathbb{Z}_{(2)}$, a local ring of dimension 1. Then $\dim R[x] = 2$. Let $p = (2x - 1)$. Then p is a maximal ideal of height 1.

Let (R, m) be local with $\dim R = d = \operatorname{ht} m$. By Prime Avoidance and the converse of the Principal Ideal Theorem, there exists $x_1, ..., x_d \in m$ such that $m = \sqrt{(x_1, ..., x_d)}$. Note that there does not exists $y_1, ..., y_t \in m$ with $t < d$ such that $m = \sqrt{(y_1, ..., y_t)}$ by the generalized Principal Ideal Theorem. A sequence $x_1, ..., x_d \in m$ such that $m = \sqrt{(x_1, ..., x_d)}$ is called a **system of parameters** for R. In fact, we can define the dimension of R by a system of parameters:

$$\dim R = \inf\{t \geq 0 | \text{ there exists } x_1, ..., x_t \in m \text{ such that } m = \sqrt{(x_1, ..., x_t)}\}.$$

Definition. *If (R, m) is local, the* ***embedding dimension*** *of R, denoted edim R, is the least number of generators for m, that is,*

$$\text{edim } R = \dim_{R/m} m/m^2 = \mu_R(m).$$

(Recall by NAK $\mu_R(M) = \dim_{R/m} M/mM$.)

By the Principal Ideal Theorem $\dim R \leq \text{edim } R$. We say R is a **regular local ring** if edim $R = \dim R$, that is, m is generated by a system of parameters.

Exercises.

(1) Let (R, m) be a local ring of dimension d and I an ideal. Show $\sqrt{I} = \sqrt{(x_1, ..., x_d)}$ for some $x_1, ..., x_d \in I$.

Proof. We first prove the following claim.

Claim. For all $i = 1, ..., d$ there exists $x_1, ..., x_i \in I$ such that for all $p \in \operatorname{Spec} R$ with $\operatorname{ht} p \leq i - 1$ we have $p \supseteq I$ if and only if $p \supseteq (x_1, ..., x_i)$.

Proof. Let $i = 1$. If I is contained in every minimal prime of R, then choose any $x_1 \in I$. Otherwise, let $p_1, ..., p_s$ be the minimal primes of R not containing I. By prime avoidance, there exists $x_1 \in I \setminus \cup p_i$. So for all minimal primes we see $p \supseteq I$ if and only if $p \supseteq (x_1)$.

Now assume $i > 1$ and proceed by induction. If I is contained in every height $i - 1$ prime containing $(x_1, ..., x_{i-1})$, choose any $x_i \in I$. Otherwise, let $q_1, ..., q_t$ be the height $i-1$ primes over $(x_1, ..., x_{i-1})$ not containing I. (Note that the q_i are minimal primes as otherwise we would have $q_i \supsetneq p \supseteq (x_1, ..., x_{i-1})$ and $\operatorname{ht} p \leq i - 2$, which implies $q_i \supseteq p \supseteq I$. As there are finitely many minimal primes over an ideal, there are finitely many q_i). By prime avoidance, there exists $x_i \in I \setminus \cup_{j=1}^{t} q_j$. Then for all height $\leq i - 1$ primes p we are done by induction and for all height $i - 1$ primes p with $p \supseteq (x_1, ..., x_i)$, we see $x_i \in p$ and $p \supseteq (x_1, ..., x_{i-1})$ says $p \neq q_j$ for any j, that is, $p \supseteq I$.

This shows that for all $p \in \operatorname{Spec} R$ with $\operatorname{ht} p \leq d - 1$ that $p \supseteq I$ if and only if $p \supseteq (x_1, ..., x_d)$. Certainly this statement also holds for m, the only prime of height d. Thus we have

$$\sqrt{I} = \bigcap_{p \supseteq I} p = \bigcap_{p \supseteq (x_1, ..., x_d)} p = \sqrt{(x_1, ..., x_d)}.$$

□

(2) Find a Noetherian ring R with a non-unit $x \in R$ such that $\dim R/(x) \leq \dim R - 2$.

Proof. Consider the ring $R = \mathbb{Z}_{(2)}[x]$. We know $\mathbb{Z}_{(2)}$ has dimension 1 and thus $\dim R = 2$. Recall $(2x - 1)$ is maximal and thus $\dim R/(2x - 1) = 0$ as it is a field. □

(3) Let (R, m) be local, $x_1, ..., x_k \in m$. Then $\dim R/(x_1, ..., x_k) \geq \dim R - k$ with equality if $(x_1, ..., x_k)$ is a regular sequence.

Proof. As $\dim R = \operatorname{ht} m$, it is enough to show $\operatorname{ht} \overline{m} \geq \operatorname{ht} m - k$. Say $\operatorname{ht} \overline{m} = d$. By the converse of the principal ideal theorem, there exists $a_1, ..., a_d \in \overline{m}$ such that $\overline{m}$ is minimal over $(a_1, ..., a_d)$. Then m is minimal over $(x_1, ..., x_k, a_1, ..., a_d)$ which implies $\operatorname{ht} m \leq k + d = k + \operatorname{ht} \overline{m}$ by the principal ideal theorem. Thus $\operatorname{ht} \overline{m} \geq \operatorname{ht} m - k$.

Now suppose $(x_1, ..., x_k)$ is a regular sequence. By induction, it is enough to show $\dim R/x \leq \dim R - 1$ for a non-zerodivisor x. Recall that if x is a non-zerodivisor, then $x \notin \cup_{p \in \operatorname{Ass} R} p$ and so, in particular, $x \notin p$ for any minimal primes p of R. So $\operatorname{ht}(x) \geq 1$ which implies $\operatorname{ht} m/(x) \leq \operatorname{ht} m - 1$. □

Recall that $\dim M = \dim R/\operatorname{Ann} M$ for an $R-$module M.

Corollary 3.6. *Let (R, m) be local, M a finitely generated $R-$module, and $x_1, ..., x_k \in m$. Then $\dim M/(x_1, ..., x_k)M \geq \dim M - k$ with equality if $x_1, ..., x_k$ is an $M-$sequence.*

Proof. Let $d = \dim M$ and $\overline{R} = R/\operatorname{Ann} M$. Then $\dim M/(x_1, ..., x_k)M = \dim R/\operatorname{Ann}_R(M \otimes R/(x_1, ..., x_k))$. Recall if M and N are finitely generated $R-$modules that $\sqrt{\operatorname{Ann}_R M \otimes N} = \sqrt{\operatorname{Ann}_R M + \operatorname{Ann}_R N}$. So

$$\begin{aligned}
\dim M/(x_1, ..., x_k)M &= \dim R/\sqrt{\operatorname{Ann}_R(M \otimes_R R/(x_1, ..., x_k))} \\
&= \dim R/\sqrt{\operatorname{Ann}_R M + \operatorname{Ann}_R R/(x_1, ..., x_k)} \\
&= \dim R/\operatorname{Ann}_R M + \operatorname{Ann}_R R/(x_1, ..., x_k) \\
&= \dim \overline{R}/(\overline{x_1}, ..., \overline{x_k}) \\
&\geq \dim \overline{R} - k = d - k.
\end{aligned}$$

□

Definition. *A Noetherian commutative ring R is called* ***catenary*** *if for all $Q \supseteq P$ primes in R, every saturated chain of primes from P to Q has the same length ($= \operatorname{ht}(Q/P)$).*

We would like that $\operatorname{ht}(Q/P) = \operatorname{ht} Q - \operatorname{ht} P$, but this is not true in general.

Example. Let $R = k[x,y,z]/(x) \cap (y,z)$. Then R is catenary by the exercise. However, if we take $Q = (x,y,z)$ and $P = (y,z)$ we see $\operatorname{ht} Q = 2, \operatorname{ht} P = 0$, yet $\operatorname{ht} Q/P = 1$.

Remark. If R is a local catenary domain (or R is catenary and equidimensional, i.e., $\dim R/p = \dim R$ for all minimal primes p), then for all $p \in \operatorname{Spec} R$, we have $\dim R/p + \operatorname{ht} p = \dim R$.

Proof. If R is local then $\dim R/p = \operatorname{ht} m/p$ and $\dim R = \operatorname{ht} m$. □

This says that the ring R from the example is not equidimensional.

Remarks.

(1) Quotients and localizations of catenary rings are catenary (as a result of the one-to-one correspondence between primes)
(2) R catenary does not imply $R[x]$ is catenary.

Definition. *Say R is* ***universally catenary (u.c.)*** *if every finitely generated $R-$algebra is catenary, that is, if $R[x_1, ..., x_n]$ is catenary for all n where x_i are indeterminants.*

Example. Fields, Cohen Macaulay rings, and completely local rings are u.c.

Notation. Let $p \in \operatorname{Spec} R$. Let $k(p) := Q(R/p) = R_p/pR_p$.

Let S/R be ring extensions, $Q \in \operatorname{Spec} S$ and $Q \cap R = p$. Then $R/p \hookrightarrow S/Q$ is an inclusion of domains. So $k(p) \hookrightarrow k(Q)$. If S is a finitely generated $R-$algebra, then $k(Q)$ is a finitely generated field extension of $k(p)$. If S is a finitely generated $R-$module, then $k(Q)$ is a finite dimensional $k(p)-$vector space. If S is integral over R, then $k(Q)$ is algebraic over $k(p)$.

Theorem 3.7 (**Dimension Theorem**)**.** *Let $R \subseteq S$ be domains where S is a finitely generated $R-$algebra. Let $Q \in \operatorname{Spec} S, p = Q \cap R$. Then*

$$\operatorname{ht} Q + \operatorname{trdeg}_{k(p)} k(Q) \leq \operatorname{ht} p + \operatorname{trdeg}_{Q(R)} Q(S) (*)$$

with equality if R is u.c.

Proof. Say $S = R[u_1, ..., u_n]$ and induct on n. Let $n = 1$ and for simplicity say $S = R[u]$.

Case 1. u is transcendental over $Q(R)$. Thus u is an indeterminant over R. Generalizing the proof that for a Noetherian ring R, $\dim R[x] = \dim R + 1$, we see that either $Q = pR[u]$ or $Q = (p, f(u))R[u]$ for some monic polynomial $f(u)$.

If $Q = pR[u]$, then $\operatorname{ht} Q = \operatorname{ht} p$. Note that $R[u]/Q = R/p[u]$ and so $k(Q) = k(p)(x)$. So $\operatorname{trdeg}_{k(p)} k(Q) = 1$. As $S = R[u]$, we know $\operatorname{trdeg}_{Q(R)} Q(S) = 1$, so equality holds in $(*)$.

If $Q = (p, f(u))R[u]$, then $\operatorname{ht} Q = \operatorname{ht} P + 1$. Note that $R[u]/Q = (R/p[u])/(f(u))$ and so $k(Q) = k(p)(\overline{u})$ where $\overline{f}(\overline{u}) = 0$. As f is monic, this says $\overline{u}$ is algebraic over $k(p)$. Thus $\operatorname{trdeg}_{k(p)} k(Q) = 0$. Again, as $\operatorname{trdeg}_{Q(R)} Q(S) = 1$, we see quality hold in $(*)$.

Case 2. u is algebraic over $Q(R)$. Let x be an indeterminant over R. Map $\phi : R[x] \to R[u]$ by $f(x) \to f(u)$. As u is algebraic over $Q(R)$, we see $q = \ker \phi \neq 0$. Note q is prime as $R[x]/q \cong R[u]$ is a domain. Now $R \to R[x]/q$ defined by $r \mapsto r$ is injective and so $q \cap R = 0$. Note $\operatorname{ht} q = \operatorname{ht} q_W$ where $W = R \setminus \{0\}$. Of course q_W is a nonzero prime in $R_W[x] = Q(R)[x]$. As $Q(R)[x]$ is a PID, we se $q = \operatorname{ht} q_W = \operatorname{ht} q$.

Let $Q' \in \operatorname{Spec} R[x]$ where $Q'/q = Q \in \operatorname{Spec} R[x]/q = S$. By case 1,

$$\operatorname{ht} Q' + \operatorname{trdeg}_{k(p)} k(Q') = \operatorname{ht} p + \underbrace{\operatorname{trdeg}_{Q(R)} Q(R)(x)}_{=1}.$$

Note that $R[x]/Q' = (R[x]/q)/(Q'/q) = S/Q$. Thus $k(Q') = k(Q)$. So we have

$$\operatorname{ht} Q' + \operatorname{trdeg}_{k(p)} k(Q) = \operatorname{ht} p + 1.$$

Now $\underbrace{\operatorname{ht} q}_{=1} + \underbrace{\operatorname{ht} Q'/q}_{\operatorname{ht} Q} \leq \operatorname{ht} Q'$ with equality if R is catenary. So $\operatorname{ht} Q + 1 \leq \operatorname{ht} Q'$ with equality if R is catenary. Thus

$$\operatorname{ht} Q + \operatorname{trdeg}_{k(p)} k(Q) \leq \operatorname{ht} p$$

with equality if R is catenary. As u is algebraic over $Q(R)$, we see $\operatorname{trdeg}_{Q(R)} Q(S) = 0$, thus $(*)$ holds.

Now suppose $A = R[x_1, ..., x_{n-1}]$. Then $Q(R) \subseteq Q(A) \subseteq Q(S)$. So

$$\operatorname{trdeg}_{Q(R)} Q(S) = \operatorname{trdeg}_{Q(A)} Q(S) + \operatorname{trdeg}_{Q(R)} Q(A).$$

Let $Q \in \operatorname{Spec} R, Q' = Q \cap A, p = Q \cap R$. Then we have $R/p \hookrightarrow A/Q' \hookrightarrow S/Q$ and so $k(p) \subseteq k(Q') \subseteq k(Q)$. So again

$$\operatorname{trdeg}_{k(P)} k(Q) = \operatorname{trdeg}_{k(Q')} k(Q) + \operatorname{trdeg}_{k(p)} k(Q').$$

Now

$$\begin{aligned} \operatorname{ht} Q + \operatorname{trdeg}_{k(p)} k(Q) &= \operatorname{ht} Q + \operatorname{trdeg}_{k(P)} k(Q') + \operatorname{trdeg}_{k(Q')} k(Q) \\ &\leq \operatorname{ht} Q' + \operatorname{trdeg}_{Q(A)} Q(S) + \operatorname{trdeg}_{k(P)} k(Q') \text{ by the } n = 1 \text{ step} \\ &\leq \operatorname{ht} p + \operatorname{trdeg}_{Q(R)} Q(A) + \operatorname{trdeg}_{Q(A)} Q(S) \text{ by induction} \\ &= \operatorname{ht} p + \operatorname{trdeg}_{Q(R)} Q(S), \end{aligned}$$

with equality if R is u.c. □

Proposition 3.8 (**Noether Normalization Lemma**). *Suppose A is a domain which is a finitely generated $k-$algebra. Then there exists $u_1, ..., u_n \in A$ algebraically independent over k such that A is integral over $k[u_1, ..., u_n]$.*

This says $\dim A = \dim k[u_1, ..., u_n] = n$. Then $\operatorname{trdeg} Q(A) = n = \dim A$. For all $Q \in \operatorname{Spec} A$, we see $Q \cap k = (0)$. So

$$\operatorname{ht} Q + \operatorname{trdeg}_k k(Q) = \underbrace{\operatorname{ht} Q \cap k}_{=0} + \underbrace{\operatorname{trdeg}_k Q(A)}_{=n}.$$

Thus $\operatorname{ht} Q = \operatorname{trdeg}_k Q(A) - \operatorname{trdeg}_k k(Q)$. If Q is maximal in A, then A/Q is a finitely generated $k-$algebra. Since A/Q is a field, the Strong Nullstellensatz says A/Q is algebraic and so $\operatorname{trdeg}_k k(Q) = 0$. So $\operatorname{ht} Q = \operatorname{trdeg}_k Q(A)$ for all maximal ideal Q of A.

3.1. **Fields are Universally Catenary.** In this section we wish to show fields are universally catenary. To do so, we must first build up a bit of machinery. The material in this section is taken from Kunz.

Lemma 3.9. *The set of all elements of K (the quotient field of R) which are integral over I, an ideal of R, is $\sqrt{I\overline{R}}$.*

Proof. Recall x is integral over I if there exists a monic polynomial $f(t) \in R[t]$ such that $f(x) = 0$ and $f(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ with $a_i \in I$.

If $x \in K$ is integral over I then there exists $f(t)$ as above. Then $x^n = -(a_1 x^{n-1} + \cdots + a_n) \in IR[x] \subset I\overline{R}$. Thus $x \in \sqrt{I\overline{R}}$.

If $x \in \sqrt{I\overline{R}}$, then there exists m such that $x^m \in IR[x_1, ..., x_n]$ for some $x_1, ..., x_n \in \overline{R}$. Since $R[x_1, ..., x_n]$ is a finitely generated $R-$module (as x_i are integral), we have x is integral over I (by the determinant trick). □

Lemma 3.10 (Krull)**.** *Let I be an ideal of R, S a multiplicatively closed subset of R with $I \cap S = \emptyset$. Then the set M of all ideals J of R with $I \subseteq J$ and $J \cap S = \emptyset$ has a maximal element which is a prime ideal.*

Recall. R is normal if it is an integral domain which is integrally closed in its field of fractions.

Lemma 3.11. *Let R be a normal ring with field of fractions K, L/K a field extension and $I \subseteq R$ a prime ideal. If $x \in L$ is integral over I then the minimal polynomial $m(t)$ of x over K has the form $m(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ where $a_i \in I$.*

Proof. Let $\{x = x_1, ..., x_n\}$ be the set of zeros of $m(t)$ in $\overline{K}$ (the algebraic closure). Then $x \in L$ is integral over I implies there exists a monic polynomial $f(t)$ with coefficients in I such that $f(x) = 0$. For each $i = 1, ..., n$ there exists an automorphism $\sigma_i \in \operatorname{Aut}(\overline{K}/K)$ such that $\sigma_i(x) = x_i$. So $f(x_i) = f(\sigma_i(x)) = \sigma_i(f(x)) = 0$. So each x_i is a zero of $f(t)$ and hence is integral over I. The coefficients a_i of $m(t)$ are the elementary symmetric functions on the x_i's and so are integral and therefore in $\sqrt{I\overline{R}}$. But R normal implies $R = \overline{R}$ and I is prime. So $\sqrt{IR} = I$. Thus $a_i \in I$ for $i = 1, ..., n$. □

Proposition 3.12 (**The Going Down Theorem**)**.** *Let S/R be an integral ring extension where R and S are integral domains and R is normal. Let $p_0 \subset p_1$ be a prime ideal chain in* $\operatorname{Spec} R$ *and Q_1 a prime ideal of S lying over p_1. Then there exists $Q_0 \in \operatorname{Spec} S$ with $Q_0 \subset Q_1$ and $Q_0 \cap R = p_0$.*

Proof. First note $N_0 = R \setminus p_1, N_1 = S \setminus Q_1$ and $N = N_0 N_1$ are multiplicatively closed subsets. We will show $p_0 S \cap N = \emptyset$. Then by Krull's Lemma, there exists $Q_0 \in \operatorname{Spec} S$ with $p_0 S \subseteq Q_0$ and $Q_0 \cap N = \emptyset$, that is, $Q_0 \subseteq Q_1$ and $Q_0 \cap R = p_0$.

Now suppose for contradiction that $x \in p_0 S \cap N$. Say $x = \sum p_i s_i$. As S is integral over R, we see $(s_i)^n + r_1 (s_i)^{n-1} + ... + r_n = 0$ for $r_i \in R$. Multiplying by p_i^n, we get $(p_i s_i)^n + r_1 p_i (p_i s_i)^{n-1} + ... + r_n p_i^n = 0$, that is, $p_i s_i$ is integral over p_0 for all i and so x is integral over p_0. By the previous lemma, the minimal polynomial over $m(t)$ for x over k has the form $m(t) = t^n + a_1 t^{n-1} + ... + a_n$ for $a_i \in p_0$. Also $x \in N$ implies $x = rs$ for some $r \in N_0, s \in N_1$. The minimal polynomial for $s = \frac{x}{r}$ over k is $h(t) = t^n + \frac{a_1}{r} t^{n-1} + ... + \frac{a_n}{r^n}$ (minimal as it is $\frac{f(rt)}{r^n}$). By the previous lemma, the coefficients of $h(t)$ lie in R since $s \in S$ is integral over R. The $p_i = \frac{a_i}{r^i} \in R$. Then $a_i = p_i r^i$ where $a_i \in p_0$ and $r^i \notin p_i$. As p_0 is prime, this says $p_i \in p_0$. Thus s is integral over p_0 which implies $s \in \sqrt{p_0 S} \subseteq Q_1$. This contradicts the fact that $s \in N_1 = S \setminus Q_1$. Therefore, $p_0 S \cap N = \emptyset$. □

Theorem 3.13 (**Strong Version of Noether Normalization Theorem**)**.** *Let A be a finitely generated $k-$algebra for a field k and $I \subsetneq A$ an ideal. Then there exists $\delta, d \in \mathbb{N}$ with $\delta \leq d$ and $Y_1, ..., Y_d \in A$ such that*

1. *$Y_1, ..., Y_d$ are algebraically independent over k*
2. *A is finitely generated as a $k[Y_1, ..., Y_d]-$module*
3. *$I \cap k[Y_1, ..., Y_d] = (Y_{\delta+1}, ..., Y_d)$.*

Proof. See Kunz, page 50. □

Proposition 3.14. *If $A \neq 0$ is a finitely generated $k-$algebra with $Y_1, ..., Y_d$ algebraically independent over k and A finitely generated as a $k[Y_1, ..., Y_d]-$module (that is, $k[Y_1, ..., Y_d] \subset A$ is a* ***Noether Normalization****), then* $\dim A = d$*. Moreover, if A is an integral domain, then all maximal chains of primes have length d.*

Proof. Let $B = k[Y_1, ..., Y_d]$. Now A a finitely generated $k-$algebra implies $A = k[x_1, ..., x_n] = B[x_1, ..., x_n]$ for some $x_1, ..., x_n \in A$. As $A = B[x_1, ..., x_n]$ is finitely generated as a $B-$module, A/B is integral. Thus $\dim A = \dim B = d$.

Let A be an integral domain and $Q_0 \subsetneq \cdots \subsetneq Q_m$ a maximal chain of primes in A. Then $Q_0 = (0)$ and Q_m is maximal. Let $p_i = Q_i \cap B$. Then $(0) = p_0 \subsetneq \cdots \subsetneq p_m$ is a prime ideal chain in B with p_m maximal. We want to show that this is also a maximal chain. So suppose we can insert another prime between p_i and p_{i+1}. Choose a Noether Normalization $k[T_1, ..., T_d] \subset B$ (there are d elements as $\dim B = d$) such that $p_i \cap k[T_1, ..., T_d] = (T_{\delta+1}, ..., T_d)$ for

$\delta \leq d$. Then $k[T_1, ..., T_\delta] = k[T_1, ..., T_d]/p_i \subset B/p_i$ is also a Noether Normalization. Since we can insert a prime between (0) and p_{i+1}/p_i, we can between (0) and $p_{i+1}/p_i \cap k[T_1, ..., T_\delta]$. Now $k[T_1, ..., T_\delta] \subset A/Q_i$ is also a Noetherian Normalization. By the Going Down Theorem, there exists a prime between Q_{i+1}/Q_i and thus between Q_i and Q_{i+1}, a contradiction to maximality. Thus $p_0 \subset \cdots \subset p_m$ is maximal.

Now, we show $m = d$ by induction. If $d = 0$, then $\dim A = 0$ and there is nothing to show. If $d > 0$ choose a Noether Normalization $k[T_1, ..., T_d] \subset B$ with $p_1 \cap k[T_1, ..., T_d] = (T_{\delta+1}, ..., T_d)$. As $\operatorname{ht} p_1 = 1$, we see $\operatorname{ht}(T_{\delta+1}, ..., T_d) = 1$ by the going down theorem. Thus $\delta = d - 1$. Now $K[T_1, ..., T_\delta] \subset B/p_1$ is a Noetherian Normalization and so by induction $(0) = p_1/p_1 \subset p_2/p_1 \subset \cdots \subset p_m/p_1$ has length $d - 1$. Thus $m = d$. □

Corollary 3.15. *For a field k, k is universally catenary.*

Proof. Let A be a finitely generated $k-$algebra, $p \subset q$ primes in A. We want to show any saturated chain of primes $p = p_0 \subset \cdots \subset p_m = q$ has the same length. Mod out by p to get an integral domain $A' = A/p$ and a chain $(0) = \overline{p_0} \subset \cdots \subset \overline{p_m} = \overline{q}$. By the proposition, the chain can be extended to a maximal chain of length $\dim A/p$. Then the part of the chain starting at $\overline{q}$ will correspond to a maximal prime ideal chain in $A'/\overline{q} = A/q$, which by the proposition has length $\dim A/q$. Thus $m = \dim A/p - \dim A/q$, that is, the length of the chain does not depend on the intermediate primes we choose. □

3.2. **Dimension of the Associated Graded Ring.** Let R be a commutative ring, I an ideal. Define $S = R[It, t^{-1}] := \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \cdots$ to be the **extended Rees Algebra**.

Let $T = R[t, t^{-1}] \supset S \supset R$. Notice $T = S_{t^{-1}}$. Let $p \in \operatorname{Spec} R$. Then $pT = \cdots \oplus pt^{-2} \oplus pt^{-1} \oplus pt^0 \oplus pt \oplus pt^2 \in \operatorname{Spec} T$ since $T/pT = T/p[t, t^{-1}]$. Thus $pT \cap S \in \operatorname{Spec} S$. Also $pT \cap R = p$. Note also for $q \in \operatorname{Spec} S, t^{-1} \notin q$ that $qT = (q)_{t^{-1}} \in \operatorname{Spec} T$ and $qT \cap S = q$.

Claim. $\operatorname{Min}_S S = \{pT \cap S | p \in \operatorname{Min}_R R\}$.

Proof. Let $q \in \operatorname{Spec} S$. Then $q \cap R \supseteq p$ for some $p \in \operatorname{Min}_R R$. So $qT \supseteq pT$ and so $q = qT \cap S \supseteq pT \cap S$. Thus the minimal primes of S are contained in $\{pT \cap S | p \in \operatorname{Min}_R R\}$. For the other containment, let $p \in \operatorname{Min}_R R$ and suppose $pT \cap S \supseteq q$ where $q \in \operatorname{Spec} S$. Then $pT \supseteq qT$. Not $pT \in \operatorname{Min}_T T$ (as if $pT \supseteq Q \in \operatorname{Spec} T$, then $p = pT \cap R \supseteq Q \cap R$, that is, $p \subseteq Q$ and so $pT \subseteq Q$). So $pT = qT$ and thus $pT \cap S = qT \cap S = q$.

For $p \in \operatorname{Spec} R$, let $\tilde{p} = pT \cap S \in \operatorname{Spec} S$.

Claim. $S/\tilde{p} \cong R/p[(I + p/p)t, t^{-1}] =: \overline{R}[\overline{I}t, t^{-1}]$.

Proof. Notice $\tilde{p} = pT \cap S = \cdots \oplus pt^{-1} \oplus p \oplus (p \cap I)t \oplus (p \cap I)^2t^2 \oplus \cdots$. So map $S \xrightarrow{\phi} \overline{R}[\overline{I}t, t^{-1}] = \oplus \cdots R/pt^{-1} \oplus R/p \oplus (I + p/p)t \oplus \cdots$ by $a_nt^n \mapsto \overline{a_n}t^n$ for $a_n \in I_n$. Then ϕ is a surjective ring map with $\ker \phi = \tilde{p}$.

Remark. $\dim S = \max\{\dim S/\tilde{p} | \tilde{p} \in \operatorname{Min}_S S\} = \max\{\dim R/p[(I + p/p)t, t^{-1}] | p \in \operatorname{Min}_R S\}$.

Theorem 3.16. *Suppose R is Noetherian, I an ideal. Then* $\dim S = \dim R + 1$. *Further, if* $\operatorname{ht} m = \dim R$ *for some maximal ideal $m \supseteq I$ of R, then* $\operatorname{ht}(m, It, t^{-1})S = \dim S$.

Proof. By the remark, we may assume R is a domain. Notice

$$\dim S = \dim R[It, t^{-1}] \geq \dim R[It, t^{-1}]_{t^{-1}} = \dim T \geq_{(*)} \dim R + 1.$$

$(*)$ If $(0) = p_0 \subsetneq \cdots \subsetneq p_d = m$ is a chain of primes in R with $d = \dim R$, then $(0) = p_0T \subsetneq \cdots \subsetneq p_dT = mT$ is a chain of primes in T. Note mT is not maximal as $T/mT = T/m[t, t^{-1}]$ is not a field. So $\dim T > \dim R + 1$.

Let M be a maximal ideal of S. We want to show $\operatorname{ht} M \leq \dim R + 1$. Let $m = M \cap R \in \operatorname{Spec} R$. By the dimension inequality

$$\operatorname{ht} M + \operatorname{trdeg}_{K(m)} K(M) \leq \operatorname{ht} m + \operatorname{trdeg}_{Q(R)} Q(S).$$

Of course $\operatorname{trdeg}_{K(m)} K(M) = \operatorname{trdeg}_{K(m)} S/M = 0$ as S/M is finitely generated over $K(m)$ and both are fields (Strong Nullstellensatz). Also $\operatorname{trdeg}_{Q(R)} Q(S) = \operatorname{trdeg}_{Q(R)} Q(R)(t) = 1$. Thus, the dimension inequality becomes $\operatorname{ht} M \leq \operatorname{ht} m + 1 \leq \dim R + 1$.

Now $(m, It, t^{-1})S = \cdots \oplus Rt^{-1} \oplus Rt^{-1} \oplus m \oplus It \oplus I^2t^2 \oplus \cdots$. So $S/(m, IT, t^{-1})S \cong R/m$, a field. Hence $M = (m, IT, t^{-1})S$ is a maximal ideal of S. Now $\operatorname{ht} M \leq \dim S = \dim R + 1$. Also $\operatorname{ht} m = \dim R = d$, so $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_d = m$ in R. Thus $\tilde{p_0} \subsetneq \tilde{p_1} \subsetneq \cdots \subsetneq \tilde{p_d} = \tilde{m}$ is a chain of primes in S where $\tilde{m} = mT \cap S = \cdots \oplus mt^{-2} \oplus mt^{-1} \oplus m \oplus It \oplus I^2t^2 \oplus \cdots$. Thus $S/\tilde{m} = R/m[t^{-1}]$ is not a field, making $\tilde{m}$ not maximal. Thus $\operatorname{ht} M = \dim R + 1$. □

Lemma 3.17. *Let R be Noetherian, I an ideal of R and $\operatorname{ht} m = \dim R$ for some maximal ideal $m \supseteq I$. Then $\dim S/t^{-1}S = \dim R$.*

Proof. We see t^{-1} is a non-zerodivisor on S and so $\dim S/t^{-1}S \leq \dim S - 1 = \dim R$. Let $M = (m, It, t^{-1})S$. By the proposition, $\operatorname{ht} M = \dim S = \dim R + 1$. Then $\dim S_M = \dim R + 1$, but t^{-1} is a non-zerodivisor on S_M. So $\dim(S/t^{-1}S)_M = \dim S_M/t^{-1}S_M = \dim S_M - 1 = \dim R$. Thus $\dim S/t^{-1}S \geq \dim R$. □

Theorem 3.18. *If $\dim R = \operatorname{ht} m$ for some maximal ideal $m \supseteq I$, then $\dim gr_I(R) = \dim R$.*

Proof. Note that $t^{-1}S = \cdots \oplus Rt^{-1} \oplus I \oplus I^2t \oplus I^3t^2 \oplus \cdots$. Thus $S/t^{-1}S = R/I \oplus I/I^2 \oplus \cdots = gr_I(R)$. By the previous lemma, done. □

4. Depth

Definition. *Let R be a Noetherian ring, M a finitely generated $R-$module. An element $x \in R$ is called $M-$**regular** if $M \neq xM$ and x is a non-zerodivisor on M. A sequence $x_1, ..., x_n \in R$ is an $M-$**sequence** if x_i is $M/(x_1, ..., x_{i-1})M-$regular for $i = 1, ..., n$ (this forces $(x_1, ..., x_i)M \neq M$). We say $x_1, ..., x_n$ is a **maximal** $M-$**sequence** if there does not exist $x_{n+1} \in R$ such that $x_1, ..., x_{n+1}$ is an $M-$sequence.*

Note. If $x_1, ..., x_n$ is an $M-$sequence then $x_1M \subsetneq (x_1, x_2)M \subsetneq \cdots \subsetneq (x_1, ..., x_n)M$.

Proof. Clearly containment is true. So suppose $(x_1, ..., x_i)M = (x_1, ..., x_{i+1})M$. Let $m \in M \setminus (x_1, ..., x_i)M$. Then $x_{i+1}m \in (x_1, ..., x_i)M$, that is, x_{i+1} is a zerodivisor on $M/(x_1, ..., x_i)M$, a contradiction. □

Hence all maximal $M-$sequences are finite by ACC.

Definition. *Let R be Noetherian, M a finitely generated $R-$module, I an ideal of R such that $IM \neq M$. The $\operatorname{depth}_I M$ is defined to be*

$$\sup\{n \mid \text{there exists } x_1, ..., x_n \in I \text{ which is an } M-\text{sequence}\}.$$

In the case $M = R$, $\operatorname{depth}_I R$ is sometimes called the grade I. If (R, m) is local, write $\operatorname{depth} M$ or $\operatorname{depth}_R M$ for $\operatorname{depth}_m M$.

We require $IM \neq M$ so that if $\operatorname{depth}_I M = 0$ we can say I consists of zerodivisors of M. Otherwise $\operatorname{depth}_I M = 0$ implies either $xM = M$ or x is a zerodivisor for each $x \in M$.

Theorem 4.1. *Let M be a finitely generated $R-$module, $I \subset R$ such that $IM \neq M$. Suppose $x_1, ..., x_n \in I$ is a maximal $M-$sequence. Then $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i < n$ and $\operatorname{Ext}^n_R(R/I, M) \neq 0$. This implies every maximal $M-$sequence contained in I has length n. Hence $\operatorname{depth}_I M < \infty$ and $\operatorname{depth}_I M = \inf\{n \mid \operatorname{Ext}^n_R(R/I, M) \neq 0\}$.*

Proof. Induct on n. If $n = 0$, then I consists of zerodivisors on M. Then $I \subset \cup_{p \in \operatorname{Ass} M} p$. By prime avoidance, $I \subset p$ for some $p \in \operatorname{Ass} M$. Say $p = (0 :_R x)$ for $x \in M \setminus \{0\}$. Thus $R/I \twoheadrightarrow R/p \hookrightarrow M$ defined by $\bar{1} \mapsto \bar{1} \mapsto x \neq 0$ is a nonzero homomorphism. Therefore $\operatorname{Ext}^0_R(R/I, M) = \operatorname{Hom}_R(R/I, M) \neq 0$. So suppose $n > 0$. Let $x_1, ..., x_n \in I$ be a maximal $M-$sequence. Then x_1 is $M-$regular and $x_2, ..., x_n$ is a maximal M/x_1M-sequence. By induction

$\operatorname{Ext}^i_R(R/I, M/x_1M) = 0$ for $i < n-1$ and $\operatorname{Ext}^{n-1}_R(R/I, M/x_1M) \neq 0$. Consider the short exact sequence $0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0$. Apply $\operatorname{Hom}_R(R/I, -)$ to get a long exact sequence on Ext :

$$\cdots \to \operatorname{Ext}^{i-1}_R(R/I, M/x_1M) \to \operatorname{Ext}^i_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, M/x_1M) \to \operatorname{Ext}^{i+1}_R(R/I, M) \xrightarrow{x_1} \cdots.$$

Notice that multiplication by x_1 in the above sequence is just the zero map as $x_1 \in I$. Thus, our long exact sequence becomes the following short exact sequence:

$$0 \to \operatorname{Ext}^i_R(R/I, M) \to \operatorname{Ext}^i_R(R/I, M/x_1M) \to \operatorname{Ext}^{i+1}_R(R/I, M) \to 0$$

Now, the middle term is zero for all $i < n-1$ and so $\operatorname{Ext}^1_R(R/I, M) = 0$ for all $i < n$. Similarly, if we consider the $i = n$ case, we see the left term is zero and the middle term is nonzero and so $\operatorname{Ext}^{n+1}_R(R/I, M) \neq 0$. □

Corollary 4.2. *If $x \in I$ is $M-$regular then* $\operatorname{depth}_I M/xM = \operatorname{depth}_I M - 1$.

Proof. If $\overline{x_2}, ..., \overline{x_n}$ is an $M/xM-$sequence, then $x, x_2, ..., x_n$ is an $M-$sequence. □

Corollary 4.3. *For all $p \in \operatorname{Spec} R$ with $p \supseteq I$ we have* $\operatorname{depth}_{I_p} M_p \geq \operatorname{depth}_I M$.

Proof. Recall that $\operatorname{Ext}^i_R(R/I, M)_p \cong \operatorname{Ext}^i_{R_p}(R_p/I_p, M_p)$. So if $\operatorname{depth}_I M = n$, then $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i < n$ which implies $\operatorname{Ext}^i_{R_p}(R_p/I_p, M_p) = 0$ for all $i < n$. So $\operatorname{depth}_{I_p} M_p \geq n$. □

Corollary 4.4. *There is a maximal ideal $m \supseteq I$ such that* $\operatorname{depth}_I M = \operatorname{depth}_{I_m} M_m$.

Proof. Let $n = \operatorname{depth}_I M$. Then $\operatorname{Ext}^n_R(R/I, M) \neq 0$. Recall $M = 0$ if and only if $M_m = 0$ for all maximal ideals m. Thus we must have $\operatorname{Ext}^n_{R_m}(R_m/I_m, M_m) \neq 0$ for some m. Therefore, $\operatorname{depth}_{I_m} M_m = n$. Note that $m \supseteq I$ as otherwise $R_m/I_m = 0$ which implies $\operatorname{Ext}^n_{R_m}(R_m/I_m, M_m) = 0$. □

Remark. x, y $M-$regular does *not* implies that y, x is $M-$regular.

- For example let k be a field and $R = k[x, y, z]/x(y-1)$. Then $z(y-1), y$ is not an $R-$sequence as $z(y-1)$ is a zero divisor. However, $y, z(y-1)$ is an $R-$sequence. As $(0) = x(y-1) = (x) \cap (y-1)$, we see the associated primes are $(x), (y-1)$ and certainly $y \notin (x) \cup (y-1)$. Also $z(y-1)$ is a non zerodivisor on $R/(y)$ as $R/(y) = k[x, y, z]/(x, y) = k[z]$ and $\overline{z(y-1)} = \overline{z}$.

However, if $x_1, ..., x_n \in J(R)$ is an $M-$sequence, then any permutation of $x_1, ..., x_n$ is an $M-$sequence. Thus, in a local ring, regular sequences permute.

Proof. If $x_1, ..., x_n$ is an $M-$sequence, consider the homology on the Koszul Complex $H_i(x_1, ..., x_n; M) = 0$ for all $i \geq 1$. If $x_1, ..., x_n \in J(R)$ then $H_1(x_1, ..., x_n; M) = 0$ and so $x_1, ..., x_n$ is an $M-$sequence. As permuting elements of the Koszul complex gives an isomorphic complex, the homology does not depend on the order. □

Proposition 4.5. *Let (R, m) be local, $M \neq 0$ a finitely generated $R-$module. Then* $\operatorname{depth} M \leq \dim R/p$ *for all* $p \in \operatorname{Ass}_R M$.

Proof. Induct on $\operatorname{depth} M$. If $\operatorname{depth} M = 0$, done. So suppose $\operatorname{depth} M > 0$. Let $x \in m$ be $M-$regular. So $\operatorname{depth} M/xM = \operatorname{depth} M - 1$ and by induction

$$\operatorname{depth} M/xM \leq \operatorname{depth} R/q \text{ for all } q \in \operatorname{Ass}_R M/xM.$$

Let $p \in \operatorname{Ass}_R M$. Then $p = (0 :_R z)$ for some $z \in M \setminus \{0\}$. Let $\Lambda = \{Ry | py = 0, y \in M \setminus \{0\}\}$. Certainly $\Lambda \neq \emptyset$ as it contains z. As M is Noetherian, there exists a maximal element $Ry \in \Lambda$.

Claim. $y \notin xM$.

Proof. Suppose $y = xu$ for some $u \in M$. Then $0 = py = pxu = x(pu)$. As x is a non-zerodivisor $pu = 0$. Then $Ru \in \Lambda$ and $Ry \subseteq Ru$. By maximality of Ry, $Ry = Ru$. So $u = ry$ for $r \in R$. Then $y = xry$ implies $(1 - xr)y = 0$. As $x \in m$, $1 - xr$ is a unit and so $y = 0$, contradiction.

Hence p consists of zerodivisors on M/xM and $p \subseteq q$ for some $q \in \operatorname{Ass}_R M/xM$. Note $x \notin p$ as x is a non-zerodivisor. So $(M/xM)_p = 0$ but $(M/xM)_q \neq 0$ (as the associated primes are contained in the support of the module). Thus $p \subsetneq q$. Therefore

$$\dim R/p \geq \dim R/q + 1 \geq \operatorname{depth} M/xM + 1 = \operatorname{depth} M.$$

□

Exercise. In any Noetherian ring R, $\operatorname{grade} I \leq \operatorname{ht} I$ for all ideals I of R.

Proof. First assume $I = p$, a prime. Then

$$\operatorname{grade} p = \operatorname{depth}_p R \leq \operatorname{depth}_{pR_p} R_p \leq \dim R_p/Q \leq \dim R_p = \operatorname{ht} p$$

for all $Q \in \operatorname{Ass}_{R_p} R_p$. For I an ideal, let $p \supseteq I$ for a prime p. Then $\operatorname{grade} I = \operatorname{depth}_I R \leq \operatorname{depth}_p R \leq \operatorname{ht} p$. Since true for all $p \supseteq I$ and $\operatorname{ht} I = \inf\{\operatorname{ht} p | p \supseteq I\}$, we see $\operatorname{grade} I \leq \operatorname{ht} I$. □

Definition. *A local ring (R, m) is **Cohen-Macaulay** (CM) if* $\operatorname{depth} R = \dim R$. *A Noetherian ring R is CM if R_m is CM for all maximal ideals m.*

Proposition 4.6. *Let (R, m) be a local CM ring. Then* $\dim R/p = \dim R$ *for all* $p \in \operatorname{Ass}_R R$.

Proof. We proved $\dim R = \operatorname{depth} R \leq \dim R/p \leq \dim R$ for all $p \in \operatorname{Ass} R$. □

Note that this says $\operatorname{Min}_R R = \operatorname{Ass}_R R$ for a CM local ring R.

Proposition 4.7. *Let R be a Noetherian CM ring, $x \in R$ $R-$regular. Then R/xR is CM.*

Proof. Let $m \supseteq (x)$ be a maximal ideal. Then x is R_m-regular and $\dim(R/xR)_m = \dim R_m - 1 = \operatorname{depth} R_m - 1 = \operatorname{depth}(R/xR)_m$. □

Proposition 4.8. *Let R be CM, I an ideal. Then* $\operatorname{grade} I = \operatorname{ht} I$.

Proof. By the above exercise, it is enough to show $\operatorname{grade} I \geq \operatorname{ht} I$. Let $m \supseteq I$ be a maximal ideal such that $\operatorname{grade} I_m = \operatorname{grade} I$. If we show $\operatorname{grade} I_m \geq \operatorname{ht} I_m \geq \operatorname{ht} I$, we'll be done. So we may assume (R, m) is a CM local ring. Induct on $h = \operatorname{ht} I$. If $h = 0$, done. So suppose $h > 0$. Then $I \not\subseteq p$ for all $p \in \operatorname{Min}_R R = \operatorname{Ass}_R R$. Hence I contains a non-zerodivisor x. Now $\operatorname{grade} I/(x) = \operatorname{grade} I - 1$ and $\operatorname{ht} I/(x) = \operatorname{ht} I - 1$ (as $\operatorname{ht} p/(x) = \dim R_p/xR_p = \dim R_p - 1 = \operatorname{ht} p - 1$). As $R/(x)$ is CM, induction implies $h - 1 \leq \operatorname{grade} I/(x) - 1$. Thus $h \leq \operatorname{grade} I$. □

Proposition 4.9. *Let R be CM, $p \in \operatorname{Spec} R$. Then R_p is CM.*

Proof. $\operatorname{depth} R_p = \operatorname{grade}_{R_p} pR_p \geq \operatorname{grade} p = \operatorname{ht} p = \dim R_p$. □

Proposition 4.10. *Let (R, m) be CM local. Then for any ideal I of R* $\operatorname{ht} I + \dim R/I = \dim R$.

Proof. Let $h = \operatorname{ht} I = \operatorname{grade} I$. Let $x_1, ..., x_n$ be an $R-$sequence. Let $\overline{R} = R/(x_1, ..., x_n)$, a CM ring of dimension $\dim R - h$, and $\overline{I} = I/(x_1, ..., x_n)$ so that $\operatorname{ht} \overline{I} = \operatorname{ht} I - h = 0$. Then $\overline{I} \subseteq \overline{p}$ where $\overline{p} \in \operatorname{Min}_{\overline{R}} \overline{R}$. As $\overline{R}$ is CM, $\dim \overline{R}/\overline{p} = \dim \overline{R}$. Now $\dim R/I = \dim \overline{R}/\overline{I} \geq \dim \overline{R}/\overline{p} = \dim \overline{R} \geq \dim R/I$. So $\dim R/I = \dim \overline{R} = \dim R - h = \dim R - \operatorname{ht} I$. □

Theorem 4.11. *Let R be CM. Then R is catenary.*

Proof. Suppose not. Then there exist primes $P \subsetneq Q$ such that there are saturated chains of primes from P to Q of unequal length. Among all such $P \subsetneq Q$, choose a pair with smallest possible saturated chain. Let s be the length of the smallest saturated chain. Note $s > 1$ as otherwise $P \subsetneq Q$ is a saturated chain of primes and thus there are no others. As we can localize at Q, we may assume (R, m) is a CM local ring and $Q = m$. Let $P = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_s = m$ and $P = q_0 \subsetneq \cdots \subsetneq q_t = m$ be saturated chains of primes with $s < t$. Then

$$\begin{aligned} \operatorname{ht} p + \dim R/p = \dim R &= \operatorname{ht} p_1 + \dim R/p_1 \\ &= \dim R_{p_1} + \dim R/p_1 \\ &= \operatorname{ht}(PR_{p_1}) + \dim R_{p_1}/PR_{p_1} + \dim R/p_1 \\ &= \operatorname{ht} P + 1 + \dim R/p_1 \text{ as } P \subsetneq p_1 \text{ is saturated} \end{aligned}$$

Thus $\dim R/P = \dim R/p_1 + 1$. Similarly, $\dim R/P = \dim R/q_1$. Thus $\dim R/p_1 = \dim R/q_1$. Now $p_1 \subseteq \cdots \subsetneq p_s = m$ is a saturated chain of primes in R of length $s-1$. By choice of s, all saturated chains from p_1 have length $s-1$. Thus $\dim R/p_1 = s-1$. Hence $\dim R/q_1 = \dim R/p_1 = s-1$. But this contradicts that there exists a saturated chain of length $t-1 > s-1$. □

Exercises.

(1) R CM implies $R[x]$ is CM.

Proof. Let $S = R[x]$. Let $n \subseteq S$ be a maximal ideal. Let $n \cap R = p$. Then $S_n = R_p[x]$. So we may assume R is local. Let $k = R/m$. Then $S/mS = k[x]$ is a PID., So n/mS is a principal ideal, say $(\phi(x))$ where $\phi(x)$ is a monic irreducible. Let $f(x) \in S$ be monic such that $\overline{f(x)} = \phi(x)$. So $n = (m, f(x))$. Let $(a_1, ..., a_d) \in m$ be a maximal $R-$sequence. So $d = \dim R$.

Claim. $(a_1, ..., a_d, f)$ is an $S-$sequence.

Proof. As S is a flat $R-$module, $0 \to R \xrightarrow{a_1} R \to R/a_1 \to 0$ exact implies $0 \to S \to S \to S/a_1S \cong (R/a_1)[x] \to 0$ is exact. So $(a_1, ..., a_d)$ is an $s-$sequence. Note $\overline{f(x)}$ is monic in $R[x]/(a_1, ..., a_d) \cong R/(a_1, ..., a_d)[x]$ and so it is not a zerodivisor. Thus $\operatorname{depth} S_n \geq d+1 = \dim S \geq S_n$. □

(2) (Depth Lemma) Let R be Noetherian, I an ideal, $0 \to L \to M \to N \to 0$ a short exact sequence of finitely generated $R-$modules.

(a) If $\operatorname{depth}_I M > \operatorname{depth}_I N$ then $\operatorname{depth}_I L = \operatorname{depth}_I N + 1$.

Proof. Recall $\operatorname{depth}_I M = \inf\{n | \operatorname{Ext}^n_R(R/I, M) \neq 0\}$. Say $n = \operatorname{depth}_I N$. Then $\operatorname{Ext}^n_R(R/I, N) \neq 0, \operatorname{Ext}^k_R(R/I, N) = 0$ for all $k < n$, and since $\operatorname{depth}_I M > n$, we have $\operatorname{Ext}^k_R(R/I, M) = 0$ for all $k \leq n$. Consider the long exact sequence on $\operatorname{Ext}_R(R/I, -)$:

$$\cdots \to \underbrace{\operatorname{Ext}^{k-1}_R(R/I, N)}_{=0 \ for \ k \leq n} \to \operatorname{Ext}^k_R(R/I, L) \to \underbrace{\operatorname{Ext}^k_R(R/I, M)}_{=0 \ for \ k \leq n} \to \cdots$$

Thus $\operatorname{Ext}^k_R(R/I, L) = 0$ for $k \leq n$. Of course, we also see

$$\cdots \to \underbrace{\operatorname{Ext}^n_R(R/I, M)}_{=0} \to \underbrace{\operatorname{Ext}^n_R(R/I, N)}_{\neq 0} \to \operatorname{Ext}^{n+1}_R(R/I, L) \to \cdots$$

and so if the last term was zero, so would the middle term be. Thus $\operatorname{Ext}^{n+1}_R(R/I, L) \neq 0$ which implies $\operatorname{depth}_I L = n + 1 = \operatorname{depth}_I N + 1$. □

(b) Given an example where $\operatorname{depth}_I L > \operatorname{depth}_I M$.

Proof. Let $R = \left(k[x, y]/(x^2, xy)\right)_{\overline{x, y}}$. Then $\operatorname{depth} R = 0$ as $\overline{x}$ kills the maximal ideal. Let $p = (x)R$. Then $p \in \operatorname{Min}_R R \subseteq \operatorname{Ass}_R R$. Consider R/p. We have proved $p \in \operatorname{Ass} R$ implies $R/p \hookrightarrow R$. Thus we have an exact sequence $0 \to R/p \xrightarrow{\phi} R \to \operatorname{coker} \phi \to 0$. Yet $\operatorname{depth} R/p = 1$ as $R/p \cong k[y]_{(y)}$. [More to the point, y is a non-zerodivisor on R/p but not on R.] □

Proposition 4.12. *Let $F.$ be a minimal free resolution of M, $k = R/m$. Then* $\operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^R(k, M) =: \beta_i(M)$, *the i^{th} Betti number.*

Proof. Take $F. \to M \to 0$ and apply $k \otimes_R -$. Then $\operatorname{Tor}_i^R(k, M) = H_i(k \otimes_R F.) = k^{n_i}$ if $\operatorname{rank} F_i = n_i$ (all the maps become the 0 map when you tensor because all of the maps had entries in m). So $\dim \operatorname{Tor}_i^R(k, M) = n_i$ if $F_i \cong R^{n_i}$. □

Corollary 4.13. $\operatorname{pd}_R M = \sup\{i | \operatorname{Tor}_i^R(k, M) \neq 0\}$ = *the length of any minimal free resolution of M.*

Proposition 4.14. *Let (R, m) be local, M finitely generated. Suppose $x \in m$ such that x is $R-$regular and $M-$regular. Let $F.$ be a minimal free resolution of M. Then $F. \otimes R/(x)$ is a minimal free $R/(x)-$resolution of M/xM. Hence* $\operatorname{pd}_R M = \operatorname{pd}_{R/(x)} M/xM$.

Proof. We know it is minimal as the entries of $\overline{\phi}$ are still in m/xm and clearly the modules are still free. So we need only check $H_i(F. \otimes R/(x))$ is 0 for $i > 0$ and M/xM for $i = 0$. Recall $H_i(F \cdot \otimes R/(x)) \cong \operatorname{Tor}_i^R(M, R/(x))$. Note $(G.)\ 0 \to R \xrightarrow{x} R \to 0$ is a free resolution of $R/(x)$. Apply $- \otimes_k M$ to get $0 \to M \xrightarrow{x} M \to 0$. Thus $H_i(G. \otimes_R M)$ is 0 for $i > 1$, is $(0 :_M x) = 0$ for $i = 1$ and $\operatorname{coker} x = M/xM$ for $i = 0$. □

Auslander-Buchsbaum Formula: Let (R, m) be a local ring and M a finitely generated $R-$module. If $\operatorname{pd}_R M < \infty$ then $\operatorname{pd}_R M + \operatorname{depth} M = \operatorname{depth} R$.

Proof. Induct on $\operatorname{depth} R$. If $\operatorname{depth} R = 0$, then $m \in \operatorname{Ass}_R R$. Therefore $\operatorname{Hom}_R(R/m, R) \neq 0$ and hence $\operatorname{Hom}_R(R/m, F) \neq 0$ for any finitely generated nonzero free $R-$module. Let $0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0$ be a minimal free resolution of M. If $n > 0$, we have $0 \to F_n \xrightarrow{\phi} F_{n-1}$ where $\operatorname{im} \phi \subseteq mF_{n-1}$. Apply $\operatorname{Hom}_R(R/m, -)$ to get $0 \to \operatorname{Hom}_R(R/m, F_n) \xrightarrow{0 \text{ map}} \operatorname{Hom}_R(R/m, F_{n-1})$. As $\operatorname{Hom}_R(R/m, -)$ is exact, this implies $\operatorname{Hom}_R(R/m, F_n) = 0$, a contradiction. Thus $n = 0$ and M is projective. Of course, finitely generated projectives over a local ring are free. Thus M is free and by the ext characterization of depth, $\operatorname{depth}_R M = \operatorname{depth} R$. Now, assume $\operatorname{depth} R > 0$ and proceed by induction.

> *Case 1.* $\operatorname{depth} M > 0$. Let $\operatorname{Ass}_R R = \{Q_1, ..., Q_\ell\}$ and $\operatorname{Ass}_R M = \{P_1, ..., P_k\}$. Now $m \notin \operatorname{Ass}_R M$ or $\operatorname{Ass}_R R$ (as otherwise the depth would be 0). So $m \nsubseteq Q_1 \cup \cdots \cup Q_\ell \cup P_1 \cup \cdots \cup P_k$ by prime avoidance. Let $x \in m \setminus (\cup Q_i \bigcup \cup P_j)$. Then x is a non-zerodivisor on M and R. By the proposition, $\operatorname{pd}_{R/(x)} M/(x)M = \operatorname{pd}_R M, \operatorname{depth} R/(x) = \operatorname{depth} R - 1$, and $\operatorname{depth} M/(x)M = \operatorname{depth} M - 1$. Done by induction.
>
> *Case 2.* $\operatorname{depth} M = 0$. Let $0 \to F_n \to \cdots \xrightarrow{\phi_1} F_0 \to 0$ be a minimal free resolution of M. So $n = \operatorname{pd}_R M$. Let $K = \operatorname{im} \phi_1 = \ker(F_0 \to M)$. Then $0 \to F_n \to \cdots \to F_1 \to 0$ is a minimal free resolution of K, making $\operatorname{pd}_R K = n - 1$. Also $0 \to K \to F_0 \to M \to 0$ and since $\operatorname{depth} F_0 = \operatorname{depth} R > \operatorname{depth} M = 0$, the Depth Lemma implies $\operatorname{depth} K = \operatorname{depth} M + 1 = 1$. By case 1, $\operatorname{pd} K + \operatorname{depth} K = \operatorname{depth} R$. Of course $\operatorname{depth} R = \operatorname{pd} K + \operatorname{depth} K = \operatorname{pd} M + \operatorname{depth} M$. □

Exercise. (Stronger version of the Prime Avoidance Lemma) Let R be a commutative ring, I an ideal, $x \in R$. Suppose $p_1, ..., p_n$ are primes and $(x, I) \nsubseteq P_i$ for all i. Then there exists $x + y \notin p_i$ for all i.

Proof. Assume $p_k \nsubseteq p_j$ for any $k, j \leq n$ as otherwise, we can just throw the smaller one out. If $x \notin p_i$ for all i, take $y = 0$. Suppose $x \in p_1, ..., p_r$ but $x \notin p_{r+1}, ..., p_n$. Then $I \notin p_1, ..., p_r$ and by the prime avoidance lemma, $I \notin p_1 \cup \cdots \cup p_r$. Choose $i \in I \setminus (p_1 \cup \cdots \cup p_r)$. If $r = n$, take $y = i$. So suppose $r < n$.

> *Claim.* $P_{r+1} \cap \cdots \cap P_n \nsubseteq P_1 \cup \cdots \cup P_r$.
>
> *Proof.* Suppose not. Then $P_{r+1} \cap \cdots \cap P_n \subset P_j$ for some $j \leq r$. Then $p_k \subseteq p_j$ for some $k \geq r + 1$ as p_j is prime.

Choose $z \in p_{r+1} \cap \cdots \cap p_n \setminus p_1 \cup \cdots \cup p_r$. Let $y = iz$. Then $x + y \notin P_i$ for all i. □

Exercise. Let R be a Noetherian local ring, $I = (x_1, ..., x_n)$ an ideal of R such that $\operatorname{grade} I = n$. Then $x_1, ..., x_n$ is a regular sequence.

Proof. For $n = 1$, we have $I = (x)$ and $\operatorname{grade} I = 1$. So there exists $y \in I$ a non-zerodivisor. But $y = xz$ for some $z \in R$ and thus x is a non-zerodivisor. Therefore, x is $R-$regular. So suppose $n > 1$. Since $\operatorname{grade} I > 0$, $I \not\subset p$ for any $p \in \operatorname{Ass}_R R$. Let $J = (x_2, ..., x_n)$. Then $(x_1, J) \not\subseteq p$ for all $p \in \operatorname{Ass}_R R$. By the previous exercise, there exists $y \in J$ such that $x_1 + y \notin p$ for all $p \in \operatorname{Ass}_R R$. Say $y = r_2x_2 + ...r_nx_n$ for $r_i \in R$. Let $x' = x_1 + y$. Then x' is a non-zerodivisor on R. Note $I = (x', x_2, ..., x_n)$. So $\operatorname{grade} I/(x') = n - 1$ and $I/(x') = (\overline{x_2}, ..., \overline{x_n})$. By induction, $\overline{x_2}, ..., \overline{x_n}$ is $R/(x')-$regular. Thus $x', x_2, ..., x_n$ and therefore $x_2, ..., x_n, x'$ are $R-$regular as R is local. Thus $\overline{x'} = \overline{x_1}$ is a non-zerodivisor on $R/(x_2, ..., x_n)$. So $x_2, ..., x_n, x_1$ and therefore $x_1, x_2, ..., x_n$ are $R-$regular. □

Thus in a Cohen-Macaulay local ring, if $x_1, ..., x_d$ is a system of parameters, then $\operatorname{grade}(x_1, ..., x_d) = \operatorname{ht}(x_1, ..., x_d) = d$ and so $x_1, ..., x_d$ is $R-$regular. So any system of parameters is an $R-$regular sequence.

Exercise. Given an example of an ideal $I = (x, y)$ such that $\operatorname{grade} I = 1$ but x, y are zerodivisors on (R, m).

Proof. Consider $R = (k[x, y]/(xy))_{(x,y)}$ and $I = (x, y)$. Then $x + y$ is a non-zerodivisor and so $\operatorname{grade} I = 1$. □

Definition. *Let R be local, I an ideal. If* $\operatorname{grade} I = \mu_R(I)$, *then I is a* ***complete intersection (c.i.)***.

By the exercise, a complete intersection ideal is just an ideal generated by a regular sequence.