YOUNG MEASURES GENERATED BY SEQUENCES IN MORREY SPACES

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $1 \leq p < \infty$ and $0 \leq \lambda < n$, we give a characterization of Young measures generated by sequences of functions $\{\mathbf{f}_j\}_{j=1}^{\infty}$ uniformly bounded in the Morrey space $L^{p,\lambda}(\Omega; \mathbb{R}^N)$ with $\{|\mathbf{f}_j|^p\}_{j=1}^{\infty}$ equiintegrable. We then treat the case that each $\mathbf{f}_j = \nabla \mathbf{u}_j$ for some $\mathbf{u}_j \in W^{1,p}(\Omega; \mathbb{R}^N)$. As an application of our results, we consider the functional

$$\mathbf{u} \mapsto \int_{\Omega} f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \mathrm{d}\mathbf{x},$$

and provide conditions that guarantee the existence of a minimizing sequence with gradients uniformly bounded in $L^{p,\lambda}(\Omega; \mathbb{R}^{N \times n})$.

1. Introduction

Fix n, $N \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $1 \leq p < \infty$ and $0 \leq \lambda < n$, we characterize Young measures that are generated by a sequence $\{\mathbf{f}_j\}_{j=1}^{\infty}$ uniformly bounded in the Morrey space $L^{p,\lambda}(\Omega; \mathbb{R}^N)$ with $\{|\mathbf{f}_j|^p\}_{j=1}^{\infty}$ equiintegrable. After first considering the easier case where no additional constraints are placed on the generating sequence $\{\mathbf{f}_j\}$, we then investigate the case that $\{\mathbf{f}_j\}$ is a sequence of weak gradients. Here and in what follows, we denote by $L^{p,\lambda}(\Omega; \mathbb{R}^N)$ the Morrey space

$$L^{p,\lambda}(\Omega; \mathbb{R}^N) := \left\{ \mathbf{f} \in L^p(\Omega; \mathbb{R}^N) : \sup_{\substack{\mathbf{x}_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega \cap Q_{\mathbf{x}_0, \rho}} |\mathbf{f}|^p \, \mathrm{d}\mathbf{x} < \infty \right\},$$

with the norm

$$\|\mathbf{f}\|_{L^{p,\lambda}} := \left(\sup_{\substack{\mathbf{x}_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega \cap Q_{\mathbf{x}_0, \rho}} |\mathbf{f}|^p \, \mathrm{d}\mathbf{x} \right)^{1/p},$$

where $Q_{\mathbf{x}_0, \rho}$ is the cube centered at $\mathbf{x}_0$ with edges of length ρ that are parallel to the coordinate axes. We also define the Sobolev-Morrey space $W^{1,(p,\lambda)}(\Omega; \mathbb{R}^N)$ by

$$W^{1,(p,\lambda)}(\Omega; \mathbb{R}^N) := \{\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^N) : \mathbf{u} \in L^{p,\lambda}(\Omega; \mathbb{R}^N) \text{ and } \nabla \mathbf{u} \in L^{p,\lambda}(\Omega; \mathbb{R}^{N \times n})\},$$

with the corresponding norm

$$\|\mathbf{u}\|_{W^{1,(p,\lambda)}} := \|\mathbf{u}\|_{L^{p,\lambda}} + \|\nabla \mathbf{u}\|_{L^{p,\lambda}}.$$

We refer the reader to [9] for additional background on Morrey and Sobolev-Morrey spaces.

D. Kinderlehrer and P. Pedregal [11] have provided a characterization of Young measures generated by sequences $\{\nabla \mathbf{v}_j\}$, where the sequence $\{\mathbf{v}_j\}$ is bounded in $W^{1,p}(\Omega; \mathbb{R}^N)$ and $\{|\nabla \mathbf{v}_j|^p\}$ is equiintegrable. Using this result, it can be shown that if $p > 1$, then any Young measure generated by a sequence $\{\nabla \mathbf{u}_j\}$ with $\{\mathbf{u}_j\}$ uniformly bounded in $W^{1,p}$ can also be generated by a sequence $\{\nabla \mathbf{v}_j\}$, with $\{\mathbf{v}_j\}$ uniformly bounded in $W^{1,p}$ and $\{|\nabla \mathbf{v}_j|^p\}$ equiintegrable. The primary purpose of this paper is to extend their results to the more general setting of Young measures generated by sequences of gradients of functions belonging to the Sobolev-Morrey space $W^{1,(p,\lambda)}$, with $0 \le \lambda < n$. Since $L^{p,n} \cong L^\infty$, the case $\lambda = n$ has already been treated in [10] and [12], so we omit it from our consideration here.

While the results in [11] provide optimal integrability results for the sequence of generating functions, our results allow one to refine the regularity properties the generating sequence can be expected to possess. For example, if a Young measure ν is homogeneous and can be generated by a sequence of gradients $\{\nabla \mathbf{u}_j\}$ bounded in $L^p(\Omega; \mathbb{R}^{N\times n})$, by the results obtained here, it can also be generated by a sequence of gradients $\{\nabla \mathbf{v}_j\}$ that is uniformly bounded in $L^{p,\lambda}(\Omega; \mathbb{R}^{N\times n})$ for each $0 \le \lambda < n$. So even when it may not be possible to get any higher integrability for a sequence of gradients generating ν, it is nevertheless possible to get much more Morrey regularity on the gradients. If the boundary of Ω is Lipschitz, the Morrey regularity on the gradients translates to Hölder continuity of the potential functions $\mathbf{v}_j$. In fact, in this case we would have that $\{\mathbf{v}_j\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(\Omega; \mathbb{R}^N)$ for each $0 \le \alpha < 1$.

I. Fonseca and S. Müller [7] have generalized the result of D. Kinderlehrer and P. Pedregal [11] in a different direction than is carried out here; they characterize Young measures generated by sequences $\{\mathbf{v}_j\}$ bounded in L^p that satisfy $\mathrm{A}\mathbf{v}_j = 0$ for some constant rank partial differential operator A. The gradient case $\mathbf{v}_j = \nabla \mathbf{u}_j$ addressed in [11] corresponds to the case $\mathrm{A} = \mathrm{curl}$. This generalization, among other things, allows one to characterize Young measures generated by sequences of higher gradients. It may be possible to combine the ideas in [7] with those in the present paper to obtain a characterization of Young measures generated by sequences $\{\mathbf{v}_j\}$ bounded in $L^{p,\lambda}$ that satisfy $\mathrm{A}\mathbf{v}_j = 0$. The main obstacle to employing the methods used here to obtain such a characterization is the absence of a way to approximate an A-free function by essentially bounded A-free functions (cf. Theorem 2.2). We would like to mention here that one can look to [2, 4, 5, 6, 10, 11, 12, 13, 15, 14, 16] for some applications of and supplementary material regarding Young measures.

The paper is organized as follows. In Section 2 we introduce further notation and state some theorems proved elsewhere that we will use throughout the paper. In Section 3, we use a constructive argument to characterize Young measures generated by p-equintegrable sequences of functions bounded in the Morrey space $L^{p,\lambda}(\Omega; \mathbb{R}^N)$. We then proceed to consider the gradient case in Section 4. We conclude the paper in Section 5 with an application of our results to the Calculus of Variations.

2. Notation and Preliminaries

If E is a Lebesgue measurable set, we use χ_E and $m(E)$ to indicate the characteristic function of E and the Lebesgue measure of E, respectively; if $0 < m(E) < \infty$ and $f \in L^1(E)$, we define the average value of f over E by

$$\fint_E f(\mathbf{x})\mathrm{d}\mathbf{x} := \frac{1}{m(E)} \int_E f(\mathbf{x})\mathrm{d}\mathbf{x}.$$

We will say that the sequence of functions $\{\mathbf{f}_j\}_{j=1}^\infty$ is p-equiintegrable if the sequence $\{|\mathbf{f}_j|^p\}_{j=1}^\infty$ is equiintegrable.

We denote by $\mathcal{M}(\mathbb{R}^N)$ the set of all Radon measures supported on $\mathbb{R}^N$. Given $\mu \in \mathcal{M}(\mathbb{R}^N)$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, we define

$$\langle \mu, \varphi \rangle := \int_{\mathbb{R}^N} \varphi(\mathbf{y}) \mathrm{d}\mu(\mathbf{y}).$$

We say that a mapping $\nu : E \to \mathcal{M}(\mathbb{R}^N)$ is a Young measure if $\nu(\mathbf{x})$ is a probability measure for almost every $\mathbf{x} \in E$ and the map $\mathbf{x} \mapsto \langle \nu(\mathbf{x}), \varphi \rangle$ is measurable for every $\varphi \in \mathcal{C}_0(\Omega)$. Complying with standard notation for Young measures, we will write $\nu_{\mathbf{x}}$ instead of $\nu(\mathbf{x})$, and will usually denote the map ν by $\{\nu_{\mathbf{x}}\}_{\mathbf{x}\in E}$.

We define $Q := (0,1)^n$, and for $\mathbf{x} \in \mathbb{R}^k$ and $\rho > 0$, we use $Q_{\mathbf{x},\rho}$ to denote the open cube in $\mathbb{R}^k$ centered at $\mathbf{x}$ with edges of length ρ parallel to the coordinate axes.

We now state a version of the fundamental theorem for Young measures; more general versions of this theorem are available (see [1, 3, 6, 7], for instance), but the one given here suffices for our purposes. We recall that a function $f : E \times \mathbb{R}^N \to \mathbb{R}$ is said to be Carathéodory if f is Borel measurable and $f(\mathbf{x}, \cdot)$ is continuous for almost every $\mathbf{x} \in E$.

Theorem 2.1 (Fundamental Theorem for Young Measures)**.** *Let $E \subset \mathbb{R}^n$ be a measurable set with finite measure, and let $\{\mathbf{z}_j\}_{j=1}^\infty$ be a sequence of measurable functions mapping E into $\mathbb{R}^N$ that generates the Young measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in E}$. Suppose $f : E \times \mathbb{R}^N \to [0,\infty)$ is Carathéodory. Then*

$$\liminf_{j\to\infty} \int_E f(\mathbf{x}, \mathbf{z}_j(\mathbf{x}))\mathrm{d}\mathbf{x} \geq \int_E \int_{\mathbb{R}^N} f(\mathbf{x}, \mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.$$

Furthermore, if $\{f(\cdot, \mathbf{z}_j(\cdot))\} \subset L^1(E)$, then $\{f(\cdot, \mathbf{z}_j(\cdot))\}$ is equiintegrable if and only if

$$\lim_{j\to\infty} \int_E f(\mathbf{x}, \mathbf{z}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \int_E \int_{\mathbb{R}^N} f(\mathbf{x}, \mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} < \infty.$$

In this case,

$$f(\cdot, \mathbf{z}_j(\cdot)) \rightharpoonup \int_{\mathbb{R}^N} f(\cdot, \mathbf{y})\mathrm{d}\nu_{(\cdot)}(\mathbf{y}) \text{ in } L^1(E).$$

The following theorem, which is essentially Theorem 4 on page 203 of [8], provides a tool for approximating functions in $W^{1,p}$ by Lipschitz functions.

Theorem 2.2. *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^N)$ with $1 \leq p < \infty$. For $T \geq 0$, define the closed set A_T by*

$$A_T := \{\mathbf{x} \in \mathbb{R}^n : M(|\nabla \mathbf{u}|)(\mathbf{x}) \leq T\},$$

where $M(f)$ denotes the maximal function of f. Then there exists a Lipschitz function $\mathbf{v}_T : \mathbb{R}^n \to \mathbb{R}^N$ such that

(i) *$\mathbf{v}_T(\mathbf{x}) = \mathbf{u}(\mathbf{x})$ and $\nabla \mathbf{v}_T(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})$ for almost every $\mathbf{x} \in A_T$;*
(ii) *$\|\nabla \mathbf{v}_T\|_{L^\infty} \leq c(N,n)T$;*
(iii) *$m(\mathbb{R}^n \setminus A_T) \leq c(n)T^{-p} \int_{\{|\nabla u| > T/2\}} |\nabla \mathbf{u}|^p \, \mathrm{d}\mathbf{x}$.*

The following theorem and its proof can be found in [6].

Theorem 2.3. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with finite measure and let $\{\mathbf{f}_j\}_{j=1}^\infty$ and $\{\mathbf{g}_j\}$ be sequences of measurable functions mapping E into $\mathbb{R}^N$. If $\{\mathbf{f}_j\}$ generates the Young measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in E}$ and $\{\mathbf{g}_j\}$ converges in measure to a measurable function $\mathbf{g} : E \to \mathbb{R}^N$, then $\{\mathbf{f}_j + \mathbf{g}_j\}$ generates the translated Young* *measure* $\tilde{\nu} = \{\tilde{\nu}_{\mathbf{x}}\}_{\mathbf{x}\in E}$ *defined by* $\langle \tilde{\nu}_{\mathbf{x}}, \varphi\rangle := \langle \nu_{\mathbf{x}}, \varphi(\cdot + \mathbf{g}(\mathbf{x}))\rangle$ *for every* $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$. *In particular, if* $\mathbf{g}_j \to 0$ *in measure, then* $\{\mathbf{f}_j + \mathbf{g}_j\}$ *generates* ν.

3. The General Case

In this section, we determine which Young measures can be generated by a sequence bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$, for $1 \le p < \infty$ and $0 \le \lambda < n$. We first consider the homogeneous case when $\nu_{\mathbf{x}} \equiv \nu$ for some probability measure ν supported on $\mathbb{R}^N$; using a similar strategy as is found in [11] and [12], we use the homogeneous result to prove the more general theorem in the nonhomogenous case. We would like to point out that the arguments in this section are entirely constructive, though it is possible to give a shorter, nonconstructive argument using some of the same techniques that are utilized in Section 4.

3.1. **Homogeneous Measures.** Let $\nu \in \mathcal{M}(\mathbb{R}^N)$ be a probability measure. We construct a sequence of functions $\{\mathbf{f}_j\}_{j=1}^\infty$ mapping Q into $\mathbb{R}^N$ that generates the homogeneous Young measure ν. Furthermore, we demonstrate that this sequence of functions is uniformly bounded in $L^{p,\lambda}(Q;\mathbb{R}^N)$ for every $0 \le \lambda < n$ if ν satisifies the condition

$$\int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}) < \infty.$$

We will use the following lemma.

Lemma 3.1. *Suppose that* $\nu \in \mathcal{M}(\mathbb{R}^N)$ *is a probability measure and that* $\{\mathbf{f}_j\}_{j=1}^\infty$ *is a sequence of measurable functions mapping a measurable set* $E \subset \mathbb{R}^n$ *into* $\mathbb{R}^N$. *If the equality*

$$\nu(G)m(D\cap E) = \lim_{j\to\infty} m(D\cap E\cap \mathbf{f}_j^{-1}(G)) \tag{1}$$

holds for each Borel set $G \subset \mathbb{R}^N$ *and every cube* $D \subset \mathbb{R}^n$, *then the sequence* $\{\mathbf{f}_j\}_{j=1}^\infty$ *generates* ν.

Proof. Recall that the sequence $\{\mathbf{f}_j\}$ generates ν if and only if

$$\lim_{j\to\infty} \int_E \xi(\mathbf{x})\varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \int_E \xi(\mathbf{x})\mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}) \tag{2}$$

for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$ and $\xi \in L^1(E)$; it actually suffices to show that (2) holds for all $\varphi \in \mathcal{S}$ and $\xi \in \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are dense subsets of $\mathcal{C}_0(\mathbb{R}^N)$ and $L^1(E)$, respectively. To this end, fix $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, and suppose that $\xi = \chi_{D\cap E}$ for some cube $D \subset \mathbb{R}^n$. We will show that (2) holds; i.e.,

$$\lim_{j\to\infty} \int_{D\cap E} \varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = m(D\cap E) \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}).$$

Since $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, for any $\varepsilon > 0$, we can find a compact $K \subset \mathbb{R}^N$ such that $|\varphi(\mathbf{z})| < \varepsilon$ for all $\mathbf{z} \in \mathbb{R}^N \setminus K$ and a $\delta > 0$ so that $|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| < \varepsilon$ whenever $|\mathbf{x} - \mathbf{y}| < \delta$. We select a finite number of disjoint cubes $\{Q_k\}_{k=1}^M$ that cover K and that each have diameter less than δ. Denote by $A_{k,j}$ the sets

$$A_{k,j} := D\cap E\cap \mathbf{f}_j^{-1}(Q_k).$$

Lastly, we choose $\mathbf{a}_k \in Q_k$ for each $1 \le k \le M$, and introduce the simple function $\mathbf{g}_j : E \to \mathbb{R}^N$ defined by

$$\mathbf{g}_j := \sum_{k=1}^{M} \varphi(\mathbf{a}_k)\chi_{A_{k,j}}.$$

We begin by adding and subtracting $\mathbf{g}_j$ inside the integral:

$$\begin{aligned}\limsup_{j\to\infty}&\left|\int_{D\cap E}\varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - m(D\cap E)\int_{\mathbb{R}^N}\varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y})\right| \\ &\le \limsup_{j\to\infty}\int_{D\cap E}|\varphi(\mathbf{f}_j(\mathbf{x})) - \mathbf{g}_j(\mathbf{x})|\,\mathrm{d}\mathbf{x} \\ &\quad + \limsup_{j\to\infty}\left|\int_{D\cap E}\mathbf{g}_j(\mathbf{x})\mathrm{d}\mathbf{x} - m(D\cap E)\int_{\mathbb{R}^N}\varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y})\right|.\end{aligned}$$

By the way we defined $\mathbf{g}_j$, the integrand in the first integral is bounded by ε for each j. Also, by (1), we see that

$$\lim_{j\to\infty} m\left(A_{k,j}\right) = m(D\cap E)\nu(Q_k).$$

Using this and the definition of $\mathbf{g}_j$ in the inequality obtained above, we see that

$$\begin{aligned}\limsup_{j\to\infty}&\left|\int_{D\cap E}\varphi(\mathbf{f}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - m(D\cap E)\int_{\mathbb{R}^N}\varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y})\right| \\ &\le m(D\cap E)\varepsilon + \left|\sum_{k=1}^{M}\varphi(\mathbf{a}_k)m(D\cap E)\nu(Q_k) - m(D\cap E)\int_{\mathbb{R}^N}\varphi(\mathbf{y})\mathrm{d}\nu(\mathbf{y})\right| \\ &\le m(D\cap E)\varepsilon + m(D\cap E)\sum_{k=1}^{K}\int_{Q_k}|\varphi(\mathbf{a}_k)-\varphi(\mathbf{y})|\,\mathrm{d}\nu(\mathbf{y}) \\ &\quad + m(D\cap E)\int_{\mathbb{R}^N\setminus\cup_{k=1}^{\infty}Q_k}|\varphi(\mathbf{y})|\,\mathrm{d}\nu(\mathbf{y}) \\ &\le 2m(D\cap E)\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain the equality in (2) when $\xi = \chi_{D\cap E}$ for some cube $D \subset \mathbb{R}^n$. Letting $\mathcal{T}$ be the set of all finite linear combinations of functions of the form $\chi_{D\cap E}$, we see that (2) also holds for every $\xi \in \mathcal{T}$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$. Since $\mathcal{T}$ is dense in $L^1(E)$, it follows that $\{\mathbf{f}_j\}$ generates ν. □

Lemma 3.2. *Let $\nu \in \mathcal{M}(\mathbb{R}^N)$ be a probability measure. There is a measurable function $\mathbf{g} : (0,1) \to \mathbb{R}^N$ such that*

$$\nu(E) = m(\mathbf{g}^{-1}(E))$$

for every Borel set $E \subset \mathbb{R}^N$. Moreover, $\mathbf{g}$ satisfies

$$\int_{(0,1)} |\mathbf{g}(x)|^p\,\mathrm{d}x = \int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu(\mathbf{y})$$

for every $1 \le p < \infty$.

Proof. First we construct the function $\mathbf{g}$. Let $\{\mathbf{a}_{1,j}\}_{j=1}^{\infty}$ be an enumeration of $\mathbb{Z}^N$, and for each $j \in \mathbb{N}$, we define $D_{1,j} := \mathbf{a}_{1,j} + [0,1)^N$. Note that $\{D_{1,j}\}_{j=1}^{\infty}$ partitions $\mathbb{R}^N$. Assuming that the cubes $\{D_{k,j}\}_{j=1}^{\infty}$ have been chosen for some $k \in \mathbb{N}$, we partition each of these cubes into 2^N subcubes, each subcube having the form

$D_{k+1,j} = \mathbf{a}_{k+1,j} + [0, 2^{-k})^N$ for some $\mathbf{a}_{k+1,j} \in 2^{-k}\mathbb{Z}^N$. We thus obtain a collection of dyadic cubes $\{D_{k+1,j}\}_{j=1}^{\infty}$ that partitions $\mathbb{R}^N$, and each cube has edges of length 2^{-k}. Furthermore, we stipulate that we first partition $D_{k,1}$ into subcubes, then $D_{k,2}$, and so on. That is, we require

$$\bigcup_{l=(j-1)2^N+1}^{j2^N} D_{k+1,l} = D_{k,j}, \tag{3}$$

for every k, $j \in \mathbb{N}$.

For each k, we partition the interval $[0,1)$ into a family of intervals $\{I_{k,j}\}_{j=1}^{\infty}$ as follows. Define

$$I_{k,j} := \left[\nu\left(\cup_{l=1}^{j-1} D_{k,l}\right), \nu\left(\cup_{l=1}^{j} D_{k,l}\right)\right). \tag{4}$$

Note that the length of $I_{k,j}$ is $\nu(D_{k,j})$. (Here we are using the convention that the interval $[a,a) := \emptyset$.)

Now we define a sequence $\{\mathbf{g}_k\}_{k=1}^{\infty}$ of functions mapping $[0,1)$ into $\mathbb{R}^N$ by

$$\mathbf{g}_k(x) := \sum_{j=1}^{\infty} \mathbf{a}_{k,j}\chi_{I_{k,j}}(x). \tag{5}$$

We will show this sequence is Cauchy in the uniform norm. To this end, fix $\varepsilon > 0$, and find $K \in \mathbb{N}$ such that $\operatorname{diam}(D_{K,j}) < \varepsilon$. Now fix $x \in (0,1)$. Then there is a unique $j_1 \in \mathbb{N}$ such that

$$x \in I_{1,j_1} = \left[\nu\left(\cup_{l=1}^{j_1-1} D_{1,l}\right), \nu\left(\cup_{l=1}^{j_1} D_{1,l}\right)\right).$$

Recalling the way that $\mathbf{g}_1$ is defined, we see that $\mathbf{g}_1(x) = \mathbf{a}_{1,j_1} \in D_{1,j_1}$. By (3) and (4), we have that

$$I_{1,j_1} = \bigcup_{j=(j_1-1)2^N+1}^{j_1 2^N} I_{2,j},$$

whence $x \in I_{2,j_2}$ for some $(j_1-1)2^N + 1 \le j_2 \le j_1 2^N$ and $\mathbf{g}_2(x) \in D_{2,j_2} \subset D_{1,j_1}$. Proceeding inductively, we obtain $\{j_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\mathbf{g}_k(x) \in D_{k,j_k}$ and $D_{1,j_1} \supset \cdots \supset D_{k,j_k} \supset \cdots$. Recalling that we chose $K \in \mathbb{N}$ so that $\operatorname{diam}(D_{K,j}) < \varepsilon$, we see that if k_1 and k_2 are both at least K, then $\mathbf{g}_{k_1}(x)$ and $\mathbf{g}_{k_2}(x)$ belong to the same cube $D_{K,j}$ for some $j \in \mathbb{N}$, and hence $|\mathbf{g}_{k_1}(x) - \mathbf{g}_{k_2}(x)| < \varepsilon$. Hence $\{\mathbf{g}_k\}$ is Cauchy in the uniform norm, and so there is some $\mathbf{g} : (0,1) \to \mathbb{R}^N$ such that $\mathbf{g}_k \to \mathbf{g}$ uniformly as $k \to \infty$.

We claim that the function $\mathbf{g}$ so constructed satisfies the conclusions of the lemma. First note that if $D = \cup_{i=1}^{J} D_{k_i,j_i}$ is a finite (disjoint) union of cubes of the form $D_{k,j}$, then for $l \ge \max\{k_i\}$, we have by (5) that

$$\mathbf{g}_l^{-1}(D) = \bigcup_{i=1}^{J} I_{k_i,j_i}.$$

Since $\mathbf{g}_l \to \mathbf{g}$ uniformly, we therefore have

$$\mathbf{g}^{-1}(\overline{D}) \supset \bigcup_{i=1}^{J} I_{k_i,j_i}. \tag{6}$$

Note that, by (4), we have $m(I_{k_i,j_i}) = \nu(D_{k_i,j_i})$, so from (6) we obtain

$$m(\mathbf{g}^{-1}(\overline{D})) \geq \nu(D). \tag{7}$$

Now, for fixed $k, j \in \mathbb{N}$, we can find a sequence of cubes $\{D_i\}_{i=1}^{\infty}$ such that for each $i \in \mathbb{N}$, the cube D_i is a finite union of cubes of the form $D_{s,t}$ with $D_i \subset D_{i+1} \subset D_{k,j}$, and

$$\bigcup_{i=1}^{\infty} \overline{D_i} = \bigcup_{i=1}^{\infty} D_i = D_{k,j}. \tag{8}$$

Using (7) along with (8), we thus obtain

$$m(\mathbf{g}^{-1}(D_{k,j})) = \lim_{i\to\infty} m(\mathbf{g}^{-1}(\overline{D_i})) \geq \lim_{i\to\infty} \nu(D_i) = \nu(D_{k,j}).$$

Since the cubes $\{D_{k,j}\}_{k,j\in\mathbb{N}}$ are a generating set for the Borel σ-algebra on $\mathbb{R}^N$, the preceding inequality implies

$$m(\mathbf{g}^{-1}(E)) \geq \nu(E) \tag{9}$$

for all Borel sets $E \subset \mathbb{R}^N$. Since ν and the image of m under $\mathbf{g}$ are both probability measures on $\mathbb{R}^N$, the inequality in (9) must in fact be an equality; that is, we must have

$$m(\mathbf{g}^{-1}(E)) = \nu(E) \tag{10}$$

for all Borel sets $E \subset \mathbb{R}^N$, which establishes the first statement of the lemma.

If $\psi : \mathbb{R}^N \to \mathbb{R}$ is a simple function, then (10) gives

$$\int_{(0,1)} \psi(\mathbf{g}(x))\mathrm{d}x = \int_{\mathbb{R}^N} \psi(\mathbf{y})\mathrm{d}\nu(\mathbf{y}).$$

By taking a sequence of simple functions $\{\psi_j\}_{j=1}^{\infty}$ increasing to the function $\mathbf{y} \mapsto |\mathbf{y}|^p$ and using the above equality and the monotone convergence theorem, we find that

$$\int_{(0,1)} |\mathbf{g}(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}),$$

which concludes the proof of the lemma. □

Now we use the function $\mathbf{g}$ given by Lemma 3.2 to build a sequence of functions uniformly bounded in $L^{p,\lambda}(Q;\mathbb{R}^N)$ that generates the measure ν.

Theorem 3.1. *Suppose that $\nu \in \mathcal{M}(\mathbb{R}^N)$ is a probability measure that satisfies*

$$\int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}) < \infty.$$

For each $0 \leq \lambda < n$, there is a p-equiintegrable sequence of functions $\{\mathbf{f}_j\}_{j=1}^{\infty}$ that generates ν, is uniformly bounded in $L^{p,\mu}(Q,\mathbb{R}^N)$ for every $0 \leq \mu < n$, and satisfies

$$\|\mathbf{f}_j\|_{L^{p,\lambda}}^p \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}).$$

Proof. Let $\mathbf{g}$ be the function given by Lemma 3.2, and extend it by periodicity to all of $\mathbb{R}$. Let $\{\mathbf{g}_j\}_{j=1}^{\infty}$ be the sequence of functions mapping Q into $\mathbb{R}^N$ defined by

$$\mathbf{g}_j(\mathbf{x}) = \mathbf{g}(jx_1).$$

Then using a change of variables, the periodicity of $\mathbf{g}$, and Lemma 3.2, we see that

$$\begin{aligned}\|\mathbf{g}_j\|^p_{L^p(Q;\mathbb{R}^N)} &= \int\limits_{(0,1)} |\mathbf{g}(jx_1)|^p \,\mathrm{d}x_1 = \frac{1}{j} \int\limits_{(0,j)} |\mathbf{g}(x_1)|^p \,\mathrm{d}x_1 = \int\limits_{(0,1)} |\mathbf{g}(x_1)|^p \,\mathrm{d}x_1 \\ &= \int\limits_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}).\end{aligned} \tag{11}$$

If $D \subset \mathbb{R}^n$ is a cube and $G \subset \mathbb{R}^N$ is a Borel set, then

$$\begin{aligned}\lim_{j\to\infty} m(D \cap Q \cap \mathbf{g}_j^{-1}(G)) &= m(D \cap Q) m(\mathbf{g}^{-1}(G)) \\ &= m(D \cap Q)\nu(G),\end{aligned}$$

where we have employed Lemma 3.2 to obtain the last equality. Therefore, by Lemma 3.1, the sequence $\{\mathbf{g}_j\}$ generates the Young measure ν.

We now define a new sequence of functions $\{\mathbf{f}_j\}_{j=1}^\infty$ that are truncations of the functions $\mathbf{g}_j$:

$$\mathbf{f}_j(\mathbf{x}) := \operatorname{sgn}(\mathbf{g}_j(\mathbf{x})) \min\left\{\frac{\log(j)}{C_\lambda}, |\mathbf{g}_j(\mathbf{x})|\right\},$$

where for each $0 \le \mu < n$ we have put

$$C_\mu := \left(\max_{j\in\mathbb{N}}\left\{\frac{\{\log(j)\}^p}{2^n j^{n-\mu} \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y})}\right\}\right)^{\frac{1}{p}}.$$

Then $\{\mathbf{f}_j\}$ generates the measure ν by Theorem 2.3, since the measure of the set where $\mathbf{f}_j \neq \mathbf{g}_j$ tends to 0 as $j \to \infty$. It is easily seen that $\{\mathbf{g}_j\}$, and hence also $\{\mathbf{f}_j\}$, is p-equiintegrable. Furthermore, $\{\mathbf{f}_j\}$ is uniformly bounded in $L^{p,\mu}$ for every $0 \le \mu < n$. To see this, fix $0 < \rho \le 1$, $\mathbf{x}_0 \in Q$, and $j \in \mathbb{N}$. We consider two cases.
Case 1: $0 < \rho \le 1/j$.
In this case, using the bound $|\mathbf{f}_j| \le \log(j)/C_\lambda$, we have

$$\begin{aligned}\rho^{-\mu}\int_{Q\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} &\le \rho^{n-\mu}\left\{\frac{\log(j)}{C_\lambda}\right\}^p \le \frac{\{\log(j)\}^p}{C_\lambda^p j^{n-\mu}} \\ &\le 2^n\left(\frac{C_\mu}{C_\lambda}\right)^p \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}).\end{aligned}$$

Case 2: $k/j < \rho \le (k+1)/j$ for some $k \in \{1, 2, \cdots, j-1\}$.
Using the inequality $|\mathbf{f}_j| \le |\mathbf{g}_j|$ and the periodicity of $\mathbf{g}_j$, we obtain

$$\begin{aligned}\rho^{-\mu}\int_{Q\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} &\le \rho^{-\mu}\int_{Q\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{g}_j(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \\ &\le \rho^{-\mu}(k+1)^n \int_{Q\cap Q_{\mathbf{x}_0,\frac{1}{j}}} |\mathbf{g}_j(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \\ &\le \rho^{-\mu}\left(\frac{k+1}{j}\right)^n \|\mathbf{g}_j\|^p_{L^p(Q;\mathbb{R}^N)}.\end{aligned}$$

But $\rho^{-\mu} \le \rho^{-n} \le (j/k)^n$; using this and (11) in the above inequality, we see that

$$\rho^{-\mu}\int_{Q\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \le 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}).$$

Collecting the estimates we obtained in each case, we have shown that

$$\|\mathbf{f}_j\|^p_{L^{p,\mu}} \leq 2^n \max\left\{\left(\frac{C_\mu}{C_\lambda}\right)^p, 1\right\} \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}),$$

which implies that $\{\mathbf{f}_j\}$ is uniformly bounded in $L^{p,\mu}(Q;\mathbb{R}^N)$ for each $0 \leq \mu < n$; taking $\mu = \lambda$ yields

$$\|\mathbf{f}_j\|^p_{L^{p,\lambda}} \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu(\mathbf{y}),$$

which concludes the proof. □

3.2. **Nonhomogeneous measures.** We now extend the result just proved to the nonhomogeneous case.

Theorem 3.2. *Let $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ be a Young measure on $\mathbb{R}^N$, and let $0 \leq \lambda < n$. There is a p-equiintegrable sequence of functions $\{\mathbf{f}_j\}_{j=1}^\infty$ that is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$ and generates the measure ν if and only if ν satisfies*

$$\sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} < \infty. \tag{12}$$

Proof. The necessity follows from Theorem 2.1. Indeed, for fixed $\mathbf{x}_0 \in \Omega$ and $\rho > 0$, define $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ by $f(\mathbf{x},\mathbf{y}) = \rho^{-\lambda}\chi_{\Omega\cap Q_{\mathbf{x}_0,\rho}}(\mathbf{x})\,|\mathbf{y}|^p$. The aforementioned theorem now yields

$$\liminf_{j\to\infty} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\mathbf{f}_j|^p \,\mathrm{d}\mathbf{x} \geq \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.$$

Since the sequence $\{\mathbf{f}_j\}$ is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$, it follows that ν satisfies (12).

We now turn to the sufficiency. For each $k \in \mathbb{N}$, let $\{\mathbf{a}_{k,i}\}_{i=1}^\infty$ be an enumeration of the set $2^{-k}\mathbb{Z}^n$, and define

$$D_{k,i} = \mathbf{a}_{k,i} + 2^{-k}[0,1)^n.$$

Note that for each k, $\{D_{k,i}\}_{i=1}^\infty$ is a set of dyadic cubes that partition $\mathbb{R}^n$ and have edges of length 2^{-k}. For each k, there will be finitely many of these cubes that are entirely contained in Ω. We will define $A_k \subset \mathbb{N}$ to be the collection of the indices of such cubes:

$$A_k := \{i \in \mathbb{N} : D_{k,i} \subset \Omega\}.$$

Since Ω is open, it is easy to see that

$$\bigcup_{k=1}^\infty \bigcup_{i\in A_k} D_{k,i} = \Omega.$$

For each $k \in \mathbb{N}$ and i in A_k, we define the probability measure $\nu_{k,i} \in \mathcal{M}(\mathbb{R}^N)$ to be the "average" of the measures $\{\nu_{\mathbf{x}}\}_{\mathbf{x}\in D_{k,i}}$. That is, we select $\nu_{k,i}$ so that for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu_{k,i}(\mathbf{y}) = \fint_{D_{k,i}} \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}. \tag{13}$$

The existence of such a measure is guaranteed by Theorem 7.1 in [12]. Note that this measure is homogeneous, and that

$$\int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{k,i}(\mathbf{y}) = \fint_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} = 2^{kn} \int_{D_{k,i}} \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x} < \infty. \tag{14}$$

Using Theorem 3.1, we find a p-equiintegrable sequence $\{\mathbf{f}_j^{k,i}\}_{j=1}^\infty \subset L^{p,\lambda}(Q;\mathbb{R}^N)$ that satisfies

$$\left\|\mathbf{f}_j^{k,i}\right\|_{L^{p,\lambda}}^p \leq 2^n \int_{\mathbb{R}^N} |\mathbf{y}|^p \, \mathrm{d}\nu_{k,i}(\mathbf{y}) \tag{15}$$

and generates the measure $\nu_{k,i}$. For a given $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$, we denote by $\overline{\varphi}$ and $\overline{\varphi_k}$ the functions defined by

$$\begin{aligned} \overline{\varphi}(\mathbf{x}) &:= \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}); \\ \overline{\varphi_k}(\mathbf{x}) &:= \sum_{i\in A_k} \chi_{D_{k,i}}(\mathbf{x}) \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu_{k,i}(\mathbf{y}). \end{aligned}$$

Notice that both $\overline{\varphi}$ and $\overline{\varphi_k}$ are functions belonging to $L^\infty(\Omega)$, with essential supremums at most $\|\varphi\|_{L^\infty(\mathbb{R}^N)}$. If $\mathbf{x} \in \Omega$, then for each k sufficiently large, there is a unique $i(k) \in A_k$ such that $\mathbf{x} \in D_{k,i(k)}$. And if $\mathbf{x}$ is a Lebesgue point of $\overline{\varphi}$, we use (13) to compute

$$\begin{aligned} \lim_{k\to\infty} \overline{\varphi_k}(\mathbf{x}) = \lim_{k\to\infty} \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu_{k,i(k)}(\mathbf{y}) &= \lim_{k\to\infty} \fint_{D_{k,i(k)}} \int_{\mathbb{R}^N} \varphi(\mathbf{y})\mathrm{d}\nu_{\hat{\mathbf{x}}}(\mathbf{y})\mathrm{d}\hat{\mathbf{x}} \\ &= \lim_{k\to\infty} \fint_{D_{k,i(k)}} \overline{\varphi}(\hat{\mathbf{x}})\mathrm{d}\hat{\mathbf{x}} = \overline{\varphi}(\mathbf{x}). \end{aligned}$$

Therefore $\overline{\varphi_k}$ converges to $\overline{\varphi}$ pointwise almost everywhere. We have already noted that $|\overline{\varphi_k}| \leq \|\varphi\|_{L^\infty}$, so we can use Lebesgue's Dominated Convergence Theorem to show that $\overline{\varphi_k}$ converges to $\overline{\varphi}$ in $L^1(\Omega)$. Therefore we have that

$$\lim_{k\to\infty} \int_\Omega \xi(\mathbf{x})\overline{\varphi_k}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega \xi(\mathbf{x})\overline{\varphi}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{16}$$

for every $\varphi \in \mathcal{C}_0(\mathbb{R}^N)$ and $\xi \in L^\infty(\Omega)$.

Let $\{\xi_k\}_{k=1}^\infty \subset L^\infty(\Omega)$ and $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{C}_0(\mathbb{R}^N)$ be countable dense subsets of $L^1(\Omega)$ and $\mathcal{C}_0(\mathbb{R}^N)$, respectively. Let $\varphi_0 \in \mathcal{C}(\mathbb{R}^N)$ and $\xi_0 \in L^\infty(\Omega)$ be defined by $\varphi_0(\mathbf{y}) := |\mathbf{y}|^p$ and $\xi_0 \equiv 1$. For each $k \in \mathbb{N}$ and $i \in A_k$, since the p-equiintegrable sequence $\{\mathbf{f}_j^{k,i}\}_{j=1}^\infty$ generates the measure $\nu_{k,i}$, we can choose $j = j(k,i)$ such that

$$\left| \int_Q \xi_s(\mathbf{a}_{k,i}+2^{-k}\mathbf{x})\varphi_t(\mathbf{f}_j^{k,i}(\mathbf{x}))\mathrm{d}\mathbf{x} - \int_Q \xi_s(\mathbf{a}_{k,i}+2^{-k}\mathbf{x})\mathrm{d}\mathbf{x} \int_{\mathbb{R}^N} \varphi_t(\mathbf{y})\mathrm{d}\nu_{k,i}(\mathbf{y}) \right| \leq \frac{2^{nk-i}}{k}$$

for $0 \leq s,t \leq k$. With j chosen in this way, we define the sequence of functions $\{\mathbf{f}_k\}_{k=1}^\infty$ in the following way:

$$\mathbf{f}_k(\mathbf{x}) := \begin{cases} \mathbf{f}_j^{k,i}(\frac{\mathbf{x}-\mathbf{a}_{k,i}}{2^{-k}}) & \text{if } \mathbf{x} \in D_{k,i} \text{ for some } i \in A_k \\ 0 & \text{if } \mathbf{x} \in \Omega \setminus \cup_{i\in A_k} D_{k,i} \end{cases}. \tag{17}$$

Fix $s,t\in\mathbb{N}$, and suppose that $k\ge\max\{s,t\}$. Using the definition of $\mathbf{f}_k$ and changing variables in the preceding inequality yields

$$\left|\int_{D_{k,i}}\xi_s(\mathbf{x})\varphi_t(\mathbf{f}_k(\mathbf{x}))\mathrm{d}\mathbf{x}-\int_{D_{k,i}}\xi_s(\mathbf{x})\mathrm{d}\mathbf{x}\int_{\mathbb{R}^N}\varphi_t(\mathbf{y})\mathrm{d}\nu_{k,i}(\mathbf{y})\right|\le\frac{2^{-i}}{k}$$

whenever $i\in A_k$. Hence

$$\left|\int_{\cup_{i\in A_k}D_{k,i}}\xi_s(\mathbf{x})\varphi_t(\mathbf{f}_k(\mathbf{x}))\mathrm{d}\mathbf{x}-\int_\Omega\xi_s(\mathbf{x})\overline{(\varphi_t)_k}(\mathbf{x})\mathrm{d}\mathbf{x}\right|\le\frac{1}{k}. \tag{18}$$

So for each $s,t\in\mathbb{N}$, using (18) and (16), we have

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega\xi_s(\mathbf{x})\varphi_t(\mathbf{f}_k(\mathbf{x}))\mathrm{d}\mathbf{x}&=\lim_{k\to\infty}\int_\Omega\xi_s(\mathbf{x})\overline{(\varphi_t)_k}(\mathbf{x})\mathrm{d}\mathbf{x}\\&=\int_\Omega\xi_s(\mathbf{x})\overline{\varphi_t}(\mathbf{x})\mathrm{d}\mathbf{x}\\&=\int_\Omega\xi_s(\mathbf{x})\int\limits_{\mathbb{R}^N}\varphi_t(\mathbf{y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.\end{aligned}$$

This implies that the sequence $\{\mathbf{f}_k\}_{k=1}^\infty$ generates $\{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$.

Using the same steps we used to arrive at (18), this time taking $s=t=0$ and recalling that $\mathbf{f}_k=0$ in $\Omega\setminus\cup_{i\in A_k}D_{k,i}$, we obtain

$$\left|\int_\Omega|\mathbf{f}_k(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}-\sum_{i\in A_k}\int_{D_{k,i}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{k,i}(\mathbf{y})\mathrm{d}\mathbf{x}\right|\le\frac{1}{k}. \tag{19}$$

Using (14), we have

$$\begin{aligned}\sum_{i\in A_k}\int_{D_{k,i}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{k,i}(\mathbf{y})\mathrm{d}\mathbf{x}&=\sum_{i\in A_k}\int_{D_{k,i}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}\\&=\int_{\cup_{i\in A_k}D_{k,i}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.\end{aligned} \tag{20}$$

By (19) and (20), we have

$$\lim_{k\to\infty}\int_\Omega|\mathbf{f}_k(\mathbf{x})|^p\,\mathrm{d}\mathbf{x}=\int_\Omega\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}<\infty,$$

and hence the p-equiintegrability of $\{\mathbf{f}_k\}$ follows from Theorem 2.1. We only have yet to show that this sequence is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^N)$.

To this end, fix $\mathbf{x}_0\in\Omega$, $\rho>0$, and $k\in\mathbb{N}$. We consider two cases.

Case 1: $0<\rho\le2^{-k}$.

Let $J:=\{i\in A_k:D_{k,i}\cap Q_{\mathbf{x}_0,\rho}\ne\emptyset\}$. Note that there are at most 2^n elements in J, since $\rho\le2^{-k}$. By changing variables and using the fact that $\mathbf{f}_k=0$ outside of $\cup_{i\in A_k}D_{k,i}$, we obtain

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}}|\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x}=\rho^{-\lambda}\sum_{i\in J}2^{-kn}\int_{Q\cap Q_{2^k(\mathbf{x}_0-\mathbf{a}_{k,i}),2^k\rho}}\left|\mathbf{f}_j^{k,i}\right|^p\mathrm{d}\mathbf{x}.$$

By (15), we have

$$\int_{Q\cap Q_{2^k(\mathbf{x}_0-\mathbf{a}_{k,i}),2^k\rho}}\left|\mathbf{f}_j^{k,i}\right|^p\mathrm{d}\mathbf{x}\le2^n(2^k\rho)^\lambda\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{k,i}(\mathbf{y}).$$

Using this inequality in the one preceding it and then employing (14), we find that

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}}|\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x}\le 2^n\sum_{i\in J}(2^k)^{\lambda-n}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{k,i}(\mathbf{y})$$
$$=2^n\sum_{i\in J}(2^{-k})^{-\lambda}\int_{D_{k,i}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}.$$

But $D_{k,i}=Q_{\mathbf{y}_{k,i},2^{-k}}$ for some $\mathbf{y}_{k,i}\in\Omega$, so the above inequality gives

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}}|\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x}\le 2^n\sum_{i\in J}M,$$

where

$$M:=\sup_{\substack{\mathbf{x}_0\in\Omega\\ R>0}}R^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,R}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}<\infty;$$

therefore, since $|J|\le 2^n$,

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}}|\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x}\le 4^nM.$$

Case 2: $\rho>2^{-k}$.

In this case, we can find a cube Q_0 containing $Q_{\mathbf{x}_0,\rho}$ that is comprised of cubes of the form $D_{k,i}$, and such that the sides of Q_0 have length of at most 2ρ. Letting $\mathbf{y}_0$ denote the center of Q_0, we can break up $Q_{\mathbf{x}_0,\rho}$ into cubes $D_{k,i}$ as we did in Case 1 and perform a similar computation to obtain

$$\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{x}_0,\rho}}|\mathbf{f}_k|^p\,\mathrm{d}\mathbf{x}\le\rho^{-\lambda}\int_{\Omega\cap Q_{\mathbf{y}_0,2\rho}}\int_{\mathbb{R}^N}|\mathbf{y}|^p\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\mathrm{d}\mathbf{x}\le 2^{\lambda}M.$$

From the estimates we obtained in each case, we see that

$$\|\mathbf{f}_k\|^p_{L^{p,\lambda}(\Omega;\mathbb{R}^N)}\le 4^nM,$$

which finishes the proof. □

4. Gradient Young Measures

We now turn our attention to Young measures generated by sequences of gradients bounded in $L^{p,\lambda}$. To simplify the statements of the theorems, for this section we will assume that Ω has Lipschitz-continuous boundary. The following lemma can be deduced from Theorem 8.16 and Lemma 8.3 in [12].

Lemma 4.1. *Suppose that $\nu\in\mathcal{M}(\mathbb{R}^{N\times n})$ is a probability measure that satisfies*

(i) $\int_{\mathbb{R}^{N\times n}}\mathbf{Y}\mathrm{d}\nu(\mathbf{Y})=\mathbf{0}$;
(ii) $\int_{\mathbb{R}^{N\times n}}\varphi(\mathbf{Y})\mathrm{d}\nu(\mathbf{Y})\ge\varphi(\mathbf{0})$ *for every quasiconvex* $\varphi:\mathbb{R}^{N\times n}\to[0,\infty)$ *satisfying* $\varphi(\mathbf{Y})\le C(1+|\mathbf{Y}|^p)$;
(iii) $\int_{\mathbb{R}^{N\times n}}|\mathbf{Y}|^p\,\mathrm{d}\nu(\mathbf{Y})<\infty$.

Then there exists a sequence of functions $\{\mathbf{u}_j\}_{j=1}^{\infty}\subset W_0^{1,p}(Q;\mathbb{R}^N)$ such that the sequence $\{\nabla\mathbf{u}_j\}_{j=1}^{\infty}$ generates the measure ν and is p-equiintegrable.

To find Morrey regular sequences generating the measure, we will need the following lemma, which allows us to generate certain Young measures by p-equiintegrable gradients of Lipschitz functions.

Lemma 4.2. *Let $\{\mathbf{u}_j\}_{j=1}^\infty$ be a bounded sequence in $W_0^{1,p}(Q;\mathbb{R}^N)$ for some $1 \leq p < \infty$, and suppose that $\{\nabla \mathbf{u}_j\}_{j=1}^\infty$ generates the measure $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in Q}$ and is p-equiintegrable. Let $\{T_j\}_{j=1}^\infty$ be a sequence of non-negative numbers such that $\lim_{j\to\infty} T_j = +\infty$. Then there is a sequence of functions $\{\mathbf{v}_j\}_{j=1}^\infty \subset W_0^{1,\infty}(Q;\mathbb{R}^N)$ such that*

(i) *$\{\nabla \mathbf{v}_j\}_{j=1}^\infty$ generates ν and is p-equiintegrable;*
(ii) *$\|\nabla \mathbf{v}_j\|_{L^\infty} \leq T_j$;*

Proof. Since $\{\mathbf{u}_j\}_{j=1}^\infty \subset W_0^{1,p}(Q;\mathbb{R}^N)$, we can extend each $\mathbf{u}_j$ by zero to all of $\mathbb{R}^n$ so that $\mathbf{u}_j \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^N)$. For each j, we define the set A_j by

$$A_j := \{\mathbf{x} \in \mathbb{R}^n : M(|\nabla \mathbf{u}_j|)(\mathbf{x}) \leq T_j\};$$

by Theorem 2.2, there is a sequence of Lipschitz functions $\{\mathbf{w}_j\}_{j=1}^\infty$ and a constant $c = c(n, N, p)$ such that

(i) $\mathbf{u}_j = \mathbf{w}_j$ and $\nabla \mathbf{u}_j = \nabla \mathbf{w}_j$ almost everywhere on A_j;
(ii) $\|\nabla \mathbf{w}_j\|_{L^\infty} \leq T_j$;
(iii) $m(\mathbb{R}^n \setminus A_j) \leq cT_j^{-p} \int_{\{|\nabla \mathbf{u}_j| > T_j/c\}} |\nabla \mathbf{u}_j|^p \,\mathrm{d}\mathbf{x}$.

Combining (ii) and (iii) gives

$$\int_{\mathbb{R}^n \setminus A_j} |\nabla \mathbf{w}_j|^p \,\mathrm{d}\mathbf{x} \leq c \int_{|\nabla \mathbf{u}_j| > T_j/c} |\nabla \mathbf{u}_j|^p \,\mathrm{d}\mathbf{x}.$$

Therefore, since $\nabla \mathbf{w}_j = \nabla \mathbf{u}_j$ almost everwhere on A_j, the above inequality and the p-equiintegability of $\{\nabla \mathbf{u}_j\}_{j=1}^\infty$ yield the p-equiintegrability of $\{\nabla \mathbf{w}_j\}_{j=1}^\infty$. If we simply restrict $\mathbf{w}_j$ to Q, we see that $\{\nabla \mathbf{w}_j\}$ generates ν, is p-equiintegrable, and satisfies the appropriate L^∞ estimates, but we do not necessarily have that $\mathbf{w}_j$ has zero trace on ∂Q. However, using the definitions of A_j and the maximal function, we have

$$A_j \supset \left\{ \mathbf{x} \in \mathbb{R}^n : \frac{L}{\operatorname{dist}(\mathbf{x}, Q)^n} \leq T_j \right\}$$

for some $L < \infty$; since $\mathbf{w}_j = \mathbf{u}_j$ almost everywhere on A_j and $\mathbf{u}_j = 0$ outside Q, we obtain a sequence $\{r_j\}_{j=1}^\infty \subset [1, \infty)$ and a sequence of cubes $\{Q_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} r_j = 1$, $Q \subset Q_j$, the sides of Q_j have length r_j, and $\mathbf{w}_j \in W_0^{1,\infty}(Q_j;\mathbb{R}^N)$. Furthermore, we can assume that Q_j has sides parallel to the axes and has center at $(1/2, 1/2, \cdots, 1/2)$. Letting $\mathbf{x}_j := ((1 - r_j)/2, (1 - r_j)/2, \cdots, (1 - r_j)/2)$ (i.e. $\mathbf{x}_j$ is the corner of Q_j in which each coordinate is minimized), we can define $\mathbf{v}_j \subset W_0^{1,\infty}(Q;\mathbb{R}^N)$ to be a rescaled version of $\mathbf{w}_j$:

$$\mathbf{v}_j(\mathbf{x}) := \frac{1}{r_j} \mathbf{w}_j\left(\mathbf{x}_j + r_j \mathbf{x}\right).$$

Note that $\nabla \mathbf{v}_j(\mathbf{x}) = \nabla \mathbf{w}_j(\mathbf{x}_j + r_j \mathbf{x})$. Thus the p-equiintegrability of $\{\nabla \mathbf{v}_j\}_{j=1}^\infty$ follows from the p-equiintegrability of $\{\nabla \mathbf{w}_j\}_{j=1}^\infty$, and $\|\nabla \mathbf{v}_j\|_{L^\infty} \leq T_j$. Therefore the only thing we have yet to show is that $\{\nabla \mathbf{v}_j\}_{j=1}^\infty$ generates the Young measure ν. It suffices to show that

$$\lim_{j\to\infty} \int_{Q_0} \{\varphi(\nabla \mathbf{w}_j(\mathbf{x})) - \varphi(\nabla \mathbf{v}_j(\mathbf{x}))\} \mathrm{d}\mathbf{x} = 0 \tag{21}$$

for every cube $Q_0 \subset Q$ and every $\varphi \in \mathcal{C}_0(\mathbb{R}^{N\times n})$, since from this it follows that $\{\nabla \mathbf{w}_j\}$ and $\{\nabla \mathbf{v}_j\}$ generate the same Young measure. Using a change of variables and letting $\tilde{Q}_j := \mathbf{x}_j + r_j Q_0$, we obtain

$$\int\limits_{Q_0} \varphi(\nabla \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - \int\limits_{Q_0} \varphi(\nabla \mathbf{v}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \int\limits_{Q_0} \varphi(\nabla \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x} - r_j^{-n} \int\limits_{\tilde{Q}_j} \varphi(\nabla \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x}.$$

Since $r_j \to 1$ and $\mathbf{x}_j \to (0, 0, \cdots, 0)$, it is easily seen that

$$m((Q_0 \setminus \tilde{Q}_j) \cup (\tilde{Q}_j \setminus Q_0)) \to 0.$$

Using this and that $r_j^{-n} \to 1$ in the above equality yields (21), which finishes the proof. $\square$

Lemma 4.3. *Suppose that $\nu \in \mathcal{M}(\mathbb{R}^{N\times n})$ is a probability measure that satisfies*

(i) $\int_{\mathbb{R}^{N\times n}} \mathbf{Y}\mathrm{d}\nu(\mathbf{Y}) = \mathbf{0}$*;*
(ii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{Y})\mathrm{d}\nu(\mathbf{Y}) \geq \varphi(\mathbf{0})$ *for every quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0, \infty)$ *satisfying* $\varphi(\mathbf{Y}) \leq C(1 + |\mathbf{Y}|^p)$*;*
(iii) $\int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu(\mathbf{Y}) < \infty$.

For each $0 \leq \lambda < n$, there is a sequence of functions $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^{1,p}(\Omega; \mathbb{R}^N)$ uniformly bounded in $W^{1,(p,\mu)}(Q; \mathbb{R}^N)$ for every $0 \leq \mu < n$ such that $\{\nabla \mathbf{u}_j\}$ is p-equiintegrable, generates the measure ν, and satisfies

$$\|\nabla \mathbf{u}_j\|^p_{L^{p,\lambda}(Q;\mathbb{R}^{N\times n})} \leq 2^{n+1} \int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu(\mathbf{Y}).$$

Remark 4.1. Using the continuous embedding

$$W^{1,(p,\mu)}(Q; \mathbb{R}^N) \hookrightarrow \mathcal{C}^{0,1-(n-\mu)/p}(Q; \mathbb{R}^N),$$

where $\mathcal{C}^{0,\alpha}(Q; \mathbb{R}^N)$ denotes the space of Hölder-continuous functions with exponent α, we see that $\{\mathbf{u}_j\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(Q; \mathbb{R}^N)$ for each $0 \leq \alpha < 1$.

Proof. By Lemma 4.1, there is a sequence $\{\mathbf{w}_j\}_{j=1}^{\infty} \subset W_0^{1,p}(Q; \mathbb{R}^N)$ such that the sequence $\{\nabla \mathbf{w}_j\}$ is p-equiintegrable and generates ν. For each $0 \leq \mu < n$, we define

$$C_\mu := \left(\max_{j\in\mathbb{N}} \left\{ \frac{\{\log(j)\}^p}{2^{n+1} j^{n-\mu} \int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu(\mathbf{Y})} \right\} \right)^{\frac{1}{p}},$$

and apply Lemma 4.2 with $T_j := \log(j)/C_\lambda$ to obtain a sequence $\{\mathbf{v}_j\}_{j=1}^{\infty} \subset W_0^{1,\infty}(Q; \mathbb{R}^N)$ such that the sequence $\{\nabla \mathbf{v}_j\}_{j=1}^{\infty}$ generates the Young measure ν, is p-equiintegrable, and satisfies

$$\|\nabla \mathbf{v}_j\|_{L^\infty} \leq \log(j)/C_\lambda.$$

Extend $\mathbf{v}_j$ by periodicity to all of $\mathbb{R}^n$ and define the new sequence $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^{1,p}(Q; \mathbb{R}^N)$ by

$$\mathbf{u}_j(\mathbf{x}) = j^{-1}\mathbf{v}_j(j\mathbf{x}).$$

Note that

$$\|\nabla \mathbf{u}_j\|_{L^p(Q;\mathbb{R}^{N\times n})} = \|\nabla \mathbf{v}_j\|_{L^p(Q;\mathbb{R}^{N\times n})}$$

and that $\{\nabla \mathbf{u}_j\}$ is also p-equiintegrable. Furthermore, it is not too difficult to see that, for any cube $D \subset Q$ and $\varphi \in \mathcal{C}_0(\mathbb{R}^{N\times n})$,

$$\lim_{j\to\infty} \left(\int_D \varphi(\nabla \mathbf{u}_j)\mathrm{d}\mathbf{x} - \int_D \varphi(\nabla \mathbf{v}_j)\mathrm{d}\mathbf{x} \right) = 0,$$

from which it follows that $\{\nabla \mathbf{u}_j\}$ generates the same Young measure as $\{\nabla \mathbf{v}_j\}$, namely ν. Since $\{\nabla \mathbf{u}_j\}$ is p-equiintegrable, Theorem 2.1 gives

$$\lim_{j\to\infty} \|\nabla \mathbf{u}_j\|_{L^p}^p = \int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu(\mathbf{Y});$$

therefore, upon taking the tail end of the sequence if necessary, we can assume without loss of generality that

$$\|\nabla \mathbf{u}_j\|_{L^p}^p \le 2\int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu(\mathbf{Y}).$$

With this observation in mind, the proof that $\{\nabla \mathbf{u}_j\}$ is uniformly bounded in $L^{p,\mu}$ for every $0 \le \mu < n$ and that $\|\nabla \mathbf{u}_j\|_{L^{p,\lambda}}^p$ satisfies the appropriate estimate proceeds in the same way as the proof of Theorem 3.1; it follows from Proposition 3.7 in [9] that $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\mu)}(\Omega;\mathbb{R}^N)$ for each $0 \le \mu < n$. ☐

Now we consider nonhomogeneous measures.

Theorem 4.1. *Suppose that Ω has Lipschitz-continuous boundary. Let $0 \le \lambda < n$ be given, and suppose that $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ is a Young measure on $\mathbb{R}^{N\times n}$ that satisfies*

(i) $\int_{\mathbb{R}^{N\times n}} \mathbf{Y}\mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y}) = \nabla\mathbf{u}(\mathbf{x})$ *for some* $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$*;*

(ii) $\int_{\mathbb{R}^{N\times n}} \varphi(\mathbf{Y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y}) \ge \varphi(\nabla\mathbf{u}(\mathbf{x}))$ *for almost every* $\mathbf{x}\in\Omega$ *and every quasiconvex* $\varphi : \mathbb{R}^{N\times n} \to [0,\infty)$ *satisfying* $\varphi(\mathbf{Y}) \le C(1+|\mathbf{Y}|^p)$*;*

(iii) $\sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y})\mathrm{d}\mathbf{x} < \infty$.

Then there is a sequence $\{\mathbf{u}_j\}_{j=1}^\infty$ uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ such that $\mathbf{u}_j - \mathbf{u} \in W_0^{1,p}(\Omega;\mathbb{R}^N)$ and the sequence of gradients $\{\nabla\mathbf{u}_j\}_{j=1}^\infty$ generates the Young measure ν and is p-equiintegrable.

Remark 4.2. It is easily seen that conditions (i), (ii), and (iii) are also necessary, and therefore these conditions characterize Young measures generated by p-equiintegrable sequences of gradients uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$.

Remark 4.3. If Ω does not have Lipschitz-continuous boundary, then the conclusion is weakened slightly; even though $\{\nabla\mathbf{u}_j\}$ is still uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$, we do not necessarily have that $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$. The rest of the conclusion remains unchanged.

Proof. First, assume that the function u appearing in (i) and (ii) is identically 0. In this case, the proof is similar to the proof of Theorem 3.2. Using the notation found there, we see that each $\nu_{k,i}$ satisfies (i), (ii), and (iii) of Lemma 4.3, and hence can be generated by a sequence of gradients $\{\nabla\mathbf{u}_j^{k,i}\}_{j=1}^\infty$ where $\{\mathbf{u}_j^{k,i}\} \subset W_0^{1,p}(Q;\mathbb{R}^N)$ is uniformly bounded in $W^{1,(p,\lambda)}(Q;\mathbb{R}^N)$. Since each $\mathbf{u}_j^{k,i} \in W_0^{1,p}(Q;\mathbb{R}^N)$, rescaling and "patching together" the gradients $\nabla\mathbf{u}_j^{k,i}$ (cf. (17) in Theorem 3.2) yields a function that is still the weak gradient of some function $\mathbf{u}_k \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. We can also show that $\{\nabla\mathbf{u}_k\}$ generates ν, is uniformly bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$, and is p-equiintegrable in the exact same way as we did in the proof of Theorem 3.2. Since Ω has Lipschitz-continuous boundary, it follows from Proposition 3.7 in [9] that $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$.

If we do not assume that $\mathbf{u} \equiv 0$, but instead that $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$, then we first note that we must in fact have that $\mathbf{u} \in W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$. Indeed, by (i) and Jensen's inequality, we have

$$\begin{aligned}\sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} |\nabla \mathbf{u}|^p \,\mathrm{d}\mathbf{x} &= \sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \left| \int_{\mathbb{R}^{N\times n}} \mathbf{Y} \mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y}) \right|^p \mathrm{d}\mathbf{x} \\ &\le \sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y})\mathrm{d}\mathbf{x},\end{aligned}$$

which is finite by (iii). Hence $\nabla\mathbf{u} \in L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$; again employing Proposition 3.7 in [9], we find that $\mathbf{u} \in W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$.

Now define the translated Young measure $\tilde{\nu} = \{\tilde{\nu}_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ by

$$\langle \tilde{\nu}_{\mathbf{x}}, \varphi\rangle := \langle \nu_{\mathbf{x}}, \varphi(\cdot - \nabla\mathbf{u}(\mathbf{x}))\rangle.$$

It is easy to check that the function $\tilde{\nu}$ is in the previous situation; i.e. $\tilde{\nu}$ satisfies (i), (ii), and (iii) with $\mathbf{u} \equiv 0$, so we obtain a sequence $\{\mathbf{v}_j\}_{j=1}^\infty \subset W_0^{1,p}(\Omega;\mathbb{R}^N)$ that is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ and such that $\{\nabla\mathbf{v}_j\}$ generates the measure $\tilde{\nu}$ and is p-equiintegrable. Now let $\mathbf{u}_j := \mathbf{v}_j + \mathbf{u}$; then $\{\mathbf{u}_j\}$ is uniformly bounded in $W^{1,(p,\lambda)}$ and $\mathbf{u}_j - \mathbf{u} = \mathbf{v}_j \in W_0^{1,p}(\Omega;\mathbb{R}^N)$. The sequence $\{\nabla\mathbf{u}_j\}$ is p-equiintegrable, and generates the original measure ν by Theorem 2.3, which completes the proof. □

5. An Application to the Calculus of Variations

Here we provide an application of Theorem 4.1 to the Calculus of Variations. Given $1 \le p < \infty$, we denote by p^* the Sobolev conjugate of p; i.e.

$$p^* := \begin{cases} \frac{np}{n-p} & \text{if } p < n; \\ +\infty & \text{if } p \ge n. \end{cases}$$

Theorem 5.1. *Suppose that Ω has Lipschitz-continuous boundary, and let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a measurable function such that $f(\mathbf{x},\cdot,\cdot)$ is continuous for almost every $\mathbf{x} \in \Omega$. Also assume that f satisfies*

$$\frac{1}{c}|\mathbf{Y}|^p - c|\mathbf{y}|^q - \alpha(\mathbf{x}) \le f(\mathbf{x},\mathbf{y},\mathbf{Y}) \le c|\mathbf{Y}|^p + c|\mathbf{y}|^r + \alpha(\mathbf{x})$$

for some $1 < p < \infty$, $0 \le q < p$, $0 \le r < p^*$, $c > 0$, and $\alpha \in L^1(\Omega)$. Let $\mathbf{u}_0 \in W^{1,p}(\Omega;\mathbb{R}^N)$ be given, and suppose that $\{\mathbf{u}_j\}_{j=1}^\infty \subset \mathcal{A} := \mathbf{u}_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizing sequence for the functional $J : \mathcal{A} \to \mathbb{R}$ defined by

$$J(\mathbf{v}) := \int_\Omega f(\mathbf{x},\mathbf{v}(\mathbf{x}),\nabla\mathbf{v}(\mathbf{x}))\mathrm{d}\mathbf{x}.$$

Let $\nu = \{\nu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega}$ be the Young measure generated by $\{\nabla\mathbf{u}_j\}_{j=1}^\infty$ (or possibly a subsequence). If ν satisfies

$$\sup_{\substack{\mathbf{x}_0\in\Omega\\ \rho>0}} \rho^{-\lambda} \int_{\Omega\cap Q_{\mathbf{x}_0,\rho}} \int_{\mathbb{R}^{N\times n}} |\mathbf{Y}|^p \,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y})\mathrm{d}\mathbf{x} < \infty \tag{22}$$

for some $0 \le \lambda < n$, then there is a minimizing sequence $\{\mathbf{v}_j\}_{j=1}^\infty \subset \mathcal{A}$ such that $\{\mathbf{v}_j\}_{j=1}^\infty$ is uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ and $\{\nabla\mathbf{v}_j\}_{j=1}^\infty$ is p-equiintegrable.

In particular, if $p+\lambda > n$*, then* $\{\mathbf{v}_j\}_{j=1}^{\infty}$ *is uniformly bounded in* $\mathcal{C}^{0,1-(n-\lambda)/p}(\Omega;\mathbb{R}^N)$*, the space of Hölder-continuous functions with exponent* $1-(n-\lambda)/p$*.*

Remark 5.1. The conclusion of the theorem also holds in the case $p = 1$ if we additionally assume that $\{\nabla \mathbf{u}_j\}_{j=1}^{\infty}$ is equiintegrable.

Proof. Because of the coercivity condition on f, any minimizing sequence will be bounded in $W^{1,p}(\Omega;\mathbb{R}^N)$. Therefore, taking a subsequence if necessary, we may assume that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$ and that $\{\nabla \mathbf{u}_j\}_{j=1}^{\infty}$ generates the Young measure ν. Since $\{\nabla \mathbf{u}_j\}$ is equiintegrable, it follows from Theorem 2.1 that

$$\nabla \mathbf{u}_j \rightharpoonup \int_{\mathbb{R}^{N\times n}} \mathbf{Y} \mathrm{d}\nu_{(\cdot)}(\mathbf{Y}) \text{ in } L^1(\Omega;\mathbb{R}^N),$$

and hence

$$\nabla \mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^{N\times n}} \mathbf{Y} \mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y})$$

for almost every $\mathbf{x} \in \Omega$. Thus condition (i) in Theorem 4.1 is satisfied. As ν is the Young measure generated by a sequence of gradients bounded in $L^p(\Omega;\mathbb{R}^{N\times n})$, it follows that (ii) in the same theorem is also fulfilled. Seeing that (22) is precisely (iii), Theorem 4.1 implies that there exists a sequence $\{\mathbf{v}_j\}_{j=1}^{\infty} \subset \mathcal{A}$ uniformly bounded in $W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N)$ such that the sequence of gradients $\{\nabla \mathbf{v}_j\}$ is p-equiintegrable and generates ν.

To see that $\{\mathbf{v}_j\}$ is a minimizing sequence for the functional J, we define the Young measure $\mu = \{\mu_{\mathbf{x}}\}_{\mathbf{x}\in\Omega} \subset \mathcal{M}(\mathbb{R}^N \times \mathbb{R}^{N\times n})$ by

$$\mu_{\mathbf{x}} = \delta_{\mathbf{u}(\mathbf{x})} \times \nu_{\mathbf{x}},$$

where $\delta_{\mathbf{u}(\mathbf{x})}$ denotes the Dirac mass centered at $\mathbf{u}(\mathbf{x})$. Define the sequences of functions $\{\mathbf{w}_j\}$ and $\{\mathbf{z}_j\}$, with $\mathbf{w}_j, \mathbf{z}_j : \Omega \to \mathbb{R}^N \times \mathbb{R}^{N\times n}$, by

$$\mathbf{w}_j(\mathbf{x}) := (\mathbf{u}_j(\mathbf{x}), \nabla \mathbf{u}_j(\mathbf{x}));$$
$$\mathbf{z}_j(\mathbf{x}) := (\mathbf{v}_j(\mathbf{x}), \nabla \mathbf{v}_j(\mathbf{x})).$$

Both $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ converge weakly to $\mathbf{u}$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, and hence also converge strongly to $\mathbf{u}$ in $L^p(\Omega;\mathbb{R}^N)$. Because of this strong convergence and because the sequences $\{\nabla \mathbf{u}_j\}$ and $\{\nabla \mathbf{v}_j\}$ generate ν, we have that each of the sequences $\{\mathbf{w}_j\}$ and $\{\mathbf{z}_j\}$ generate the measure μ. Furthermore, using the growth conditions on f, we see that the sequence $\{f(\cdot, \mathbf{z}_j(\cdot))\}$ is equiintegrable, so using Theorem 1, we obtain

$$\begin{aligned}\lim_{j\to\infty} J(\mathbf{v}_j) = \lim_{j\to\infty} \int_{\Omega} f(\mathbf{x}, \mathbf{z}_j(\mathbf{x}))\mathrm{d}\mathbf{x} &= \int_{\Omega} \int_{\mathbb{R}^{N\times n}} f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \mathbf{Y})\mathrm{d}\nu_{\mathbf{x}}(\mathbf{Y}) \\ &\leq \liminf_{j\to\infty} \int_{\Omega} \int_{\Omega} f(\mathbf{x}, \mathbf{w}_j(\mathbf{x}))\mathrm{d}\mathbf{x} = \liminf_{j\to\infty} J(\mathbf{u}_j).\end{aligned}$$

Since $\{\mathbf{u}_j\}$ is a minimizing sequence for J, it follows that $\{\mathbf{v}_j\}$ is also a minimizing sequence. This concludes the proof. $\square$

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