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Counting Lower Hessenberg Matrices

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Patterns of Zeroes

Linear Algebra: triangular and diagonal matrices

Combinatorics: counting matrices that follow a specific zero pattern

Given row and column sums

Nonnegative integer entries

Patterns of Zeroes

Example Pattern

$$\begin{pmatrix} 0 & X & 0 & X \\ X & 0 & X & X \\ X & 0 & X & 0 \\ 0 & X & X & X \end{pmatrix}$$

If $c = (3, 2, 5, 1)$, there is a unique solution.

$$\begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Patterns of Zeroes

Example Pattern

$$\begin{pmatrix} 0 & X & 0 & X \\ X & 0 & X & X \\ X & 0 & X & 0 \\ 0 & X & X & X \end{pmatrix}$$

If $c = (3, 1, 5, 1)$, there are no solutions.

$$\begin{aligned} x_{1,2} \leq \quad & c_2 & = 1 \\ x_{1,4} \leq \quad & c_4 & = 1 \\ c_1 = x_{1,2} \quad + \quad & x_{1,4} & \leq 2 \end{aligned}$$

Lower Hessenberg Form

Definition. A n-by-n matrix $A = [a_{i,j}]$ is called **"lower Hessenberg"** if $a_{i,j} = 0$ for all $j > i + 1$, i.e. every entry above the "super-diagonal" is a 0.

Example.

$$\begin{pmatrix} 0 & 5 & 0 & 0 & 0 \\ 3 & 8 & 6 & 0 & 0 \\ 0 & 1 & 10 & 2 & 0 \\ 6 & 9 & 3 & 0 & 11 \\ 4 & 0 & 3 & 6 & 4 \end{pmatrix}$$

For every $c = (c_1, \ldots, c_n)$ where c_i is the i^{th} row and column sum, there is at least one solution.

The Original Problem

The number of nonnegative integer lower Hessenberg matrices with given row-column sums $c = (t_1, \ldots, t_n)$ where $t_i = \binom{i+1}{2}$, is equal to $\chi_1 \chi_2 \ldots \chi_n$ where χ_i is the i^{th} Catalan number, i.e. $\chi_i = \frac{1}{i+1}\binom{2i}{i}$.

Open Problem: Proof by Bijection.

A Few Definitions

Definition. $H(c)$ = the set of lower Hessenberg matrices with row and column sums $c = (c_1, c_2, \ldots, c_n)$ and nonnegative integer entries.

Let $h(c) = |H(c)|$.

Example. When $c = (1, 2, 2)$, $h(c) = 7$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \quad = H(c)$$

A Few Definitions

Definition. Let M be an $n-by-n$ matrix. Let I and J be two subsets of $\{1,2,\ldots,n\}$. Then $M(I;J)$ is the submatrix of M formed by deleting all the rows in I and all the columns in J. If $|I|=|J|=1$, then $I=\{i\}$ and $J=\{j\}$, and we can write $M(I;J)=M(i;j)$. Note that $M(i;j)$ will be $(n-1)-by-(n-1)$.

Example.

$$M=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 1 & 2 & 6 \end{pmatrix} \qquad M(1;3)=\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 6 \end{pmatrix}$$

A Few Definitions

Definition. Let $M \in H(c)$ for some c. Let $e = [1, 1, \ldots, 1]^T$. Then we define $S_{i,j}(M) = e^T M(i; j)e$, which is the sum of the entries of M that are not in the i^{th} row or j^{th} column. We call $S_{i,j}(M)$ the i, j-subsum of M.

Example. $S_{4.4}(A_1) = 10$ and $S_{4.4}(A_2) = 11$; $\quad A_1, A_2 \in H(1, 4, 8, 9)$

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 1 & 2 & 6 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 4 & 2 \\ 0 & 0 & 2 & 7 \end{pmatrix}$$

Partitioning $H(c)$

Definition. Let $c = (c_1, \ldots, c_n)$. Then $P_{i,j}(c; s) = \{M \in H(c); S_{i,j}(M) = s\}$. As s ranges over all possibilities, $P_{i,j}(c; s)$ is what defines our partition of $H(c)$. Call this the $S_{i,j}$-partition of $H(c)$.

We are particularly interested in the numbers $p_{i,j}(c; s) = |P_{i,j}(c; s)|$.

Example. $c = (1, 4, 8, 9)$; $p_{4,4}(c; 13) = 13$

$$\begin{pmatrix} 1&0&0&0\\0&4&0&0\\0&0&8&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&3&1&0\\0&1&7&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&2&2&0\\0&2&6&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&3&0\\0&3&5&0\\0&0&0&9 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&0\\0&0&4&0\\0&4&4&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&3&0&0\\0&0&8&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&2&1&0\\0&1&7&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&1&2&0\\0&2&6&0\\0&0&0&9 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0&0\\1&0&3&0\\0&3&5&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&3&1&0\\1&0&7&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&2&2&0\\1&1&6&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&1&3&0\\1&2&5&0\\0&0&0&9 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0&0\\0&0&4&0\\1&3&4&0\\0&0&0&9 \end{pmatrix} = P_{4,4}(c; 13)$$

Partitioning $H(c)$

Theorem. Let $A = (a_{i,j})$. Then, $S_{i,j}(A)$ depends only on $a_{i,j}$, i.e. every matrix in the set $P_{i,j}(c;s)$ has the same i,j entry.

Example. $c = (1,4,8,9)$; $p_{4,4}(c;13) = 13$

$$\begin{pmatrix}1&0&0&0\\0&4&0&0\\0&0&8&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&3&1&0\\0&1&7&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&2&2&0\\0&2&6&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}1&0&0&0\\0&1&3&0\\0&3&5&0\\0&0&0&9\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\0&0&4&0\\0&4&4&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}0&1&0&0\\1&3&0&0\\0&0&8&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}0&1&0&0\\1&2&1&0\\0&1&7&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}0&1&0&0\\1&1&2&0\\0&2&6&0\\0&0&0&9\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\1&0&3&0\\0&3&5&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}0&1&0&0\\0&3&1&0\\1&0&7&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}0&1&0&0\\0&2&2&0\\1&1&6&0\\0&0&0&9\end{pmatrix} \begin{pmatrix}0&1&0&0\\0&1&3&0\\1&2&5&0\\0&0&0&9\end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\0&0&4&0\\1&3&4&0\\0&0&0&9\end{pmatrix} = P_{4,4}(c;13)$$

Here, $a_{4,4} = 9$ for all the matrices in $P_{4,4}(c;13)$.

The Distribution

How does $P_{i,j}(c; s)$ partition $H(c)$, specifically $P_{n,n}(c; s)$?

The Distribution

How does $P_{i,j}(c;s)$ partition $H(c)$, specifically $P_{n,n}(c;s)$?

Surprisingly, it does so in a very ordered fashion!

The Distribution

The distribution of $H(c)$, where $c = (1, 4, 8, 9)$:

s	$p_{4,4}(c; s)$
5	13
6	27
7	37
8	43
9	45
10	43
11	37
12	27
13	13

The Distribution

Theorem. Let $m_1 = min\,\{i,j\}$ and $m_2 = max\,\{i,j\}$. In the $S_{i,j}$-partition of any $H(c)$, the range of s that gives $p_{i,j}(c;s) > 0$ is bounded above by $\sum_{\alpha \neq m_2} c_\alpha$ and below by $\sum_{\alpha \neq i,j} c_\alpha$ if $m_1 < n$, or $\sum_{\alpha=1}^{n-2} c_\alpha$ if $m_1 = n$.

s	$p_{4,4}(c;s)$
5	13
6	27
7	37
8	43

9	45
10	43
11	37
12	27
13	13

$$c = (1,4,8,9)$$

The Distribution

Theorem As a function of c_n, $h(c)$ is constant as long as $c_n \geq c_{n-1}$.

$$H(1,1,1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$H(1,1,2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

The Distribution

Theorem. $h(c_1, \ldots, c_{n-1}) = p_{n,n}(c; \sum_{i=1}^{n-1} c_i)$, i.e. the last entry in the $S_{n,n}$-partition distribution is equal to the total number of $(n-1) - by - (n-1)$ matrices with $c = (c_1, \ldots, c_{n-1})$.

Example. $c = (1, 4, 8, 9)$; $p_{4,4}(c; 13) = 13$

$$\begin{pmatrix} 1&0&0&0\\0&4&0&0\\0&0&8&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&3&1&0\\0&1&7&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&2&2&0\\0&2&6&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&1&3&0\\0&3&5&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 1&0&0&0\\0&0&4&0\\0&4&4&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&3&0&0\\0&0&8&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&2&1&0\\0&1&7&0\\0&0&0&9 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0&0\\1&1&2&0\\0&2&6&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\1&0&3&0\\0&3&5&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&3&1&0\\1&0&7&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&2&2&0\\1&1&6&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&1&3&0\\1&2&5&0\\0&0&0&9 \end{pmatrix} \begin{pmatrix} 0&1&0&0\\0&0&4&0\\1&3&4&0\\0&0&0&9 \end{pmatrix} = P_{4,4}(c; 13)$$

$c' = (1, 4, 8)$; $h(c') = 13$

$$\begin{pmatrix} 1&0&0\\0&4&0\\0&0&8 \end{pmatrix} \begin{pmatrix} 1&0&0\\0&3&1\\0&1&7 \end{pmatrix} \begin{pmatrix} 1&0&0\\0&2&2\\0&2&6 \end{pmatrix} \begin{pmatrix} 1&0&0\\0&1&3\\0&3&5 \end{pmatrix} \begin{pmatrix} 1&0&0\\0&0&4\\0&4&4 \end{pmatrix} \begin{pmatrix} 0&1&0\\1&3&0\\0&0&8 \end{pmatrix} \begin{pmatrix} 0&1&0\\1&2&1\\0&1&7 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0\\1&1&2\\0&2&6 \end{pmatrix} \begin{pmatrix} 0&1&0\\1&0&3\\0&3&5 \end{pmatrix} \begin{pmatrix} 0&1&0\\0&3&1\\1&0&7 \end{pmatrix} \begin{pmatrix} 0&1&0\\0&2&2\\1&1&6 \end{pmatrix} \begin{pmatrix} 0&1&0\\0&1&3\\1&2&5 \end{pmatrix} \begin{pmatrix} 0&1&0\\0&0&4\\1&3&4 \end{pmatrix} = H(c')$$

The Distribution

Theorem. Let $c = (c_1, c_2, \ldots, c_n)$ and $m = c_1 + c_2 + \ldots + c_{n-2} + \frac{c_{n-1}}{2}$. Then for any radius r, $p_{n,n}(c; m - r) = p_{n,n}(c; m + r)$.

Example. The distribution of $H(c)$, where $c = (1, 4, 8, 9)$:

s	$p_{4,4}(c; s)$
5	13
6	27
7	37
8	43
9	45
10	43
11	37
12	27
13	13

The Distribution

The following is a sketch of our proof of the symmetry of the distribution.

$c = (1, 4, 8, 9)$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 8 \\ 1 & 1 & 6 & 1 \end{pmatrix} \in P_{4,4}(c; 5) \qquad \longleftrightarrow \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 6 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix} \in P_{4,4}(c; 13)$$

The Distribution

Conjecture. The distribution is unimodal (increasing towards the middle entry where $s = c_1 + \ldots + c_{n-2} + \frac{c_{n-1}}{2}$, and then decreasing). It is also log concave in this respect, i.e. $[p_{n,n}(c;s)]^2 \geq p_{n,n}(c;s-1) * p_{n,n}(c;s+1)$. We also believe that the stronger statement $2 * [p_{n,n}(c;s)] \geq p_{n,n}(c;s-1) + p_{n,n}(c;s+1)$ holds.

Example. The distribution of $H(c)$, where $c = (1,4,8,9)$:

s	$p_{4,4}(c;s)$
5	13
6	27
7	37
8	43

9	45
10	43
11	37
12	27
13	13

Determinate Forms

Definition. By a Hessenberg-determinate form, we mean a subpattern S of the Hessenberg pattern such that for any c, when values of the entries in the subpattern are consistent with a matrix in $H(c)$, they are consistent with only one.

Definition. By a PD-pattern, we mean a pattern S of zeroes, D's, and P's such that S is a subpattern of the Hessenberg pattern.

Determinate Forms

Example.

$$\begin{pmatrix} D & D & 0 & 0 \\ P & D & D & 0 \\ P & P & D & D \\ P & P & P & D \end{pmatrix}$$

$$\begin{aligned}
a_{1,1} &= c_1 - a_{2,1} - a_{3,1} - a_{4,1} \\
a_{1,2} &= a_{2,1} + a_{3,1} + a_{4,1} \\
a_{2,2} &= c_2 - a_{2,1} - a_{3,1} - a_{3,2} - a_{4,1} - a_{4,2} \\
a_{2,3} &= a_{3,1} + a_{3,2} + a_{4,1} + a_{4,2} \\
a_{3,3} &= c_3 - a_{3,1} - a_{3,2} - a_{4,1} - a_{4,2} - a_{4,3} \\
a_{3,4} &= a_{4,1} + a_{4,2} + a_{4,3} \\
a_{4,4} &= c_4 - a_{4,1} - a_{4,2} - a_{4,3}
\end{aligned}$$

Determinate Forms

Suppose $c = (1, 2, 4, 4)$ and we had the previous PD-pattern with the following parameters:

Example.

$$\begin{pmatrix} D & D & 0 & 0 \\ 0 & D & D & 0 \\ 1 & 0 & D & D \\ 0 & 1 & 2 & D \end{pmatrix}$$

Then the only solution is the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Determinate Forms

Example. Let $c = (1, 2, 4, 4)$.

$$\begin{pmatrix} D & P & 0 & 0 \\ P & D & P & 0 \\ D & P & D & D \\ P & P & D & D \end{pmatrix} \qquad \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 2 & 0 \\ D & 1 & D & D \\ 0 & 1 & D & D \end{pmatrix}$$

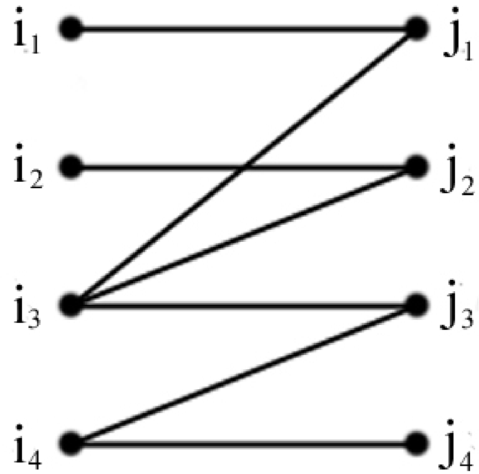
Then there are three solutions, or completions, to this matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Determinate Forms

Bipartite Graph Method

$$\begin{pmatrix} D & P & 0 & 0 \\ P & D & P & 0 \\ D & D & D & P \\ P & P & D & D \end{pmatrix}$$



Theorem. Let S be an $n-by-n$ PD-pattern. Let B be a bipartite graph with nodes $\{i_1, i_2, \ldots, i_n\}$ and $\{j_1, j_2, \ldots, j_n\}$ such that there is an edge between i_k and j_m if there is a D in the $s_{k,m}$ position of S. Then, S is a Hessenberg-determinate form if and only if B is acyclic, i.e. it contains no cycles.

Applications of Determinate Forms

$$\begin{pmatrix} x_1 & y_2 & 0 & 0 \\ x_2 & x_4 & y_4 & 0 \\ x_3 & x_5 & x_6 & y_6 \\ y_1 & y_3 & y_5 & y_7 \end{pmatrix}$$

Consider the system of linear inequalities:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq c_1 \\ x_4 + x_5 - x_1 &\leq c_2 - c_1 \\ x_6 - x_2 - x_4 &\leq c_3 - c_2 \\ x_i &\geq 0 \qquad i = 1, \ldots, 6 \end{aligned}$$

Theorem. For $c_4 \geq c_3$, $|\mathcal{S}| = h(c_1, c_2, c_3, c_4)$.

Applications of Determinate Forms

$$\begin{pmatrix} x_1 & y_2 & 0 & 0 \\ x_2 & x_4 & y_4 & 0 \\ x_3 & x_5 & x_6 & y_6 \\ y_1 & y_3 & y_5 & y_7 \end{pmatrix}$$

For $n = 4$, $h(c_1, c_2, c_3, c_4) =$

$= \sum_{x_1=0}^{c_1} \sum_{x_2=0}^{c_1-x_1} \sum_{x_3=0}^{c_1-x_1-x_2} \sum_{x_4=0}^{c_2-c_1+x_1} \sum_{x_5=0}^{c_2-c_1+x_1-x_4} \sum_{x_6=0}^{c_3-c_2+x_2+x_4} 1$

$= -\frac{1}{360}(6+11c_1+6{c_1}^2+{c_1}^3)(6{c_1}^2-3c_1(7+5c_2)+10(2+3c_2+{c_2}^2))(2c_2-3(1+c_3))$

Applications of Determinate Forms

For $n = 2$, $h(c_1, c_2) = c_1 + 1$.

For $n = 3$, $h(c_1, c_2, c_3) = \sum_{x_1=0}^{c_1} \sum_{x_2=0}^{c_1-x_1} \sum_{x_3=0}^{c_2-c_1+x_1} 1$
$= -\frac{1}{6}\left[2{c_1}^3 - 3{c_1}^2 c_2 + 3{c_1}^2 - 9c_1c_2 - 5c_1 - 6c_2 - 6\right]$

For $n = 5$, the linear inequalities give a formula of $\binom{5}{2} = 10$ nested summands which can be simplified to $h(c_1, c_2, c_3, c_4, c_5) = \frac{1}{362880}((24 + 50c_1 + 35{c_1}^2 + 10{c_1}^3 + {c_1}^4)(14{c_1}^5 - 7{c_1}^4(11 + 9c_2) - 42(6 + 11c_2 + 6{c_2}^2 + {c_2}^3)(6{c_2}^2 - 3c_2(7 + 5c_3) + 10(2 + 3c_3 + {c_3}^2)) + {c_1}^3(252{c_2}^2 - 18c_2(13 + 20c_3) + 16(37 + 45c_3 + 15{c_3}^2)) - {c_1}^2(3127 + 588{c_2}^3 + 4680c_3 + 1560{c_3}^2 - 252{c_2}^2(2 + 5c_3) + 3c_2(-437 + 60c_3 + 280{c_3}^2)) + 6c_1(1105 + 105{c_2}^4 + 1668c_3 + 556{c_3}^2 - 28{c_2}^3(-2 + 9c_3) + 21{c_2}^2(-31 - 22c_3 + 8{c_3}^2) + c_2(125 + 1098c_3 + 644{c_3}^2)))(2c_3 - 3(1 + c_4)))$.

Equal Row and Column Sums

Let $h^n(x) = h(c_1, \ldots, c_n)$ where $c_i = x$ for all c_i.

By substituting x for each of the c_i in the previous equations, and then simplifying, we arrive at the following formulas:

$$
\begin{aligned}
h^2(x) &= \binom{x+1}{x} \\
h^3(x) &= \binom{x+3}{x} \\
h^4(x) &= \frac{1}{3}(x+3)\binom{x+5}{x} \\
h^5(x) &= \frac{1}{3^2}(x+3)^2\binom{x+8}{x} \\
h^6(x) &= \frac{1}{2 \cdot 3^2 \cdot 13}(x+3)^2(x^2+12x+26)\binom{x+11}{x}
\end{aligned}
$$

Equal Row and Column Sums

$h^3(x) = \binom{x+3}{x}$

$$\begin{pmatrix} x_1 & y_2 & 0 \\ x_2 & x_3 & y_4 \\ y_1 & y_3 & y_5 \end{pmatrix}$$

Proof by Induction since $\binom{x+3}{x} = \binom{x+2}{x-1} + \binom{x+2}{2}$.

Initial Step: $h^3(1) = 4 = \binom{4}{1}$

Inductively, $\binom{x+2}{x-1} = h^3(x-1)$.

Add 1 to y_1, y_2, and y_4.

This bijection will give us every matrix in $H^3(x)$ that has $y_1 > 0$.

The elements in $H^3(x)$ not covered by $\binom{x+2}{x-1}$ are precisely those with $y_1 = 0$.

There are $\binom{x+2}{2}$ different possibilities for $x_2 + x_3 \leq x$.

Rising Binomial Coefficients

We were able to express $h^n(x)$ for $n \leq 6$ in this way, but the pattern is not clear for higher degrees. Here are the formulas we derived:

$$h^2(x) = \binom{x+1}{x} = 1 + \binom{x}{1}$$

$$h^3(x) = \binom{x+3}{x} = 1 + \binom{x}{1} + \binom{x+1}{2} + \binom{x+2}{3}$$

$$h^4(x) = \tfrac{1}{3}(x+3)\binom{x+5}{x} = 1 + \binom{x}{1} + \binom{x+1}{2} + \binom{x+2}{3} + \binom{x+3}{4} + \binom{x+4}{5} + 2\binom{x+5}{6}$$

$$h^5(x) = \tfrac{1}{3^2}(x+3)^2\binom{x+8}{x} = 1 + \binom{x}{1} + \binom{x+1}{2} + \binom{x+2}{3} + \ldots + \binom{x+7}{8} - 3\binom{x+8}{9} + 10\binom{x+9}{10}$$

$$h^6(x) = \tfrac{1}{2 \cdot 3^2 \cdot 13}(x+3)^2(x^2+12x+26)\binom{x+11}{x} = 1 + \binom{x}{1} + \binom{x+1}{2} + \binom{x+2}{3} + \ldots + \binom{x+10}{11} - 8\binom{x+11}{12} + 84\binom{x+12}{13} - 196\binom{x+13}{14} + 140\binom{x+14}{15}$$

Questions

Is there a nicer way to count $H(c)$ for higher degrees?

What about for special cases, such as $c_i = x$ for all i?

What other patterns exist in the subsum distributions for $i, j \neq n$?

What about our conjecture of log concavity of the distribution?

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