

THE LINEAR EXTENSION DIAMETER OF GRIDS

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ABSTRACT. A linear extension of a partially ordered set is simply a total ordering of the poset that is consistent with the original ordering. The linear extension diameter is a measure of how different two linear extensions could be, that is, the number of pairs of elements that are ordered differently by the two extensions. We will calculate the linear extension diameter of grids, generalizing a method of focusing on sub-cubes that was developed in a 2011 paper by Felsner and Massow. This also gives us a nice characterization of the linear extensions that are the farthest from each other, and allows us to conclude that grids are diametrically reversing.

1. INTRODUCTION

A *linear extension* L of a finite poset P is a total ordering of the elements that respects the initial partial order. So $x <_P y$ in the poset implies $x <_L y$ in the linear extension. How different can two linear extensions be? In other words, what is the greatest number of pairs of elements that could be ordered differently in two linear extensions? This parameter is known as the *linear extension diameter*, $\text{led}(P)$, because it is the diameter of the *linear extension graph*, $G(P)$, of the poset. This graph has as vertices all linear extensions of P . If we think of a linear extension as a listing of the elements of P from smallest to largest, then two linear extensions are adjacent in $G(P)$ iff they swap two adjacent elements. For example, when P is an antichain, any permutation is a linear extension and the distance between π and σ is the number of inversions of $\pi\sigma^{-1}$. As an additional example, Figure 1 shows $G(2 \times 3)$.

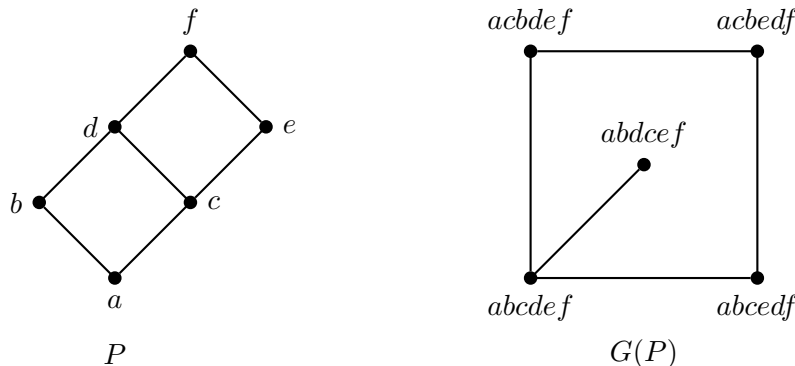


FIGURE 1. The linear extension graph of the grid 2×3 .

The graph $G(P)$ was introduced in a 1991 paper by Pruesse and Ruskey [8], and its diameter was first studied in 1999 by Felsner and Reuter [5]. The linear extension diameter of the Boolean lattice, 2^n , was not computed until 2011 though, when Felsner and Massow [4] developed a new

Date: December 6, 2011.

method that involved looking at sub-cubes of the lattice. In this paper, I generalize their method to extend their results to grids, i.e. products of chains.

The linear extension graph of a grid is a natural and interesting graph to study. While this graph contains a Hamiltonian path [9], and its minimum and maximum degrees are known [3], there are still some mysteries to solve. For example, we do not know the number of vertices. Brightwell and Tetali [2] gave an asymptotic count for the case of the Boolean lattice, namely

$$\frac{\log_2 |V(G(2^n))|}{2^n} = \log_2 \binom{n}{\lfloor n/2 \rfloor} - \frac{3}{2} \log_2 e + O\left(\frac{\log_2 n}{n}\right),$$

using the entropy method of Kahn [6].

To begin, we will introduce the standard notation to be used throughout the paper. Then, we will prove the main result in Section 3, calculating the linear extension diameter of grids. Building on this foundation, we will characterize all diametral pairs of $G(P)$ in Section 4. Ultimately, we will show that all grids have the property of being diametrically reversing in Section 5.

2. NOTATION AND DEFINITIONS

The *linear extension graph* of P , $G(P)$, is the graph that has as its vertices all linear extensions of P , and two vertices are adjacent iff the linear extensions differ in an adjacent transposition. The *linear extension diameter* of P , $\text{led}(P)$, is the diameter of $G(P)$.

The *grid poset* $m_1 \times \cdots \times m_n$ is the product of n chains of lengths m_1, m_2, \dots, m_n , respectively. We set $m_1 \times \cdots \times m_n = \{(x_i)_1^n : 0 \leq x_i < m_i\}$. For the duration of the paper, we will refer to this grid as $P(m_1, \dots, m_n)$, or simply P . We may occasionally omit the parentheses and commas when writing elements of P if the context is clear. Writing $X = (x_i)_1^n$ and $Y = (y_i)_1^n$, we have that the grid is a distributive lattice with

$$X \wedge Y = (\min(x_i, y_i))_1^n$$

and

$$X \vee Y = (\max(x_i, y_i))_1^n.$$

We write $\bar{0}$ for $(0, \dots, 0)$ and let $d(X, Y) := |\{i : x_i \neq y_i\}|$ be the Hamming metric.

A *diametral pair* of vertices (u, v) of a graph G is a pair of vertices such that $\text{dist}_G(u, v) = \text{diam}(G)$. A diametral pair (L_1, L_2) of vertices in $G(P)$ would be two linear extensions that are the most different.

Suppose $\sigma = \sigma_1 \dots \sigma_n$ is a linear ordering of $[n]$, the index set of P . We define L_σ to be the σ -lexicographic order of P (a linear extension), namely $(x_i)_1^n < (y_i)_1^n$ in L_σ iff there exists $k \in [n]$ with $x_{\sigma_i} = y_{\sigma_i}$ for $i < k$ and $x_{\sigma_k} < y_{\sigma_k}$. Hence, we call the σ_1 index the σ -most important index, and the σ_n the σ -least. Let $\text{rev}(\sigma) := \sigma_n \dots \sigma_1$, the reversed priority ordering.

Example 2.1. Consider the grid $2 \times 2 \times 3$, and let $\sigma = 312$. Then L_σ is the ordering

$$000 < 010 < 100 < 110 < 001 < 011 < 101 < 111 < 002 < 012 < 102 < 112$$

and $L_{\text{rev}(\sigma)} = L_{213}$ is given by

$$000 < 001 < 002 < 100 < 101 < 102 < 010 < 011 < 012 < 110 < 111 < 112.$$

A *reversal* between two linear extensions L and L' is a pair of elements (x, y) such that $L(x) < L(y)$ but $L'(x) > L'(y)$. We let $R(L, L')$ be the set of reversals between L and L' . The following is a standard result.

Lemma 2.2. *The number of reversals between L and L' equals the distance between L and L' in $G(P)$, i.e. $|R(L, L')| = \text{dist}_{G(P)}(L, L')$.*

Proof. See, for instance [5]. □

Let $f : [n] \rightarrow \mathcal{P}(\mathbb{N})$ be a function such that $f(k) \subseteq m_k$. In other words, f is a function that restricts each index k to a subset of $\{0, 1, \dots, m_k - 1\}$. We define the subgrid $P[f]$ to be $\{(x_k)_1^n \in P : x_k \in f(k) \text{ for all } k\}$. Similarly, we write $L[f]$ for the restriction of a linear extension L of P to $P[f]$. Note that $P[f]$ is isomorphic to the lattice $|f(1)| \times \dots \times |f(n)|$.

3. CALCULATING THE DIAMETER

We will first define a partition of P^2 in order to show that the pairs of linear extensions $\{(L_\sigma, L_{\text{rev}(\sigma)})\}$ are exactly the diametral pairs of the linear extension graph of the grid. Then, we will calculate the distance between them.

Definition 3.1. Let $B, T \in P$ with $B \leq T$. The *corner set* with bottom B and top T is the set of pairs $C_{B,T} = \{(X, Y) \in P^2 : X \wedge Y = B \text{ and } X \vee Y = T\}$. In other words, $C_{B,T}$ is the set of ordered pairs of opposite corners of the “sub-cube” with minimum element B and maximum element T . This sub-cube is $P[f]$ where $f(k) = \{b_k, t_k\}$.

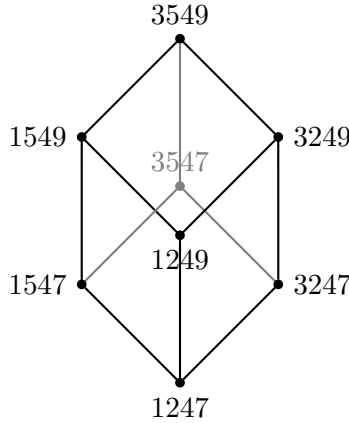


FIGURE 2. The “sub-cube” $C_{1247,3549}$ in 10^4 .

Note that $|C_{B,T}| = 2^{d(B,T)}$. (Recall that $d(B, T) := |\{k : b_k \neq t_k\}|$.) Also,

$$\{C_{B,T} : B, T \in P, B \leq T\}$$

is a partition of P^2 since $(X, Y) \in C_{B,T}$ iff $B = X \wedge Y$ and $T = X \vee Y$.

For any two linear extensions L_1 and L_2 , the set $\{C_{B,T} \cap R(L_1, L_2) : B, T \in P, B \leq T\}$ is a partition of the reversals between them. We will now show that when $L_1 = L_\sigma$ and $L_2 = L_{\text{rev}(\sigma)}$, the sets $C_{B,T} \cap R(L_1, L_2)$ have a predictable cardinality.

Lemma 3.2. *Let $B, T \in P$ with $B \leq T$, and let σ be a linear ordering of $[n]$. Then,*

$$|C_{B,T} \cap R(L_\sigma, L_{\text{rev}(\sigma)})| = \begin{cases} 0 & \text{if } d(B, T) \leq 1 \\ 2^{d(B,T)-2} & \text{otherwise} \end{cases}.$$

Proof. Fix $d = d(B, T)$. If $d = 0$, then $B = T$ and our sub-cube is simply a point, so there can be no reversals. If $d = 1$, then our sub-cube is a chain with two elements, and the ordering is forced for all linear extensions, so once again there can be no reversals.

Now suppose $d \geq 2$. Let i be the σ -most important index where B and T differ, and similarly j the σ -least important. Because B and T differ in at least 2 indices, we have that $i \neq j$. Also, we see that $X <_{\sigma} Y$ iff $x_i < y_i$, and $X <_{\text{rev}(\sigma)} Y$ iff $x_j < y_j$. So a pair (X, Y) is a reversal exactly if $x_i < y_i$ and $x_j > y_j$. Therefore, two of our choices are fixed but the other $d - 2$ are free, and we have 2^{d-2} reversals in each $C_{B,T}$ with $d \geq 2$. \square

In fact, we will show that for any linear extensions L and L' and any $B \leq T$ with $B, T \in P$, the corner set $C_{B,T}$ cannot intersect $R(L, L')$ in a larger set than this. Therefore, L_{σ} and $L_{\text{rev}(\sigma)}$ must be a diametral pair. To do so, we will need the following theorem.

Theorem 3.3 (Daykin's Extension of Kleitman's Lemma [1]). *If \mathcal{P} is a distributive lattice and $A, B \subseteq \mathcal{P}$, then*

$$|A||B| \leq |A \vee B||A \wedge B|,$$

where $A \vee B = \{a \vee b : a \in A, b \in B\}$ and $A \wedge B = \{a \wedge b : a \in A, b \in B\}$. In particular, if $A, B \subseteq P = P(m_1, \dots, m_n)$ are downsets, then

$$|A||B| \leq |P||A \cap B|$$

since $A \wedge B = A \cap B$.

Lemma 3.4. *Let L_1 and L_2 be two linear extensions of P . For every $B, T \in P$ with $B < T$ and $d(B, T) \geq 2$,*

$$|C_{B,T} \cap R(L_1, L_2)| \leq 2^{d(B,T)-2}.$$

In addition, $C_{B,T}$ cannot contain any reversals when $d(B, T) < 2$.

Proof. As in the proof of Lemma 3.2, the claim is immediate when $d(B, T) < 2$.

Suppose $d(B, T) = d \geq 2$, and fix T . Since $C_{B,T} = B + C_{\bar{0}, T-B}$, we may assume that $B = \bar{0}$ without loss of generality. So now, $C_{B,T} = \{(X, T - X) : x_k = 0 \text{ or } x_k = t_k\}$. Following [4], we say X is " T -down" in a linear extension if $X < T - X$ in the extension. Let \mathcal{F}_1 be the family of such X which are T -down in L_1 , and \mathcal{F}_2 those that are T -down in L_2 . Then $(X, T - X) \in C_{B,T}$ is a reversal between L_1 and L_2 exactly if X is T -down in one L_i but not in the other. So, we want to find an upper bound for $|\mathcal{F}_1 \Delta \mathcal{F}_2|$.

Note that these are downsets, because of transitivity. So, Theorem 3.3 shows that $|\mathcal{F}_1||\mathcal{F}_2| \leq 2^d |\mathcal{F}_1 \cap \mathcal{F}_2|$. Now, for all L_i and X , either X or $T - X$ is T -down in L_i . Hence, $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{d-1}$. So, $|\mathcal{F}_1 \cap \mathcal{F}_2| \geq 2^{d-2}$ by the previous inequality. Also, if X is T -down in L_1 and L_2 , then $T - X$ is T -down in neither, i.e. $X \in \mathcal{F}_1 \cap \mathcal{F}_2$ iff $T - X \in (\mathcal{F}_1 \cup \mathcal{F}_2)^c$. Therefore, $|\mathcal{F}_1 \cap \mathcal{F}_2| = |(\mathcal{F}_1 \cup \mathcal{F}_2)^c|$.

Because there are 2^d choices for X , we have

$$|\mathcal{F}_1 \Delta \mathcal{F}_2| = 2^d - |\mathcal{F}_1 \cap \mathcal{F}_2| - |(\mathcal{F}_1 \cup \mathcal{F}_2)^c| \leq 2^d - 2^{d-2} - 2^{d-2} = 2^{d-1}.$$

Note that each reversal $(X, T - X)$ counts doubly towards the cardinality of $|\mathcal{F}_1 \cap \mathcal{F}_2|$. Hence,

$$|C_{B,T} \cap R(L_1, L_2)| = \frac{1}{2} |\mathcal{F}_1 \cap \mathcal{F}_2| \leq 2^{d(B,T)-2}. \quad \square$$

Corollary 3.5. *L_1 and L_2 are diametral pairs of $G(P)$ iff $|C_{B,T} \cap R(L_1, L_2)| = 2^{d(B,T)-2}$ for all $B, T \in P$ with $B \leq T$ and $d(B, T) \geq 2$.*

Proof. We have just shown that this is the most number of reversals a corner set could contain, and from Lemma 3.2 we know that this number is achievable. \square

Corollary 3.6. *Let σ be a linear ordering of $[n]$. Then, L_σ and $L_{\text{rev}(\sigma)}$ are a diametral pair for P .*

Proof. From Lemma 3.2, we know $|C_{B,T} \cap R(L_\sigma, L_{\text{rev}(\sigma)})| = 2^{d(B,T)-2}$ for all $B < T$ with $d(B,T) \geq 2$. Applying Corollary 3.5, we see that L_σ and $L_{\text{rev}(\sigma)}$ must be as far apart as possible. \square

We have just shown that the diametral pairs of $G(P)$ must look like L_σ and $L_{\text{rev}(\sigma)}$, and so we can now count the number of reversals between these to calculate the linear extension diameter of P .

Theorem 3.7. *The linear extension diameter of the grid $P(m_1, \dots, m_n)$ is*

$$\frac{1}{4} \left(\prod_k m_k \right) \left(\prod_k m_k - \sum_k m_k + n - 1 \right).$$

Proof. From Corollary 3.5, we know that

$$|C_{B,T} \cap R(L_\sigma, L_{\text{rev}(\sigma)})| = 2^{d(B,T)-2}$$

for all $B < T$ with $d(B,T) \geq 2$. Let's count these pairs (B,T) with $\{i : b_i \neq t_i\} = D$. We need to pick a pair of elements from $\{0, 1, \dots, m_i - 1\}$ for each $i \in D$, and a single element from $\{0, 1, \dots, m_i - 1\}$ for each $i \notin D$. Therefore, there are

$$\sum_{\substack{D \subseteq [n] \\ |D|=d}} \left[\prod_{k \notin D} m_k \cdot \prod_{k \in D} \binom{m_k}{2} \right]$$

possible pairs for each fixed $d \geq 2$.

Then, for each of these pairs with difference d , we have 2^{d-2} reversals, from Lemma 3.2. Summing over all $d \geq 2$, we get

$$\begin{aligned} \text{led}(P) &= \sum_{d=2}^n 2^{d-2} \sum_{\substack{D \subseteq [n] \\ |D|=d}} \left[\prod_{k \notin D} m_k \cdot \prod_{k \in D} \binom{m_k}{2} \right] \\ &= \sum_{d=2}^n 2^{d-2} \frac{1}{2^d} \sum_{\substack{D \subseteq [n] \\ |D|=d}} \left[\prod_{k \in [n]} m_k \cdot \prod_{k \in D} (m_k - 1) \right] \\ &= \frac{1}{4} \left(\prod_{k \in [n]} m_k \right) \sum_{d=2}^n \sum_{\substack{D \subseteq [n] \\ |D|=d}} \left[\prod_{k \in D} (m_k - 1) \right] \\ &= \frac{1}{4} \left(\prod_{k \in [n]} m_k \right) \left(\prod_{k \in [n]} (1 + (m_k - 1)) - \sum_{k \in [n]} (m_k - 1) - 1 \right) \\ &= \frac{1}{4} \left(\prod_{k \in [n]} m_k \right) \left(\prod_{k \in [n]} m_k - \sum_{k \in [n]} m_k + n - 1 \right) \end{aligned} \quad \square$$

4. DIAMETRAL PAIRS OF $G(P)$

For a fixed permutation σ , we have shown that L_σ and $L_{\text{rev}(\sigma)}$ form a diametral pair of P . This gives us $n!/2$ diametral pairs, and we will show this is all of them. To do so, we must expand to looking at all sub-grids of P , rather than simply the sub-cubes. Recall from the notation section, that $P[f]$ and $L[f]$ are the sub-grid and linear extension, respectively, generated by the restriction function f .

Lemma 4.1. *Let $f : [n] \rightarrow \mathcal{P}(\mathbb{N})$ such that $f(k) \subseteq m_k$. If L_1 and L_2 are a diametral pair of $G(P)$, then $L_1[f]$ and $L_2[f]$ are a diametral pair of the linear extension graph of $P[f]$.*

Proof. We need to show

$$|C_{B,T}[f] \cap R(L_1[f], L_2[f])| = \begin{cases} 0 & \text{if } d(B,T) \leq 1 \\ 2^{d(B,T)-2} & \text{otherwise} \end{cases}$$

for each $B, T \in P[f]$ with $B \leq T$.

Fix such a B and T , and let $d := d(B, T)$. Notice that $C_{B,T} = C_{B,T}[f]$ because $B, T \in P[f]$ implies $b_k, t_k \in f(k)$ for all k , and so any vector in $C_{B,T}$ has b_k or t_k as each of its entries. Therefore, each vector that appears as part of a pair in $C_{B,T} \cap R(L_1, L_2)$ is still in our sub-grid $P[f]$. We also know that $|C_{B,T} \cap R(L_1, L_2)| = 2^{d-2}$ if $d \geq 2$ and 0 otherwise, because L_1 and L_2 are a diametral pair. Moreover, the linear extension restriction keeps the restricted elements in the same order, so any pair in $C_{B,T} \cap R(L_1, L_2)$ is also a reversal between $L_1[f]$ and $L_2[f]$. Hence, $|C_{B,T}[f] \cap R(L_1[f], L_2[f])| = |C_{B,T} \cap R(L_1, L_2)|$. \square

The previous lemma allows us to use induction and transitivity to show that as each dimension of a grid is built up, the diametral linear extensions maintain the same structure. In other words, for a diametral pair (L_1, L_2) to remain a diametral pair under any restriction, as the lemma requires, we must have that L_1 is the σ -lex ordering for some σ , and L_2 is its counterpart $L_{\text{rev}(\sigma)}$.

Theorem 4.2. *If L_1 and L_2 are a diametral pair of linear extensions of P and σ is the order of the atoms in L_1 , then $L_1 = L_\sigma$ and $L_2 = L_{\text{rev}(\sigma)}$.*

Proof. From Corollary 3.5, we know $|C_{B,T} \cap R(L_1, L_2)| = 2^{d(B,T)-2}$ for $d(B, T) \geq 2$. We can also assume $m_k > 1$ for all k , because the cases where some $m_k = 1$ are isomorphic to lattices with a smaller dimension.

We proceed by induction on $M := \sum_{k=1}^n (m_k - 1)$. When $M = 1$, there is only the trivial linear extension, to which both L_1 and L_2 must be equal. When $M > 1$, let σ_1 be the σ -most important index. If $m_{\sigma_1} > 2$, then we can piece together two grid restrictions. Let

$$f(k) := \begin{cases} \{0, 1\} & \text{if } k = \sigma_1 \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g(k) := \begin{cases} [m_k - 1] & \text{if } k = \sigma_1 \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases} .$$

By induction we have that $L_1[f]$ and $L_1[g]$ are in σ -lex order. Because $10\dots 0 \in P[f] \cap P[g]$, the result that $L_1 = L_\sigma$ follows from transitivity.

Similarly, let σ_n be the σ -least important index. If $m_{\sigma_n} > 2$, then we can piece together two grid restrictions once again to show that $L_2 = L_{\text{rev}(\sigma)}$. Let

$$f(k) := \begin{cases} \{0, 1\} & \text{if } k = \sigma_n \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g(k) := \begin{cases} [m_k - 1] & \text{if } k = \sigma_n \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}.$$

By induction we have that $L_2[f]$ and $L_2[g]$ are in $\text{rev}(\sigma)$ -lex order. Because $0 \dots 01 \in P[f] \cap P[g]$, the result that L_2 is in $\text{rev}(\sigma)$ -lex order follows from transitivity.

We must do a little more work when $m_{\sigma_1} = 2$ or $m_{\sigma_n} = 2$. Let us first tackle the case where $m_{\sigma_1} = 2$. By considering the sub-grids with the restriction functions

$$f(k) := \begin{cases} \{0\} & \text{if } k = \sigma_1 \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g(k) := \begin{cases} \{1\} & \text{if } k = \sigma_1 \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases},$$

in essence decreasing the dimension by 1, we know that elements with the σ_1 index fixed are in σ -lex order in L_1 and $\text{rev}(\sigma)$ -lex order in L_2 . We only need to show that the maximum element of $L_1[f]$ is less than the minimum element of $L_1[g]$, and again we will be done by transitivity. For simplicity, we can assume $\sigma_1 = 1$ and $\sigma_n = n$. So we would like to show that

$$0(m_2 - 1) \dots (m_n - 1) < 10 \dots 0$$

in L_1 . We will follow a similar proof to that of Felsner and Massow from [4].

First, we will show that the atoms are in $\text{rev}(\sigma)$ -lex order in L_2 . Note that they are in the right order in L_1 by definition. Let X and Y both be atoms and assume $X < Y$ in L_1 . Consider $C_{B,T}$ where $T = X + Y$ and $B = \bar{0}$. Since $d(B, T) = 2$, this particular $C_{B,T}$ must contain exactly one reversal between L_1 and L_2 . But

$$C_{B,T} = \{(0, T), (X, Y), (Y, X), (T, 0)\}.$$

Clearly, $(0, T)$ and $(T, 0)$ cannot be reversals, and $Y < X$ in L_1 implies that (Y, X) cannot be a reversal either. Hence, (X, Y) must be a reversal. Therefore, $Y < X$ in L_2 , and it follows that all the atoms in L_2 are in $\text{rev}(\sigma)$ -lex order.

Now consider $C_{B,T}$ where $T = (m_k - 1)_{k=1}^n$ and $B = \bar{0}$. We will show that $(X, Y) \in C_{B,T}$ cannot be a reversal if $x_1 = x_n = 0$, or similarly if $y_1 = y_n = 0$. Suppose the first were true, so lexicographically we have $X < Y$. In P , we have $X < X + 0 \dots 01$, and by the function restriction

$$f(k) := \begin{cases} [m_k - 1] & \text{if } k = n \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and induction, we have $X + 0 \dots 01 < Y$ in L_1 . Similarly, in L_2 , we have $X < X + 10 \dots 0 < Y$. Therefore (X, Y) cannot be a reversal between L_1 and L_2 .

We give a counting argument to show that all the rest must be reversals. We know $C_{B,T}$ must contain $2^{d(B,T)-2}$ reversals, and that there are $2^{d(B,T)}$ pairs in $C_{B,T}$. We have just shown that $2^{d(B,T)-1}$ pairs cannot be reversals. So the other $2^{d(B,T)} - 2^{d(B,T)-1} = 2^{d(B,T)-1}$ pairs must all be reversals.

In particular, $0(m_2 - 1) \dots (m_n - 1)$ and $10 \dots 0$ must be a reversal. But in L_2 we have already shown that $10 \dots 0 < 0 \dots 01$, so

$$10 \dots 0 < 0(m_2 - 1) \dots (m_n - 1)$$

in L_2 by transitivity. Hence, we must have

$$0(m_2 - 1) \dots (m_n - 1) < 10 \dots 0$$

in L_1 .

Similarly, suppose that $m_{\sigma_n} = 2$. This time, consider the sub-grids with the restriction functions

$$f(k) := \begin{cases} \{0\} & \text{if } k = \sigma_n \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}$$

and

$$g(k) := \begin{cases} \{1\} & \text{if } k = \sigma_n \\ \{0, 1, \dots, m_k - 1\} & \text{otherwise} \end{cases}.$$

We know that elements with the σ_n index fixed are in $\text{rev}(\sigma)$ -lex order in L_2 and σ -lex order in L_1 . We only need to show that the maximum element of $L_2[f]$ is less than the minimum element of $L_2[g]$, and once more we will be done by transitivity. Again assuming $\sigma_1 = 1$ and $\sigma_n = n$, we need to show that

$$(m_1 - 1) \dots (m_{n-1} - 1)0 < 0 \dots 01$$

in L_2 .

However, we still know that for $T = (m_k - 1)_{k=1}^n$ and $B = \bar{0}$, any $(X, Y) \in C_{B,T}$ with $x_1 = x_n = 0$ or $y_1 = y_n = 0$ cannot be a reversal between L_1 and L_2 . Therefore, by the counting argument given above, we must have that $(m_1 - 1) \dots (m_{n-1} - 1)0$ and $0 \dots 01$ are a reversal. But by assumption we have that $0 \dots 01 < 10 \dots 0$ in L_1 and so by transitivity,

$$0 \dots 01 < (m_1 - 1) \dots (m_{n-1} - 1)0.$$

Therefore, we must have

$$(m_1 - 1) \dots (m_{n-1} - 1)0 < 0 \dots 01$$

in L_2 and by transitivity, L_2 is in $\text{rev}(\sigma)$ -lex order. \square

5. CRITICAL PAIRS

Definition 5.1. A *critical pair* of a poset \mathcal{P} is an ordered pair (x, y) of incomparable elements of \mathcal{P} such that

- (1) $a < x$ implies $a < y$, and
- (2) $b > y$ implies $b > x$.

Equivalently, (x, y) is a critical pair if the addition of $x < y$ to the relations of \mathcal{P} does not transitively force any other additional relation. A third way to characterize critical pairs is that $y < x$ cannot be forced by adding any other relation.

If (x, y) is a critical pair of \mathcal{P} , then we say that a linear extension L of \mathcal{P} *reverses* (x, y) if $y < x$ in L . Note that then nothing can appear between y and x in L . A poset is *diametrically reversing* if every linear extension contained in a diametral pair reverses a critical pair. In [4], Felsner and Massow showed that Boolean lattices are diametrically reversing, and we will now extend their result to grids.

Corollary 5.2. *Grids are diametrically reversing.*

Proof. Let L be a linear extension of P that is contained in a diametral pair of $G(P)$. By Theorem 4.2, we know that $L = L_\sigma$ for some ordering σ of $[n]$.

Let σ_1 be the σ -most important index. Then consider $(x_k)_1^n$ where $x_k = 0$ for all $k \neq \sigma_1$ and $x_{\sigma_1} = 1$, and $(y_k)_1^n$ where $y_k = m_k - 1$ for all $k \neq \sigma_1$ and $y_{\sigma_1} = 0$. It's clear that this is a critical pair of P . In L_σ , we have $(x_k)_1^n > (y_k)_1^n$, and so this is actually a reversed critical pair. \square

Remark 5.3. We can easily extend this proof to show there are at least $m_{\sigma_1} - 1$ reversed critical pairs between L_σ and $L_{\text{rev}(\sigma)}$. For each $i \in [m_{\sigma_1} - 1]$, we can let $x_{\sigma_1} = i$ and $y_{\sigma_1} = i - 1$, and keep $x_k = 0$ and $y_k = m_k - 1$ for every $k \neq \sigma_1$. Then $(x_k)_1^n$ and $(y_k)_1^n$ are also reversed.

In the following proposition, we characterize all of the critical pairs of the grid. From the previous Remark, we can then conclude that there are exactly $m_{\sigma_1} - 1$ reversed critical pairs between L_σ and $L_{\text{rev}(\sigma)}$.

Proposition 5.4. *Let $X = (x_k)_1^n$ and $Y = (y_k)_1^n$. If (X, Y) is a critical pair of P , then there exists k_0 such that $x_k = 0$ and $y_k = m_k - 1$ for all $k \neq k_0$, and $x_{k_0} - y_{k_0} = 1$.*

Proof. Suppose (X, Y) is a critical pair. Because X and Y are incomparable by definition, there must be an index, say k_0 , where $x_{k_0} > y_{k_0}$. If $x_{k_0} - y_{k_0} > 1$, then the element $X - 0 \dots 1 \dots 0$, where we subtract the atom with a 1 in index k_0 , would be less than X but not less than Y , which contradicts the definition of a critical pair. So we must have $x_{k_0} - y_{k_0} = 1$.

Similarly, suppose $x_{k_1} \neq 0$ for some $k_1 \neq k_0$. Then we could decrease index k_1 in X by 1, and we would have an element that is less than X , but this element is not less than Y , because index k_0 is still larger than in Y . So we must have $x_k = 0$ for all $k \neq k_0$.

Finally, suppose $y_{k_1} < m_{k_1} - 1$ for some $k_1 \neq k_0$. Then the element $Y + 0 \dots 1 \dots 0$ where we add the atom with a 1 in index k_1 is greater than Y but not greater than X . Hence, $y_k = m_k - 1$ for all $k \neq k_0$. \square

6. SUMMARY

We have calculated the linear extension diameter of grids, and characterized all of the linear extensions contained in a diametral pair. The method used was to study sub-grids, and in particular, sub-cubes of the poset. It is a natural question to ask whether this method could apply to other posets, particularly those that are grid-like such as the poset of partially defined functions or Young's lattices. What do the extremal linear extensions of these posets look like?

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