The Magic of Universal Cycles

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Introduction

To begin, a magic trick!

Remember: Ace is *low*.

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Each binary sequence of length 4 appears *exactly* once!

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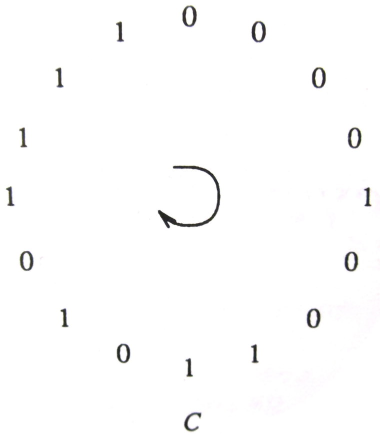
Introduction

Another way to picture this sequence:



De Bruijn Cycles

Definition

A de Bruijn cycle C_n of order n is defined to be a cyclic sequence $(x_0, x_1, \ldots, x_{2^n-1})$ where $x_i \in \{0, 1\}$ and each possible binary sequence of length n occurs uniquely as $(x_i, \ldots, x_{i+n-1})$ for some i, where index addition is performed modulo 2^n.

De Bruijn Cycles

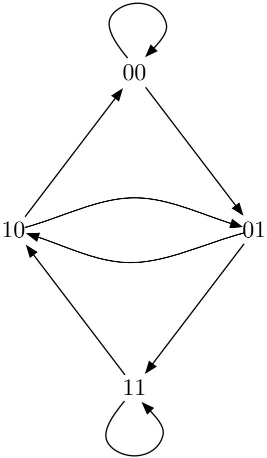
Some questions to consider:

- Do de Bruijn cycles always exist for each n?
- If so, how many are there?
- How does one construct them?
- How can one 'invert' this process in C? That is, for each given block, where is it in C?
- For what other combinatorial structures can we find universal cycles?

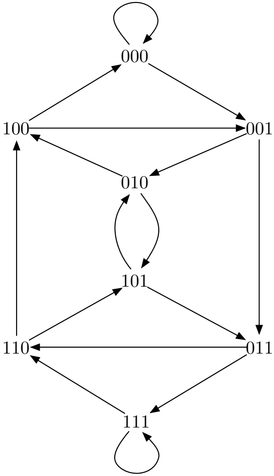
We want to consider the first question, whether universal cycles exist for the set $B_n = \{0,1\}^n$.

Definition

We define the transition digraph G_n for B_n in the following manner. The vertices of G_n are all the n-tuples $\{0,1\}^n$, and there is an edge from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_n)$ iff $x_2 = y_1,\ x_3 = y_2,\ \ldots, x_n = y_{n-1}$, i.e. if it's possible for $\mathbf{y}$ to "follow" $\mathbf{x}$ in a universal cycle.



G_2



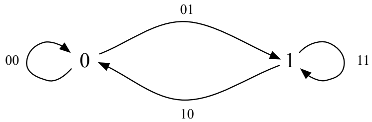
G_3

Because Hamiltonian circuits are NP-hard to find, we'll define a different (related) graph that accomplishes the same goal by looking for Eulerian circuits (which are easy to find!).

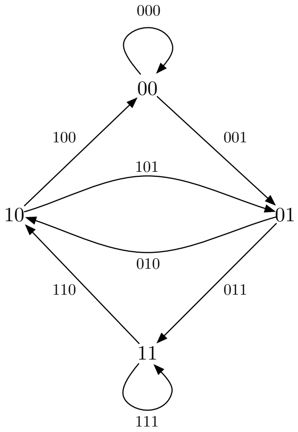
Definition

We define the arc digraph G_n^* of G_n in the following manner. The vertices of G_n^* are just the edges of G_n, which can also be thought of as "what they have in common." The edges are then the vertices of G_n, where the directed edge $(x_1, \ldots, x_n)$ in G_n^* goes from $(x_1, \ldots, x_{n-1})$ to $(x_2, \ldots, x_n)$.

De Bruijn Cycles
The Standard Approach



G_2^*



G_3^*

In general, when looking for universal cycles for a set of objects, if we can create an arc digraph that is

- *balanced* - every vertex has $\deg_{-}(v) = \deg_{+}(v)$
- *strongly connected* - for any pair of vertices, there is a directed path between them

then we're (mostly) done!

G_n^* is balanced:

There is an immediate bijection between incoming edges and outgoing edges by moving the first number to the end.

G_n^* is strongly connected:

Take 2 vertices $(x_1, x_2, \ldots, x_{n-1})$ and $(y_1, y_2, \ldots, y_{n-1})$ in G_n^*. How does one get from $\mathbf{x}$ to $\mathbf{y}$?

De Bruijn Cycles

The Standard Approach

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Consider the sequence formed by concatenating the vectors: $x_1 x_2 \ldots x_{n-1} y_1 \ldots y_{n-1}$.

Example

Suppose that in G_5^*, we want to go from the vertex 0101 to the vertex 1101. Consider the sequence 01011101, which gives this path:

$$0101 \xrightarrow{01011} 1011 \xrightarrow{10111} 0111 \xrightarrow{01110} 1110 \xrightarrow{11101} 1101$$

Permutations

Let S_n be the set of all $n!$ permutations of $[n]$. Do universal cycles of S_n exist?

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Say we're looking at S_4 and we begin the sequence with

$$3142.$$

What must come next?

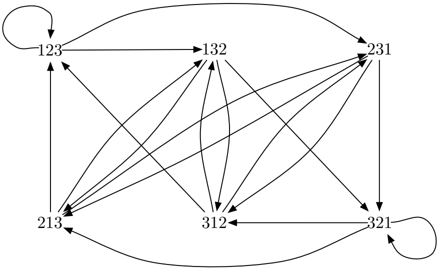
Definition

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be n-tuples of distinct integers. Then, we say $\mathbf{a}$ and $\mathbf{b}$ are order-isomorphic if $a_i < a_j \Leftrightarrow b_i < b_j$ for all $i, j \in [n]$.

Example

7395 is order-isomorphic to 3142.

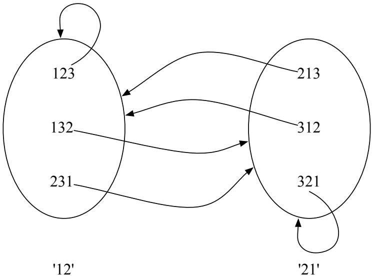
Here is the transition digraph for S_3:



For example, there is an arrow from 123 to 132, because this transition could be represented by the sequence 1243.

So even once we find a hamiltonian cycle for this graph, we must assign values to the 1s, 2s, and 3s, to *realize* the elements of S_3.

Here is another way to represent the transition digraph:



Since each permutation now has exactly one arc leaving it, it is enough to find an Eulerian circuit of this graph (which will produce a Hamiltonian circuit of the original graph).

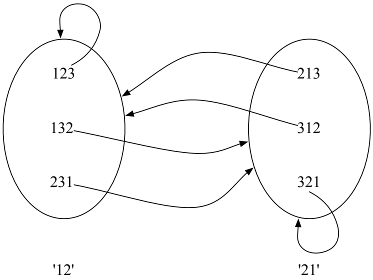
Note: it is always possible to represent the transition digraph for S_n in this way.

Is this graph Eulerian?

balanced: each permutation $x_1 \ldots x_n$ has exactly n incoming arcs and outgoing arcs

strongly connected: let $\mathbf{x}$ and $\mathbf{y}$ be two vertices in the (simplified) transition digraph. Consider the sequence created by concatenating $\mathbf{x}$ with $\mathbf{y} + n$:

$$x_1 x_2 \ldots x_n (y_1 + n) \ldots (y_n + n).$$



You can check that an Eulerian circuit for S_3 is:

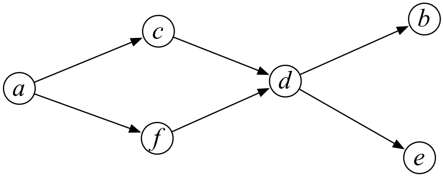
$$\ldots \to 132 \to 312 \to 123 \to 231 \to 321 \to 213 \to \ldots$$

Because there are six permutations, we know a universal cycle will look like $abcdef$. Now, we just need to find values for $a, \ldots, f$ that maintain the above ordering.

$$\ldots \to 132 \to 312 \to 123 \to 231 \to 321 \to 213 \to \ldots$$

$$\ldots \to abc \to bcd \to cde \to def \to efa \to fab \to \ldots$$

The previous Eulerian circuit generates the following partial order:



We can then let $a = 1$, $c = f = 2$, $d = 3$, and $b = e = 4$, to get the universal cycle 142342 for S_3.

It's not hard to show that we can always realize a partial order generated in this way. But how many numbers do we need to use?

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Theorem (R. Johnson, 2009)

It is possible to construct a universal cycle for S_n that uses only $n+1$ distinct integers.

Partitions

How will we represent partitions?

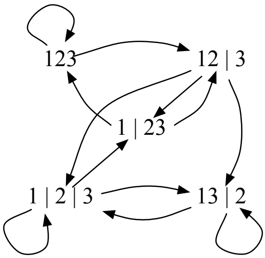
- use an alphabet, say $\{a, b, c, \ldots\}$
- put i and j in the same block of the partition iff the i^{th} and j^{th} symbols are the same

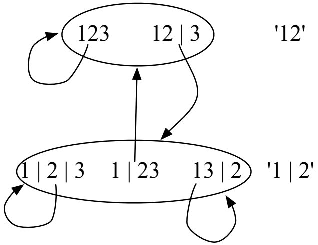
Example

$abacbccd \longleftrightarrow 13 \mid 25 \mid 467 \mid 8$

Partitions

The transition digraph and simplified transition digraph of the set of partitions of $[3]$:





Partitions

An Eulerian circuit of the simplified graph is:

$$\ldots \to 123 \to 12 \mid 3 \to 13 \mid 2 \to 1 \mid 2 \mid 3 \to 1 \mid 23 \to \ldots$$

"Lifting" the circuit to a realization:

$U_4:$	x_1	x_2	x_3	x_4	x_5			
	1	2	3					$\Rightarrow x_1 = x_2 = x_3$
		1	2 |	3				$\Rightarrow x_4 \neq x_3$
			1	3 |	2			$\Rightarrow x_5 = x_3$
				1 |	2 |	3		$\Rightarrow x_1 \neq x_5$
					1 |	2	3	

Contradiction: $x_5 \neq x_1 = x_3 = x_5$

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			1	$3 \mid$	2			$\Rightarrow x_5 = x_3$
				$1 \mid$	$2 \mid$	3		$\Rightarrow x_1 \neq x_5$
					$1 \mid$	2	3	

Contradiction: $x_5 \neq x_1 = x_3 = x_5$

There are no universal cycles for P_3! ☹

Partitions

Theorem (Chung, Diaconis, Graham, 1992)

There exists a universal cycle for P_n, $n \geq 4$.

In particular, there are exactly 52 partitions of $[5]$, and the universal cycle can be realized with 5 letters, say $\{H, D, C, S, J\}$, where J appears just once and the other letters appear 12 or 13 times.

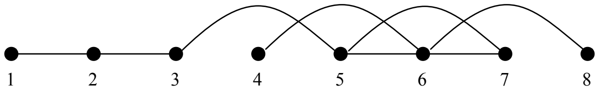
$$DDDDDCHHHCCDDCCCHCHCSHHSD$$

$$SSDSSHSDDCHSSCHSHDHSCHSJCDC$$

In order to study universal cycles for classes of graphs, we must think about what a sequence of graphs should look like.

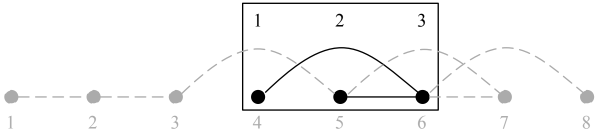
- Number the vertices of the sequence from 1 to M, and place them in a line sequentially.
- Define a k-window of the sequence by looking at the induced subgraph of vertices $v_i, v_{i+1}, \ldots, v_{i+k-1}$, where index addition is modulo M.
- In each k-window, re-number the vertices from 1 to k, in order.
- A graph sequence will include (cover) the set of all k-windows.

A 3-window of an 8 vertex graph:



This graph is actually a universal cycle for simple (labeled) graphs on 3 vertices.

A 3-window of an 8 vertex graph:



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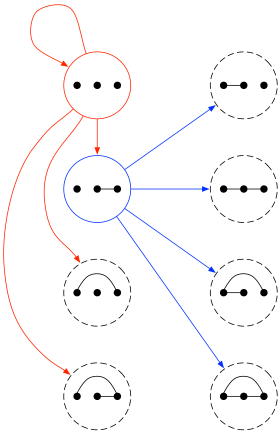
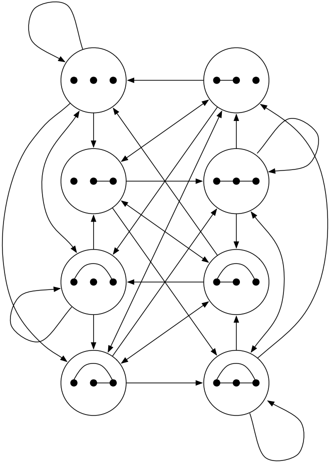
Labeled Graphs
Transition Digraph

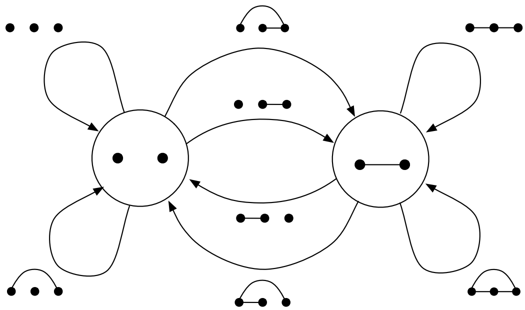
For simplicity, a partial sketch of the transition digraph for the set of simple graphs on 3 vertices:



The full transition digraph:

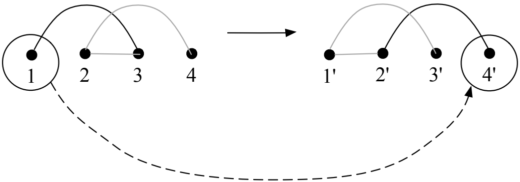


The arc digraph:



To show this arc digraph is Eulerian, we only need apply the standard method.

- *balanced* - count outgoing/incoming edges or create a bijection by moving the first vertex to the end
- *strongly connected* - since this class is relatively unrestricted, “concatenate” the graphs



Similar methods work for the following classes of labeled graphs on k vertices:

- graphs with loops
- graphs with multiple edges (up to m duplications of each edge)
- directed graphs
- hypergraphs
- j-uniform hypergraphs
- cycle-free graphs (forests)

What about more restrictive classes?

Labeled Graphs
Restricted Classes

Theorem (Brockman, Kay, Snively, 2010)

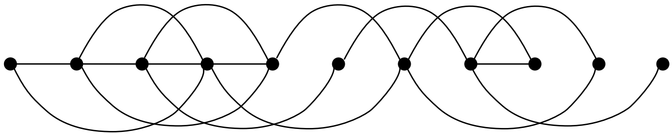
Universal cycles exist for graphs with precisely m edges and k vertices.

Conjecture

Universal cycles exist for trees on k vertices with $k \geq 3$.

Conjecture

Universal cycles exist for isomorphism classes of graphs on k vertices.



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