Young Measures Generated by Sequences in Morrey Spaces

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Introduction to Young Measures

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

Definition of a Young measure

We say that $\nu = \{\nu_x\}_{x\in\Omega}$ is a Young measure if

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Definition of a Generating Sequence

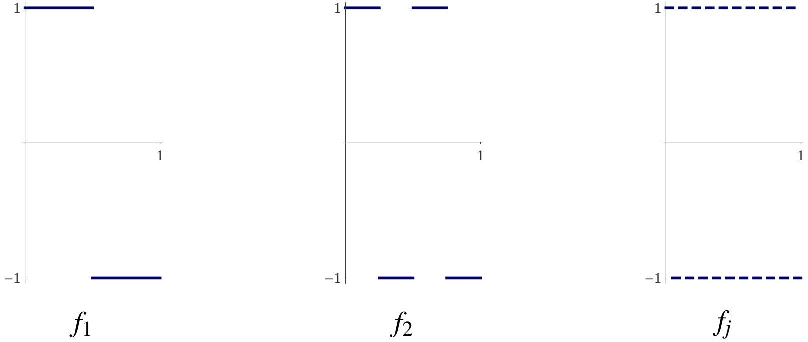
A sequence of functions $\{f_j\}_{j=1}^{\infty}$ mapping Ω into $\mathbb{R}^N$ generates ν if

$$\varphi(f_j) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^N} \varphi(y) \mathrm{d}\nu_x(y) \text{ in } L^{\infty}(\Omega)$$

for every $\varphi \in C_0(\mathbb{R}^N)$.

Example 1

Define $f_j : (0,1) \to \mathbb{R}$ by $f_j(x) := \begin{cases} 1 & \text{if } 0 < jx - \lfloor jx \rfloor \leq 1/2; \\ -1 & \text{if } 1/2 < jx - \lfloor jx \rfloor < 1. \end{cases}$



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The sequence f_j generates a Young measure with $\nu_x \equiv \nu := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

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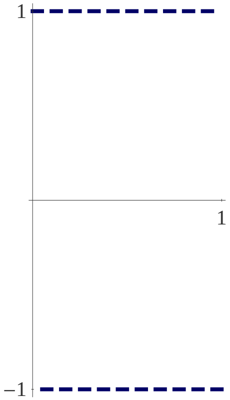

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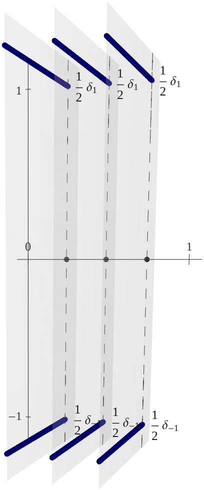

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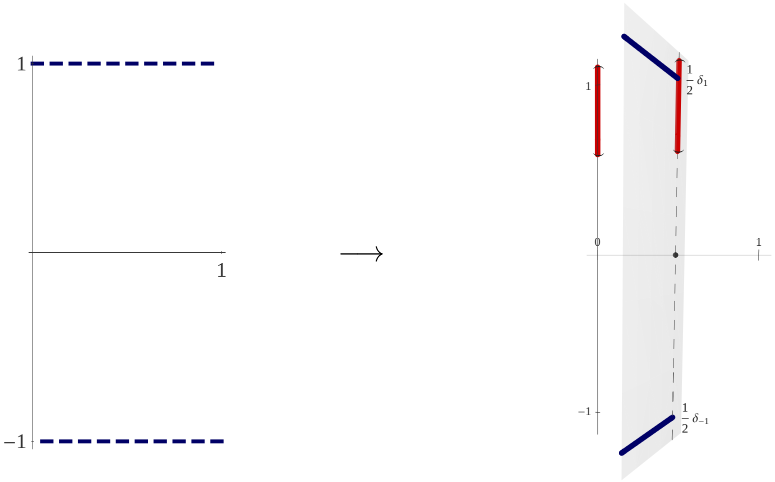

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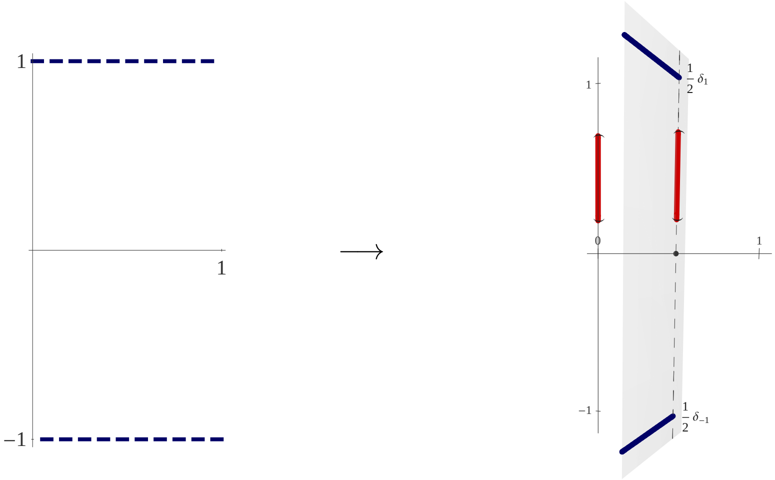

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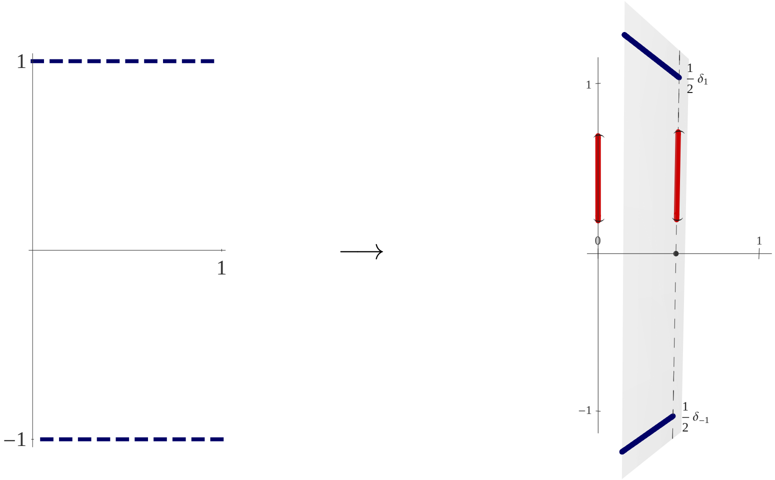
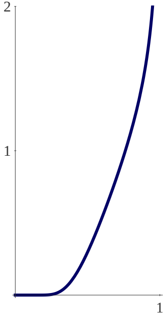


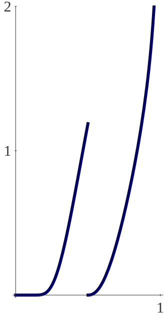
$$\varphi(f_j) \overset{*}{\rightharpoonup} \int_{\mathbb{R}} \varphi(y)\mathrm{d}\nu_x(y) = \frac{1}{2}\varphi(-1) + \frac{1}{2}\varphi(1) \text{ in } L^\infty \text{ for any continuous } \varphi.$$

Example 2

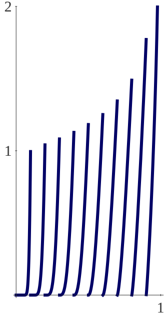
Define $f_j : (0,1) \to \mathbb{R}$ by $f_j(x) := (1-x)^{-\frac{1}{4}} \left(jx - \lfloor jx \rfloor \right)^{\frac{1}{x}}$.



f_1



f_2



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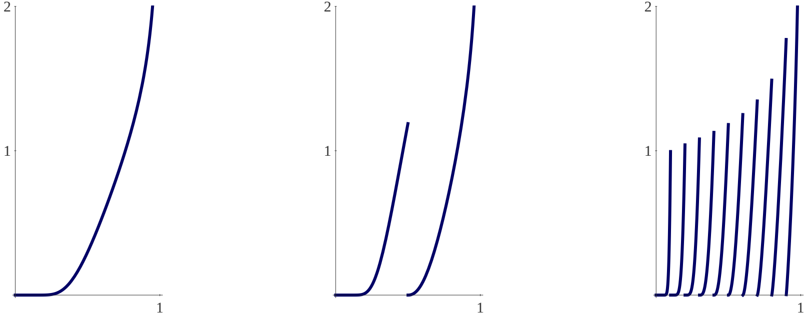

f_1 f_2 f_j

The sequence f_j generates a Young measure with

$$\frac{\mathrm{d}\nu_x}{\mathrm{d}y} = x(1-x)^{\frac{x}{4}} y^{x-1} \chi_{(0,(1-x)^{-1/4})}(y).$$

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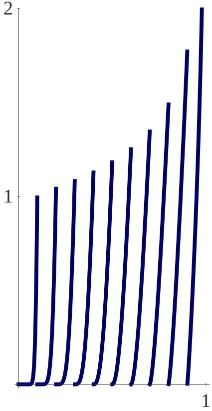

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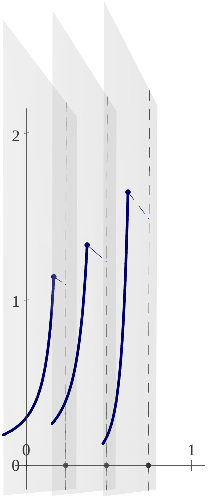

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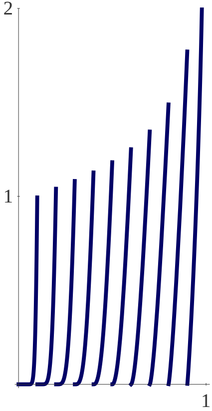

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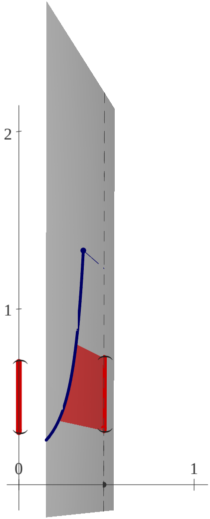

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$$f_j^2 \rightharpoonup \frac{x}{(x+2)(1-x)^{\frac{1}{2}}} \text{ in } L^p \text{ for } p < 2$$

Young's Tacking Problem

Statement of the Problem

Minimize the functional J defined by

$$J[u] = \int_{(0,1)} \left\{ \left(|u'(x)| - 1 \right)^2 + u(x)^2 \right\} \mathrm{d}x$$

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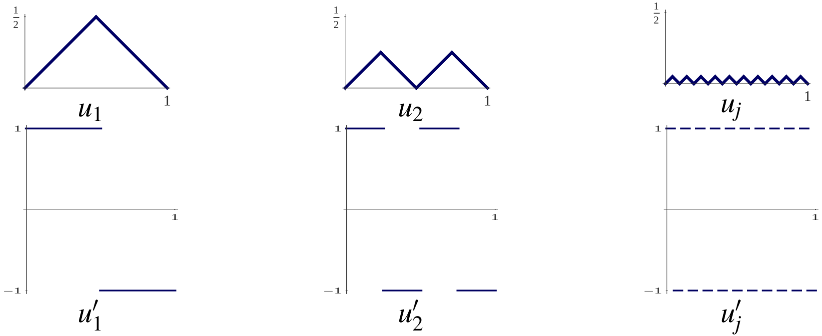
The natural place to look for minimizers is $W^{1,2}$, but the competition between properties 1 and 2 precludes the existence of a minimizer in this space.

Young's Tacking Problem (cont.)

A minimizing sequence for the functional:



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$$\int_{\mathbb{R}} y \mathrm{d}\nu_x(y) = u'(x).$$

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The minimizer is the pair $(0, \nu := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)$.

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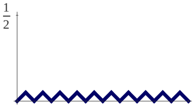
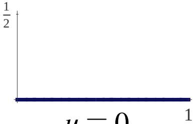
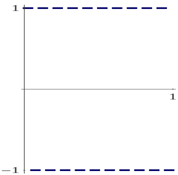


u_j

$\longrightarrow$

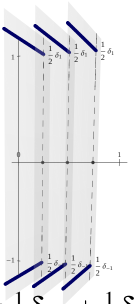


$u \equiv 0$



u_j'

$\longrightarrow$



$v_x \equiv \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$

Known Results

Let $p \in (1, \infty)$, and given a Young measure $\nu = \{\nu_x\}_{x \in \Omega}$ on $\mathbb{R}^d$, define the function $\psi_\nu : \Omega \to \mathbb{R}$ by

$$\psi_\nu(x) := \int_{\mathbb{R}^d} |y|^p \, \mathrm{d}\nu_x(y).$$

General Case (L. Tartar, 1979)

A Young measure $\nu = \{\nu_x\}_{x \in \Omega}$ on $\mathbb{R}^N$ can be generated by a sequence $\{f_j\}$ bounded in $L^p(\Omega; \mathbb{R}^N)$ if and only if $\psi_\nu \in L^1(\Omega)$.

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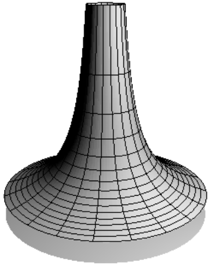
1. $\psi_\nu \in L^1(\Omega)$;
2. $\int_{\mathbb{R}^{N \times n}} Y \mathrm{d}\nu_x(Y) = \nabla u(x)$ for $x \in \Omega$.
3. $\int_{\mathbb{R}^{N \times n}} \varphi(Y) \mathrm{d}\nu_x(Y) \geq \varphi(\nabla u(x))$ for every quasiconvex φ that is bounded below and satsifies $\varphi(Y) \leq C(1 + |Y|^p)$.

Morrey Spaces

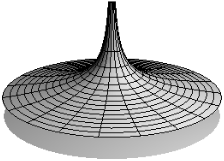
For each $p \in [1, +\infty)$ and $\lambda \in [0, n]$, define the

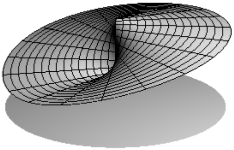
Morrey Space

$$L^{p,\lambda}(\Omega;\mathbb{R}^N) := \left\{ u \in L^p(\Omega;\mathbb{R}^N) : \sup_{\substack{x_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega \cap B_{x_0,\rho}} |u(x)|^p \, \mathrm{d}x < \infty \right\}.$$



$L^{p,0} \cong L^p$





$L^{p,n} \cong L^\infty$

Fact: If $\lambda < n$, then $L^{p,\lambda} \not\subseteq L^q$ for any $q > p$.

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A Young measure $\nu = \{\nu_x\}_{x \in \Omega}$ can be generated by a sequence $\{f_j\}$ bounded in $L^{p,\lambda}(\Omega; \mathbb{R}^N)$ if and only if $\psi_\nu \in L^{1,\lambda}(\Omega; \mathbb{R}^N)$.

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Observation: If ν is homogeneous (i.e. $\nu_x \equiv \nu$), then ψ_ν is constant, and hence $\psi_\nu \in L^{1,\lambda}$ if $\psi_\nu \in L^1$.

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Corollary

If ν is a homogeneous Young measure that can be generated by a sequence bounded in L^p, then ν can also be generated by a sequence bounded in $L^{p,\lambda}$ for every $0 \le \lambda < n$.

The Gradient Case

Let $p \in (1,\infty)$ and $\lambda \in [0,n)$. Let $\nu = \{\nu_x\}_{x\in\Omega}$ be a Young measure on $\mathbb{R}^{N\times n}$. Recall $\psi_\nu(x) := \int_{\mathbb{R}^{N\times n}} |Y|^p \, d\nu_x(Y)$.

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There is a sequence of weak gradients $\{\nabla u_j\}$ bounded in $L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n})$ that generates ν if and only if there is a $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ such that

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Every homogeneous Young measure that can be generated by a sequence of weak gradients bounded in L^p can also be generated by a sequence of weak gradients bounded in $L^{p,\lambda}$ for every $0 \leq \lambda < n$.

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Suppose $p \in (1, \infty)$ and $\lambda \in [0, n)$.

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If $p+\lambda > n$, then $\{w_j\}$ is bounded in $C^{0,1-\frac{n-\lambda}{p}}(\overline{\Omega};\mathbb{R}^N)$.
If the minimizing Young measure is homogeneous, it is possible to get a minimizing sequence bounded in $C^{0,\alpha}$ for every $0 \le \alpha < 1$.