MORREY REGULARITY FOR ALMOST MINIMIZERS OF ASYMPTOTICALLY CONVEX FUNCTIONALS WITH NONSTANDARD GROWTH

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ABSTRACT. We prove some global Morrey regularity results for almost minimizers of functionals of the form

$$\mathbf{u} \mapsto \int_\Omega f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \mathrm{d}\mathbf{x}.$$

This regularity is valid up to the boundary, provided the boundary data is sufficiently regular. The main assumption on f is that for each $\mathbf{x}$ and $\mathbf{u}$, the function $f(\mathbf{x}, \mathbf{u}, \cdot)$ behaves asymptotically like $h(|\cdot|)^{\alpha(\mathbf{x})}$, where h is an N-function with th'' comparable to h'. We provide a couple of applications of this result: an application to a broad class of PDEs; and a result which, for a large class of functionals, provides a minimizing sequence with uniform regularity.

KEYWORDS: Morrey regularity · Asymptotic convexity · Nonstandard growth · Variable exponent · Systems of partial differential equations · Minimizing sequences

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1. INTRODUCTION

We will provide Morrey regularity for the gradient of almost minimizers for functionals of the form

$$\mathbf{u} \mapsto \int_\Omega f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \mathrm{d}\mathbf{x}, \tag{1}$$

where for each $\mathbf{x} \in \Omega$ and $\mathbf{u} \in \mathbb{R}^N$, the integrand $\mathbf{F} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ behaves asymptotically like the function $\mathbf{F} \mapsto h(|\mathbf{F}|)^{\alpha(\mathbf{x})}$, where h is an N-function such that th'' is comparable to h' and α satisfies a continuity assumption. We recall that a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an N-function if $h(0) = 0$ and there exists a right-continuous nondecreasing derivative h' satisfying $h'(0) = 0$ and $h'(t) > 0$ for all $t > 0$, with $\lim_{t\to\infty} h'(t) = \infty$. With this in mind, we make the following definition, which characterizes more precisely the assumptions on h (cf. Definition 2.6 in [12]).

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Definition 1. If h is an N-function satisfying $h \in W^{2,1}(0,T)$ for each $T > 0$ and

$$(p-1)h'(t) \leq th''(t) \leq (q-1)h'(t) \text{ for a.e. } t > 0,$$

for some $1 < p \leq q < \infty$, then we will say that h has (p,q)-*structure.*

Roughly speaking, a function h with (p,q)-structure grows between the functions $t \mapsto t^p$ and $t \mapsto t^q$. (See Lemma 1 for some basic consequences of Definition 1.) Provided that there is a function h with (p,q)-structure such that the map $\mathbf{F} \mapsto f(\mathbf{x},\mathbf{u},\mathbf{F})$ behaves like $\mathbf{F} \mapsto h(|\mathbf{F}|)^{\alpha(\mathbf{x})}$ when $|\mathbf{F}|$ is large, and provided that $\mathbf{u}$ is an almost minimizer for the functional defined in (1), we prove that $\mathbf{x} \mapsto h(|\nabla\mathbf{u}(\mathbf{x})|)^{\alpha(\mathbf{x})}$ belongs to the Morrey space $L^{1,\lambda}(\Omega)$. The regularity we obtain is global and full; i.e. we have $\|h(|\nabla\mathbf{u}|)^{\alpha}\|_{L^{1,\lambda}(\Omega)} < \infty$.

To give the reader a sense of the scope of the functionals considered in this paper, we give a few examples of N-functions h that possess (p,q)-structure. For any $1 < p \leq q < \infty$ and $\beta \geq 1$, all of the following functions mapping $[0,\infty)$ into $[0,\infty)$ have $(\tilde{p},\tilde{q})$-structure for some $1 < \tilde{p} \leq \tilde{q} < \infty$:

$$\begin{aligned} &h_1(t) := t^p; \\ &h_2(t) := t^p[\log(t+e)]^{\beta}; \\ &h_3(t) := \begin{cases} t^p & \text{if } 0 \leq t \leq t_0 \\ t^{\frac{p+q}{2}+\frac{p-q}{2}\sin\log\log\log t} & \text{if } t \geq t_0 \end{cases}. \end{aligned}$$

In the last example, t_0 must be chosen sufficiently large so that h_3 is strictly convex and $\sin\log\log\log(t_0) = 1$. (The function h_3 was first given as an example in [8], and also in [21] and [12].) Then, for each $\mathbf{x} \in \Omega$ and $\mathbf{u} \in \mathbb{R}^N$, the corresponding integrands for the functional in (1) would behave asymptotically like the functions $\mathbf{F} \mapsto h_i(|\mathbf{F}|)^{\alpha(\mathbf{x})} =: f_i(\mathbf{x},\mathbf{F})$:

$$\begin{aligned} &f_1(\mathbf{x},\mathbf{F}) := |\mathbf{F}|^{p(\mathbf{x})}, \\ &f_2(\mathbf{x},\mathbf{F}) := |\mathbf{F}|^{p(\mathbf{x})}\log(e+|\mathbf{F}|)^{\gamma p(\mathbf{x})}, \\ &f_3(\mathbf{x},\mathbf{F}) := \begin{cases} |\mathbf{F}|^{p(\mathbf{x})} & \text{if } 0 \leq |\mathbf{F}| \leq t_0 \\ |\mathbf{F}|^{p(\mathbf{x})\frac{1+Q}{2}+p(\mathbf{x})\frac{1-Q}{2}\sin\log\log\log|\mathbf{F}|} & \text{if } |\mathbf{F}| \geq t_0 \end{cases}, \end{aligned}$$

where we have put $p(\mathbf{x}) = p\alpha(\mathbf{x})$, $\gamma = \beta/p$, and $Q = q/p$. The integrand f_1 is a particularly important case: integrands of this form have been utilized in image restoration (e.g. [5]), and also arise from problems in mathematical physics, specifically those associated with electro-rheological fluids (e.g. [3]) and thermistors (e.g. [24]).

Our main result is given in Section 5. We present here the following simplified version to give the reader a sense of the spirit of this result.

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and that $\alpha : \Omega \to [1,\infty)$ is Hölder continuous and h is a function with (p,q)-structure. Let $\beta : \Omega \to (0,\infty)$ be uniformly continuous in Ω, with $\frac{1}{\beta} \in L^{\infty}(\Omega)$. Suppose that there is $\lambda \in [0,n)$ such that $\gamma : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies*

$$|\gamma(\mathbf{x},\mathbf{u},\mathbf{F})| \leq C\left(\xi(\mathbf{x}) + h(|\mathbf{u}|)^{\alpha(x)} + h(|\mathbf{F}|)^{\alpha(\mathbf{x})-\varepsilon}\right)$$

for some $\xi \in L^{1,\lambda}(\Omega)$ *and* $0 < \varepsilon \leq C < \infty$. *Let* $\mathcal{A}$ *be defined by*

$$\mathcal{A} := \left\{ \mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N) : h(|\nabla \mathbf{u}(\cdot)|)^{\alpha(\cdot)} \in L^1(\Omega) \text{ and } \mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,1}(\Omega; \mathbb{R}^N) \right\},$$

where $\overline{\mathbf{u}} \in W^{1,1}(\Omega; \mathbb{R}^N)$ *and* $h(|\nabla \overline{\mathbf{u}}(\cdot)|)^{\alpha(\cdot)} \in L^{1,\lambda}(\Omega)$. *If* $\mathbf{u} \in \mathcal{A}$ *is a minimizer for the functional defined by*

$$\mathbf{u} \mapsto \int_\Omega \left\{ \beta(\mathbf{x}) h(|\nabla \mathbf{u}(\mathbf{x})|)^{\alpha(\mathbf{x})} + \gamma(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x})) \right\} \mathrm{d}\mathbf{x},$$

then $h(|\nabla \mathbf{u}(\cdot)|)^{\alpha(\cdot)} \in L^{1,\lambda}(\Omega)$.

As was mentioned earlier, the full results in Sections 4 and 5 are actually established for a very general class of almost minimizers, which allows their application to a variety of problems, some examples of which are supplied in Section 6. Similarly to [12] and [15], we furnish an application of our variational results that gives Morrey regularity for the gradient of weak solutions to PDEs of the form

$$\mathrm{div} \mathbf{A}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = \mathbf{b}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}),$$

where $\mathbf{A}$ and $\mathbf{b}$ are required to satisfy some growth conditions, and for each $\mathbf{x} \in \Omega$ and $\mathbf{u} \in \mathbb{R}^N$, the mapping $\mathbf{F} \mapsto \mathbf{A}(\mathbf{x}, \mathbf{u}, \mathbf{F})$ behaves asymptotically like the mapping $\mathbf{F} \mapsto \frac{\partial}{\partial \mathbf{F}}[h(|\mathbf{F}|)^{\alpha(\mathbf{x})}]$. As another application, provided that the functional is coercive, our results can be used to show the existence of a minimizing sequence with gradients uniformly bounded in a Morrey space, as is also carried out in [15]. The integrands of the functionals we consider are not necessarily globally convex, and are only required to be convex "at infinity." Therefore minimizers generally do not exist. Nevertheless, as we show, one can generate a minimizing sequence which is comprised of almost minimizers of the functional, and therefore our results yield Morrey regularity for each element of that minimizing sequence.

We now discuss how our results fit into the broader framework of regularity theory. As was previously noted, a primary special case of the functionals we consider are those whose integrands possess $p(\mathbf{x})$-growth asymptotically. As one might expect, the continuity of the exponent plays an important role in the type of regularity possessed by minimizers. In this paper, we henceforth denote the modulus of continuity for the exponent by ω. Under the assumption that $p(\cdot)$ satisfies a log-continuity hypothesis, namely that

$$\limsup_{r \to 0^+} \omega(r) \log\left(\frac{1}{r}\right) = L_0 < \infty,$$

V. Zhikov [24] provided a proof of higher integrability for the gradient of minimizers, and also provided an example that showed that if the exponent $p(\cdot)$ is merely continuous, and does not satisfy the above log-continuity assumption, then the higher integrability need not hold. If $L_0 = 0$, which is the assumption we make in this paper, E. Acerbi and G. Mingione [1], working within the scalar setting, showed that minimizers belong to $\mathcal{C}^{0,\alpha}$ locally for every $\alpha < 1$, and if $p(\cdot)$ is Hölder continuous, that the gradient also possesses some Hölder continuity. Again in the scalar setting, Hölder continuity of quasiminimizers, under more stringent assumptions on $p(\cdot)$, was proved in [6]. In [2], partial Hölder continuity for the gradient is obtained in the vectorial case with quasiconvex integrands. For a more extensive discussion of the case of $p(\mathbf{x})$-growth, we refer the interested reader to Section 7 of [22], and the references contained therein.

Another special case enveloped by our results is that of functionals whose integrands have growth specified by a function h with (p,q)-structure; in the notation of the present paper, this corresponds to the case $\alpha \equiv 1$. Various types of regularity have been studied in this setting; see [9], [10], and [12], for example.

The type of regularity we obtain, namely global Morrey regularity for the gradient of almost minimizers, is the same as that obtained in [12], [14], and [15]; indeed, the results in the present paper generalize those previous results to allow for variable exponent growth. We make a couple of comments about the method of proof. The first step is obtaining local Lipschitz regularity for minimizers of functionals that have integrands with (p,q)-structure, which was carried out by the authors in [12]. Then, using the estimate for this Lipschitz regularity, in combination with the techniques employed in [1], we prove that if $\mathbf{v}$ is a minimizer for a functional with integrand that looks like $(\mathbf{x},\mathbf{F}) \mapsto h(|\mathbf{F}|)^{\alpha(\mathbf{x})}$, then $h(|\nabla\mathbf{v}|)^{\alpha} \in L^{1,\kappa}_{\mathrm{loc}}(\Omega)$ for every $\kappa \in [0,n)$. Next, under the assumption that $\mathbf{u}$ is an almost minimizer for the functional, we make a comparison between $\mathbf{u}$ and $\mathbf{v}$ to obtain some Morrey regularity for $\mathbf{u}$. It is at this stage where we also incorporate the boundary values into the functional to show that $\mathbf{u}$ is in fact globally Morrey regular. All of the aforementioned work is done in the setting of functionals with convex integrands that only depend on $\mathbf{x}$ and $|\nabla\mathbf{u}|$, and constitutes Section 4. In Section 5, we extend the scope of these results to include functionals that are only asymptotically convex and possess integrands that can depend on $\mathbf{x}$, $\mathbf{u}$, and $\nabla\mathbf{u}$. The main idea required to treat this more general case is embodied in Lemma 6, where we show that an almost minimizer for the asymptotically convex functional with dependence on $\mathbf{u}$ will also be an almost minimizer for an appropriate convex functional with no dependence on $\mathbf{u}$. Hence the regularity obtained for almost minimizers of convex functionals is passed on to almost minimizers of asymptotically convex functionals.

2. Definitions and Notation

Fix n, $N \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be open and bounded. We denote by $\mathbb{R}_+$ the interval $[0,\infty)$ and by $\mathbb{R}^*$ the set $\mathbb{R}\cup\{+\infty\}$. For a measurable set $E \subset \mathbb{R}^n$, we let $|E|$ denote the Lebesgue measure of E, and use $\overline{E}$ to signify the closure of E in the usual Euclidean topology. If $0 < |E| < \infty$ and $\mathbf{f} \in L^1(E;\mathbb{R}^N)$, we define

$$(\mathbf{f})_E \equiv \fint_E \mathbf{f}(\mathbf{x})\mathrm{d}\mathbf{x} := \frac{1}{|E|}\int_E \mathbf{f}(\mathbf{x})\mathrm{d}\mathbf{x}.$$

We will use $\mathbf{x}$ and $\mathbf{y}$ to denote points in $\mathbb{R}^n$, and $\mathbf{F}$ to denote a point in $\mathbb{R}^{N\times n}$. The open ball of radius ρ centered at the point $\mathbf{x}$ is represented by $\mathcal{B}(\mathbf{x},\rho)$. For brevity, $\mathcal{B}$ denotes $\mathcal{B}(0,1)$. For a measurable set $E \subset \mathbb{R}^n$ not equal to $\mathcal{B}$, we use $E(\mathbf{x},\rho)$ to abbreviate $E \cap \mathcal{B}(\mathbf{x},\rho)$. (We refrain from using the previously mentioned notation when $E = \mathcal{B}$ in order to avoid ambiguity.) We define $\mathcal{H}^+ := \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n > 0\}$, and given a set $\mathcal{U} \subset \mathbb{R}^n$, we use $\mathcal{U}^+$ to stand for $\mathcal{H}^+ \cap \mathcal{U}$.

For a given N-function $h : \mathbb{R}_+ \to \mathbb{R}_+$, we define the Young conjugate as follows.

Definition 2. If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an N-function, then the function $h^* : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$h^*(s) := \sup_{t\in\mathbb{R}_+} \{st - h(t)\}$$

is called the *Young conjugate* of h.

It is easily seen that h^* is an N-function, and that for any s, $t \in \mathbb{R}_+$, we have

$$st \le h(s) + h^*(t),$$

which is known as Young's Inequality. Furthermore, if h' is continuous and strictly increasing, as is the case if h has (p,q)-structure, then the function $t \mapsto st - h(t)$ has a unique maximum at $t = (h')^{-1}(s)$, and hence

$$h^*(s) = s(h')^{-1}(s) - h((h')^{-1}(s)).$$

We now introduce notation for certain vector spaces that are well-suited for our situation in the present paper; these spaces are special cases of Musielak-Orlicz spaces. For a development of Musielak-Orlicz spaces, see, for example, [23]. Our notation is similar to that used for Orlicz and Orlicz-Sobolev spaces in [4].

Definition 3. Suppose that $E \subset \mathbb{R}^n$ is open. Let $g : E \times \mathbb{R}_+ \to \mathbb{R}_+$ be such that there is a constant $c < \infty$ so that for almost every $\mathbf{x} \in E$, the function $g(\mathbf{x}, \cdot)$ is convex and nondecreasing with $g(\mathbf{x}, 0) = 0$, and $g(\mathbf{x}, 2t) \le c g(\mathbf{x}, t)$ for every $t \in \mathbb{R}_+$. We then define

$$L_g(E; \mathbb{R}^N) := \left\{ \mathbf{u} : E \to \mathbb{R}^N : \mathbf{u} \text{ is measurable and } \int_E g(\mathbf{x}, |\mathbf{u}(\mathbf{x})|) \mathrm{d}\mathbf{x} < \infty \right\},$$

and the corresponding Sobolev-type space

$$W^1 L_g(E; \mathbb{R}^N) := \left\{ \mathbf{u} \in L_g(E; \mathbb{R}^N) : \nabla \mathbf{u} \in L_g(E; \mathbb{R}^{N\times n}) \right\},$$

where $\nabla \mathbf{u}$ denotes the weak gradient of the mapping $\mathbf{u}$.

Remark 1. If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a function with (p,q)-structure, we will use $L_h(E; \mathbb{R}^N)$ and $W^1 L_h(E; \mathbb{R}^N)$ to denote the spaces $L_{\tilde{h}}(E; \mathbb{R}^N)$ and $W^1 L_{\tilde{h}}(E; \mathbb{R}^N)$, respectively, where we have defined $\tilde{h} : E \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\tilde{h}(\mathbf{x}, t) := h(t)$.

The space $L_g(E; \mathbb{R}^N)$ can be equipped with the Luxemborg norm defined by

$$\|\mathbf{u}\|_{L_g} := \inf \left\{ s > 0 : \int_E g\left(\mathbf{x}, \frac{|\mathbf{u}(\mathbf{x})|}{s}\right) \mathrm{d}\mathbf{x} \le 1 \right\},$$

and is complete under this norm. The norm on $W^1 L_h(E; \mathbb{R}^N)$ is then defined by

$$\|\mathbf{u}\|_{W^1 L_g} := \|\mathbf{u}\|_{L_g} + \|\nabla \mathbf{u}\|_{L_g}.$$

We will use $W_0^1 L_g(E; \mathbb{R}^N)$ to denote the closure of $\mathcal{C}_c^\infty(E; \mathbb{R}^N)$ in $W^1 L_g(E; \mathbb{R}^N)$.

Finally, we introduce the Morrey space $L^{1,\lambda}(\Omega)$. We refer the reader to [18] for a development of some of the properties of Morrey spaces.

Definition 4. Suppose that $E \subset \mathbb{R}^n$ is measurable and bounded, and that $\lambda \in [0, n]$. We define the *Morrey space*

$$L^{1,\lambda}(E) := \left\{ \mathbf{u} \in L^1(E) : \|\mathbf{u}\|_{L^{1,\lambda}} := \sup_{\mathbf{x}_0 \in E} \sup_{\rho > 0} \rho^{-\lambda} \int_{E(\mathbf{x}_0, \rho)} |\mathbf{u}| \, \mathrm{d}\mathbf{x} < \infty \right\}.$$

The following defines the type of almost minimizers for which we will establish regularity.

Definition 5. Let $\Omega \subset \mathbb{R}^n$ be open and $\mathcal{A} \subset W^{1,1}(\Omega; \mathbb{R}^N)$ be given, and suppose that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$. Define the functional $K : \mathcal{A} \to \mathbb{R}^*$ by

$$K(\mathbf{w}) := \int_\Omega f(\mathbf{x}, \mathbf{w}, \nabla \mathbf{w}) \mathrm{d}\mathbf{x}.$$

Suppose that $\mathbf{u} \in \mathcal{A}$ with $K(\mathbf{u}) < \infty$, and that there are functions $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ and nondecreasing functions $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ satisfying $\gamma_\varepsilon(0) = 0$ for each $\varepsilon > 0$ such that for every $\varepsilon > 0$ and $0 < \rho < \operatorname{diam}(\Omega)$, we find that

$$K(\mathbf{u}) \leq K(\mathbf{v}) + (\gamma_\varepsilon(\rho) + \varepsilon) \int_{\Omega(\mathbf{x}_0,\rho)} \{|f(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})| + |f(\mathbf{x}, \mathbf{v}, \nabla \mathbf{v})| + \nu_\varepsilon(\mathbf{x})\}\, \mathrm{d}\mathbf{x} \tag{2}$$

whenever $\mathbf{v} \in \mathcal{A}$ and $\mathbf{v} - \mathbf{u} \in W_0^{1,1}(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. Then we say that $\mathbf{u}$ is a *$(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer over $\mathcal{A}$.*

3. Preliminaries

Throughout the paper, we will assume that $\alpha : \Omega \to [1, \infty)$ satisfies

$$1 \leq \alpha(\mathbf{x}) \leq \alpha_+ < \infty \text{ for every } \mathbf{x} \in \Omega, \tag{3}$$

and with $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ denoting the modulus of continuity for α, we also assume that

$$\omega(R)\, |\log{(R)}| \leq M, \tag{4}$$

and moreover that

$$\lim_{R \to 0^+} \omega(R) \log{(R)} = 0. \tag{5}$$

With α as just defined, we suppose that the function $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following for some $1 < p \leq q < \infty$, some $h : \mathbb{R}_+ \to \mathbb{R}_+$ with (p, q)-structure, and a nondecreasing $\delta \in \mathcal{C}(\mathbb{R}_+)$ with $\delta(0) = 0$:

$$g(\mathbf{x}, \cdot) \text{ has } (p\alpha(\mathbf{x}), q\alpha(\mathbf{x}))\text{-structure for each } \mathbf{x} \in \Omega; \tag{A$_1$}$$

$$h(t)^{\alpha(\mathbf{x})} \leq g(\mathbf{x}, t) \leq M h(t)^{\alpha(\mathbf{x})}; \tag{A$_2$}$$

$$\begin{aligned} |g(\mathbf{x}, t) - g(\mathbf{y}, t)| \leq M\omega(|\mathbf{x} - \mathbf{y}|)\, \{g(\mathbf{x}, t) + g(\mathbf{y}, t)\} \log(e + h(t)) \\ + M\delta(|\mathbf{x} - \mathbf{y}|)\{1 + g(\mathbf{x}, t) + g(\mathbf{y}, t)\}. \end{aligned} \tag{A$_3$}$$

To explain the last condition, with the model case $g(\mathbf{x}, t) = \beta(\mathbf{x}) h(t)^{\alpha(\mathbf{x})}$, the functions ω and δ represent the moduli of continuity for the exponent α and the function β, respectively. Note that we only require uniform continuity of β, whereas we require the stronger continuity assumption (5) for the exponent α.

In the next lemma, we collect together some basic properties of functions $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (A$_1$) and (A$_2$).

Lemma 1. *Let $\alpha : \Omega \to [1, \infty)$ and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (3) and (A$_1$)-(A$_2$). Then for every $\mathbf{x} \in \Omega$ and every $s, t \in \mathbb{R}_+$, the following hold:*

- (i) $p\alpha(\mathbf{x}) g(\mathbf{x}, t) \leq t g_t(\mathbf{x}, t) \leq q\alpha(\mathbf{x}) g(\mathbf{x}, t)$;
- (ii) $c^{p\alpha(\mathbf{x})} g(\mathbf{x}, t) \leq g(\mathbf{x}, ct) \leq c^{q\alpha(\mathbf{x})} g(\mathbf{x}, t)$ *for every* $c \geq 1$;
- (iii) $\varepsilon^{q\alpha(\mathbf{x})} g(\mathbf{x}, t) \leq g(\mathbf{x}, \varepsilon t) \leq \varepsilon^{p\alpha(\mathbf{x})} g(\mathbf{x}, t)$ *for every* $\varepsilon \in (0, 1]$;
- (iv) $\varepsilon^{q\alpha(\mathbf{x})-1} g_t(\mathbf{x}, t) \leq g_t(\mathbf{x}, \varepsilon t) \leq \varepsilon^{p\alpha(\mathbf{x})-1} g_t(\mathbf{x}, t)$ *for every* $\varepsilon \in (0, 1]$;
- (v) $g(\mathbf{x}, s + t) \leq 2^{q\alpha(\mathbf{x})}(g(\mathbf{x}, s) + g(\mathbf{x}, t))$;
- (vi) $t g_t(\mathbf{x}, s) \leq g(\mathbf{x}, t) + (q - 1) g(\mathbf{x}, s)$;
- (vii) $g_t(\mathbf{x}, |\mathbf{A}|)\, |\mathbf{B} - \mathbf{A}| \leq C \left(g(\mathbf{x}, |\mathbf{B}|) - g_t(\mathbf{x}, |\mathbf{A}|) \frac{\mathbf{A}}{|\mathbf{A}|} \cdot (\mathbf{B} - \mathbf{A}) \right)$ *for some* $C = C(p, q\alpha_+)$ *and all* $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{N \times n}$;

Proof. Since $g(\mathbf{x},\cdot)$ has $(p\alpha(\mathbf{x}),q\alpha(\mathbf{x}))$-structure for every $\mathbf{x}\in\Omega$, parts (i), (ii), (v), and (vi) follow from Lemma 3.1 in [12], and (iii) follows quickly from (ii) letting $c=\varepsilon^{-1}$. To prove (iv), we note that since $g(\mathbf{x},\cdot)$ has $(p\alpha(\mathbf{x}),q\alpha(\mathbf{x}))$-structure, we have

$$\frac{p\alpha(\mathbf{x})-1}{s}\leq\frac{g_{tt}(\mathbf{x},s)}{g_t(\mathbf{x},s)}\leq\frac{q\alpha(\mathbf{x})-1}{s}$$

for almost every $s>0$. Integrating the above inequality from εt to t and using properties of logarithms gives (iv). The proof of (vii) is essentially that of Lemma 5.1 in [12]. (Actually, the quantities in (vii) are the integrands of the estimate provided by the lemma mentioned, but by examining the proof, we see that the estimate actually holds pointwise, which is exactly (vii).) □

The following lemma establishes that the Euler-Lagrange equations hold in the weak sense for minimizers of appropriate functionals. It can be proved using the same method that Evans uses to prove Theorem 4 on page 451 in [11]. The main modification required in the proof is to use part (vi) of Lemma 1 instead of the standard version of Young's inequality.

Lemma 2. *Suppose that* $\alpha:\Omega\to[1,\infty)$ *satisfies* (3)-(5) *and* $g:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ *satisfies* (A_1)-(A_3). *Let* $\mathbf{G}_0\in\mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{w}\in W^1L_g(\Omega;\mathbb{R}^N)$ *is the minimizer for the functional*

$$\mathbf{u}\mapsto\int_\Omega g(\mathbf{x},|\nabla\mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

among all mappings $\mathbf{u}\in\mathbf{w}+W^1_0L_g(\Omega;\mathbb{R}^N)$. *Then*

$$\int_\Omega g_t(\mathbf{x},|\nabla\mathbf{w}\mathbf{G}_0|)\frac{\nabla\mathbf{w}\mathbf{G}_0}{|\nabla\mathbf{w}\mathbf{G}_0|}\cdot\nabla\boldsymbol{\varphi}\mathbf{G}_0\mathrm{d}\mathbf{x}=0$$

for every $\boldsymbol{\varphi}\in W^1_0L_g(\Omega;\mathbb{R}^N)$.

The following theorem gives a type of Sobolev-Poincaré inequality for functions in $W^1L_h(\Omega;\mathbb{R}^N)$.

Theorem 2. *Suppose* $h:\mathbb{R}_+\to\mathbb{R}_+$ *is a function with* (p,q)*-structure and that* $\Omega\subset\mathbb{R}^n$ *is open and bounded with no external cusps. If* $r_0\in(1,n/(n-1))$, *then there exists a constant* C, *which depends on* n, N, p, q, r_0, *and* Ω, *such that if* $0<R<\operatorname{diam}(\Omega)$, *then*

$$\fint_{\Omega(\mathbf{x}_0,R)}h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{r_0}\mathrm{d}\mathbf{x}\leq C\left(\fint_{\Omega(\mathbf{x}_0,R)}h(|\nabla\mathbf{u}|)\mathrm{d}\mathbf{x}\right)^{r_0}$$

for all $\mathbf{u}\in W^1L_h(\Omega;\mathbb{R}^N)$, *where* $\boldsymbol{\xi}=(\mathbf{u})_{\Omega(\mathbf{x}_0,R)}$. *If* $\mathbf{u}\in W^1_0L_h(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$, *then the above inequality also holds with* $\boldsymbol{\xi}=\mathbf{0}$.

Remark 2. The manner in which C depends on Ω is only by the quantity

$$\sup_{\mathbf{x}_0\in\Omega}\ \sup_{R\in(0,\operatorname{diam}(\Omega)]}\frac{|\mathcal{B}(\mathbf{x}_0,R)|}{|\Omega(\mathbf{x}_0,R)|}.$$

Proof. We initially suppose that $\mathbf{w}\in W^1L_{h^{r_0}}(\Omega;\mathbb{R}^N)$. By Lemma 7.14 in [17] (for $\boldsymbol{\xi}=\mathbf{0}$ and $\mathbf{w}\in W^1_0L_{h^{r_0}}(\Omega;\mathbb{R}^N)\subset W^{1,1}_0(\Omega;\mathbb{R}^N)$) and Lemma 1.50 in [20] (for

$\boldsymbol{\xi} = (\mathbf{w})_{\Omega(\mathbf{x}_0,R)}$), there is a constant $C = C(n,N)$ so that

$$|\mathbf{w}(\mathbf{x}) - \boldsymbol{\xi}| \leq C \int_{\Omega(\mathbf{x}_0,R)} \frac{|\nabla \mathbf{w}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{n-1}} \mathrm{d}\mathbf{y},$$

for almost every $\mathbf{x} \in \Omega(\mathbf{x}_0,R)$. Following the proof of Theorem 7 in [9], we obtain

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{w}-\boldsymbol{\xi}|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{w}|)\mathrm{d}\mathbf{x}\right)^{r_0}. \tag{6}$$

So we have obtained the desired estimate under the additional hypothesis that $\mathbf{u} \in W^1L_{h^{r_0}}(\Omega;\mathbb{R}^N)$. To remove this extra assumption, we use an approximation scheme. First assume that $\boldsymbol{\xi} = (\mathbf{u})_{\Omega(\mathbf{x}_0,R)}$. By Theorem 8.31(e) in [4], there is a sequence $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset C^{\infty}(\overline{\Omega(\mathbf{x}_0,R)})$ with $\mathbf{u}_j$ converging to $\mathbf{u}$ in $W^1L_h(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$. Without loss of generality, we can assume that $(\mathbf{u}_j)_{\Omega(\mathbf{x}_0,R)} = \boldsymbol{\xi}$. We also note here that $\{\mathbf{u}_j\} \subset W^1L_{h^{r_0}}(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$ Putting $\mathbf{w} = \mathbf{u}_j - \mathbf{u}_k$ into (6) gives

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}_j-\mathbf{u}_k|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{u}_j - \nabla \mathbf{u}_k|)\mathrm{d}\mathbf{x}\right)^{r_0}.$$

We therefore see that the sequence $\{\mathbf{u}_j\}_{j=1}^{\infty}$ is Cauchy in $L_{h^{r_0}}(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$. Using this and that $\mathbf{u}_j$ converges to $\mathbf{u}$ in $L^1(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$, we must have that actually $\mathbf{u}_j$ converges to $\mathbf{u}$ in $L_{h^{r_0}}(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$. But by (6) with $\mathbf{w} = \mathbf{u}_j$, we have that

$$\fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}_j-\boldsymbol{\xi}|}{R}\right)^{r_0} \mathrm{d}\mathbf{x} \leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla \mathbf{u}_j|)\mathrm{d}\mathbf{x}\right)^{r_0}.$$

Taking limits yields the inequality which was to be shown.

To obtain the desired inequality with $\boldsymbol{\xi} = \mathbf{0}$ for $\mathbf{u} \in W_0^1L_h(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$, we just note that in this case it is possible to take $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset \mathcal{C}_c^{\infty}(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$. This gives that $\{\mathbf{u}_j\}_{j=1}^{\infty} \subset W_0^1L_{h^{r_0}}(\Omega(\mathbf{x}_0,R);\mathbb{R}^N)$, and so the same arguments can be employed, using (6) with $\boldsymbol{\xi} = \mathbf{0}$. □

Next we prove a version of a Sobolev-Poincaré inequality for functions belonging to $W^1L_g(\Omega;\mathbb{R}^N)$; our method of proof follows that of Zhikov [24], with suitable modifications.

Theorem 3. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded and has no external cusps. Suppose also that $\alpha : \Omega \to [1,\infty)$ satisfies (3) and (4), and that $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (A$_1$) and (A$_2$). Then there is some $r_1 > 1$ and $R_0 > 0$, both of which depend on p, q, α_+, and ω, so that for any $\mathbf{u} \in W^1L_g(\Omega;\mathbb{R}^N)$, it follows that $\mathbf{u} \in L_{g^{r_1}}(\Omega;\mathbb{R}^N)$. Furthermore, if $\boldsymbol{\xi} := (\mathbf{u})_{\mathcal{B}(\mathbf{x}_0,R)}$, then*

$$\fint_{\Omega(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right) \mathrm{d}\mathbf{x} \leq C\left\{1 + \left(\fint_{\Omega(\mathbf{x}_0,R)} g(\mathbf{x},|\nabla \mathbf{u}|)^{\frac{1}{r_1}} \mathrm{d}\mathbf{x}\right)^{r_1}\right\}$$

whenever $0 < R < R_0$. The constant C depends on n, N, α_+, M, ω, Ω, and $\int_{\Omega}(1 + g(\mathbf{x},|\nabla \mathbf{u}|))\mathrm{d}\mathbf{x}$. If $\mathbf{u} \in W_0^1L_g(\mathcal{B}(\mathbf{x}_0,R);\mathbb{R}^N)$, then the same inequality holds with $\boldsymbol{\xi} = \mathbf{0}$.

Remark 3. By examining the proof, we see that we also have the inequality

$$\left(\fint_{\Omega(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{r_2} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_2}} \leq C\fint_{\Omega(\mathbf{x}_0,R)} \{1 + g(\mathbf{x},|\nabla \mathbf{u}|)\}\,\mathrm{d}\mathbf{x},$$

for some $r_2 \geq r_1$ and for all $R \in (0, R_0]$, where again $\boldsymbol{\xi} = (\mathbf{u})_{\mathcal{B}(\mathbf{x}_0,R)}$, or $\boldsymbol{\xi} = \mathbf{0}$ if $\mathbf{u} \in W^1_0 L_g(\mathcal{B}(\mathbf{x}_0, R); \mathbb{R}^N)$.

Proof. Fix $\mathbf{x}_0 \in \Omega$, and define

$$\begin{aligned} \alpha_1(R) &:= \inf_{\mathbf{x}\in\Omega(\mathbf{x}_0,R)} \alpha(\mathbf{x}), \\ \alpha_2(R) &:= \sup_{\mathbf{x}\in\Omega(\mathbf{x}_0,R)} \alpha(\mathbf{x}). \end{aligned}$$

For ease of notation, we will henceforth write α_1 and α_2 for $\alpha_1(R)$ and $\alpha_2(R)$, respectively, keeping in mind that α_1 and α_2 vary as R varies. Find $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\overline{\Omega(\mathbf{x}_0, R)}$ such that $\alpha(\mathbf{x}_1) = \alpha_1$ and $\alpha(\mathbf{x}_2) = \alpha_2$. Let $r_0 \in (1, n/(n-1))$ be such that $1 > r_0^{-1/2} \geq 1-(p-1)/(2q\alpha_+)$, so that $h^{\alpha_1/\sqrt{r_0}}$ has $((p+1)/2, q\alpha_+)$-structure. Then select $r_1 > 1$ and $R_0 > 0$ so that for $0 < R \leq R_0$, we have

$$1 \leq \frac{\alpha_2 r_1}{\alpha_1} \leq \sqrt{r_0}, \quad \text{and} \quad \frac{r_1\sqrt{r_0}(\alpha_2-\alpha_1)}{\alpha_1} \leq C\omega(2R) \leq 1. \tag{7}$$

We use (7), Hölder's Inequality and Theorem 2 to obtain

$$\begin{aligned} \fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{\alpha_2 r_1} \mathrm{d}\mathbf{x} &\leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} \left(h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{\frac{\alpha_1}{\sqrt{r_0}}}\right)^{r_0} \mathrm{d}\mathbf{x}\right)^{\frac{\alpha_2 r_1}{\alpha_1\sqrt{r_0}}} \\ &\leq C\left(\fint_{\Omega(\mathbf{x}_0,R)} h\left(|\nabla\mathbf{u}|\right)^{\frac{\alpha_1}{\sqrt{r_0}}} \mathrm{d}\mathbf{x}\right)^{\frac{\alpha_2 r_1\sqrt{r_0}}{\alpha_1}}. \end{aligned}$$

After writing $\frac{\alpha_2 r_1\sqrt{r_0}}{\alpha_1} = \frac{(\alpha_2-\alpha_1)r_1\sqrt{r_0}}{\alpha_1} + r_1\sqrt{r_0}$, Hölder's inequality and (7) yield

$$\begin{aligned} &\left(\fint_{\Omega(\mathbf{x}_0,R)} h\left(|\nabla\mathbf{u}|\right)^{\frac{\alpha_1}{\sqrt{r_0}}} \mathrm{d}\mathbf{x}\right)^{\frac{\alpha_2 r_1\sqrt{r_0}}{\alpha_1}} \\ &\qquad \leq CR^{-Cn\omega(2R)}\left(\int_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\alpha_1}\mathrm{d}\mathbf{x}\right)\left(\fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\frac{\alpha_1}{\sqrt{r_0}}}\mathrm{d}\mathbf{x}\right)^{r_1\sqrt{r_0}}. \end{aligned}$$

By (4), we have that $R^{-Cn\omega(2R)}$ is uniformly bounded, so by setting

$$\tilde{C} := 1 + \sup_{R>0} CR^{-Cn\omega(2R)} \int_\Omega (1 + g(\mathbf{x}, |\nabla\mathbf{u}|))\mathrm{d}\mathbf{x}$$

and combining our previous inequalities, then using Hölder's inequality, we find that

$$\begin{aligned} \fint_{\Omega(\mathbf{x}_0,R)} h\left(\frac{|\mathbf{u}-\boldsymbol{\xi}|}{R}\right)^{\alpha_2 r_1} \mathrm{d}\mathbf{x} &\leq \tilde{C}\left(1 + \fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\frac{\alpha(\mathbf{x})}{\sqrt{r_0}}}\mathrm{d}\mathbf{x}\right)^{r_1\sqrt{r_0}} \\ &\leq \tilde{C}\left(1 + \fint_{\Omega(\mathbf{x}_0,R)} h(|\nabla\mathbf{u}|)^{\frac{\alpha(\mathbf{x})}{r_1}}\mathrm{d}\mathbf{x}\right)^{r_1^2}. \end{aligned} \tag{8}$$

Because Ω is bounded, this implies that $\mathbf{u} \in L_{g^{r_1}}(\Omega; \mathbb{R}^N)$. Furthermore, the estimate in the statement of the theorem follows from (8), (A$_2$), and Hölder's inequality. □

We also have the following Caccioppoli inequality.

Theorem 4. *Suppose that* $\alpha : \Omega \to [1,\infty)$ *satisfies* (3)-(5) *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A_1) *and* (A_2). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{w} \in W^1L_g(\Omega;\mathbb{R}^N)$ *is a minimizer for*

$$\mathbf{v} \mapsto \int_\Omega g(\mathbf{x}, |\nabla \mathbf{v}(\mathbf{x})\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

among all mappings $\mathbf{v} \in \mathbf{w} + W^1_0L_g(\Omega;\mathbb{R}^N)$. *Then there is a constant* $C = C(p, q\alpha_+, |\mathbf{G}_0^{-1}|)$ *such that*

$$\int_{\mathcal{B}(\mathbf{x}_0,\frac{R}{2})} g(\mathbf{x}, |\nabla \mathbf{w}|)\mathrm{d}\mathbf{x} \leq C \int_{\mathcal{B}(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{w}-\boldsymbol{\xi}|}{R}\right)\mathrm{d}\mathbf{x}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$ *and all balls* $\mathcal{B}(\mathbf{x}_0,R) \subset \Omega$.

Proof. First, we show that for every $\mu \in (0,1]$, it holds that

$$g^*(\mathbf{x}, \mu g_t(\mathbf{x},t)) \leq q\alpha_+ \mu^{\frac{q\alpha_+}{q\alpha_+-1}} g(\mathbf{x},t), \tag{9}$$

where $g^*(\mathbf{x},\tau)$ denotes the Young conjugate of $g(\mathbf{x},\cdot)$ evaluated at τ. To demonstrate (9), we first note that for every $\tau \geq 0$, we have

$$g^*(\mathbf{x},\tau) = \tau g_t^{-1}(\mathbf{x},\tau) - g(\mathbf{x}, g_t^{-1}(\mathbf{x},\tau)) \leq \tau g_t^{-1}(\mathbf{x},\tau),$$

where $g_t^{-1}(\mathbf{x},\tau)$ denotes the inverse of $g_t(\mathbf{x},\cdot)$ evaluated at τ. Putting $\tau = \mu g_t(\mathbf{x},t)$ in the above inequality, we have that

$$g^*(\mathbf{x}, \mu g_t(\mathbf{x},t)) \leq \mu g_t(\mathbf{x},t) g_t^{-1}(\mathbf{x}, \mu g_t(\mathbf{x},t)). \tag{10}$$

To estimate $g_t^{-1}(\mathbf{x}, \mu g_t(\mathbf{x},t))$, we apply part (iv) of Lemma 1 with $\varepsilon = \mu^{1/(q\alpha_+-1)}$ and use that $\alpha(\mathbf{x}) \leq \alpha_+$ to obtain $\mu g_t(\mathbf{x},t) \leq g_t(\mathbf{x}, \mu^{1/(q\alpha_+-1)}t)$. Applying $g_t^{-1}(\mathbf{x},\cdot)$ to both sides of this inequality yields $g_t^{-1}(\mathbf{x}, \mu g_t(\mathbf{x},t)) \leq \mu^{\frac{1}{q\alpha_+-1}} t$. Putting this inequality into (10) and using part (i) of Lemma 1, we obtain (9).

With (9) available, we can now prove the result. By the Euler-Lagrange equations, we have that

$$\int_\Omega g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\frac{\nabla \mathbf{w}\mathbf{G}_0}{|\nabla \mathbf{w}\mathbf{G}_0|} \cdot \nabla \boldsymbol{\varphi}\mathbf{G}_0 \mathrm{d}\mathbf{x} = 0, \tag{11}$$

for any $\boldsymbol{\varphi} \in W^1_0L_g(\Omega;\mathbb{R}^N)$. For a ball $\mathcal{B}(\mathbf{x}_0,R) \subset \Omega$, let $\eta \in C_c^\infty(\mathcal{B}(\mathbf{x}_0,R))$ be such that $\chi_{\mathcal{B}(\mathbf{x}_0,R/2)} \leq \eta \leq \chi_{\mathcal{B}(\mathbf{x}_0,R)}$ and $|\nabla\eta| \leq \frac{4}{R}$. For a fixed $\boldsymbol{\xi} \in \mathbb{R}^N$, putting $\boldsymbol{\varphi} = (\mathbf{w}-\boldsymbol{\xi})\eta^{q\alpha_+}$ in (11), we have

$$\begin{aligned}\int_{\mathcal{B}(\mathbf{x}_0,R)} &\eta^{q\alpha_+} g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\,|\nabla \mathbf{w}\mathbf{G}_0|\,\mathrm{d}\mathbf{x} \\ &= -q\alpha_+ \int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+-1} g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\frac{\nabla \mathbf{w}\mathbf{G}_0}{|\nabla \mathbf{w}\mathbf{G}_0|} \cdot (\mathbf{w}-\boldsymbol{\xi}) \otimes \nabla\eta \mathrm{d}\mathbf{x}.\end{aligned}$$

Using Lemma 1 part (i), Young's inequality, and the bound $|\nabla\eta| \leq 4/R$, we have

$$\begin{aligned}p\int_{\mathcal{B}(\mathbf{x}_0,R)} &\eta^{q\alpha_+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x} \\ &\leq 4q\alpha_+\varepsilon \int_{\mathcal{B}(\mathbf{x}_0,R)} \left\{ g^*\left(\mathbf{x}, \eta^{q\alpha_+-1} g_t(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\right) + g\left(\mathbf{x}, \frac{|\mathbf{w}-\boldsymbol{\xi}|}{R\varepsilon}\right)\right\}\mathrm{d}\mathbf{x},\end{aligned}$$

for any $\varepsilon > 0$. Now employing (9) with $\mu = \eta^{q\alpha+-1}$ yields

$$\begin{aligned}
&p\int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x} \\
&\le 4(q\alpha_+)^2\varepsilon \int_{\mathcal{B}(\mathbf{x}_0,R)} \eta^{q\alpha_+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x} + 4q\alpha_+\varepsilon \int_{\mathcal{B}(\mathbf{x}_0,R)} g\left(\mathbf{x}, \frac{|\mathbf{w}-\boldsymbol{\xi}|}{R\varepsilon}\right)\mathrm{d}\mathbf{x}.
\end{aligned}$$

Choosing $\varepsilon = (p-1)/(4q^2\alpha_+^2)$, subtracting the first integral on the right from both sides of the inequality, and utilizing parts (ii) and (iii) of Lemma 1, we obtain the desired result. ☐

Combining the results of Theorems 3 and 4, we use Proposition 5.1 in [16] to obtain the following.

Theorem 5. *Suppose that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (3)-(5) *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A$_1$) *and* (A$_2$). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{w} \in W^1L_g(\Omega; \mathbb{R}^N)$ *is a minimizer for*

$$\mathbf{v} \mapsto \int_\Omega g(\mathbf{x}, |\nabla \mathbf{v}(\mathbf{x})\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

among all mappings $\mathbf{v} \in \mathbf{w} + W_0^1L_g(\Omega; \mathbb{R}^N)$. *Then there is a constant* $C < \infty$ *and* $r_3 > 1$ *such that*

$$\left(\fint_{\mathcal{B}(\mathbf{x}_0,\frac{R}{2})} g(\mathbf{x}, |\nabla \mathbf{w}|)^{r_3}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r_3}} \le C\left(\fint_{\mathcal{B}(\mathbf{x}_0,R)} \{1 + g(\mathbf{x}, |\nabla \mathbf{w}|)\}\,\mathrm{d}\mathbf{x}\right)$$

for all balls $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$. *The constants* C *and* r_3 *depend only on* n, N, p, q, M, α_+, ω, $\left|\mathbf{G}_0^{-1}\right|$, *and* $\int_\Omega (1 + g(\mathbf{x}, |\nabla \mathbf{w}|))\mathrm{d}\mathbf{x}$.

Using the argument provided in [7], along with Theorem 4 and Lemma 1, we have the following boundary version of the above result.

Theorem 6. *Suppose that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (3)-(5) *and that* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A$_1$)-(A$_3$). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and assume* $\mathcal{B}(\mathbf{x}_0, R) \subset \Omega$. *Suppose that* $\mathbf{w}$ *is a minimizer for*

$$\mathbf{u} \mapsto \int_{\mathcal{B}(\mathbf{x}_0,R/4)} g(\mathbf{x}, |\nabla \mathbf{u}(\mathbf{x})\mathbf{G}_0|)\mathrm{d}\mathbf{x}$$

satisfying $\mathbf{w} - \mathbf{v} \in W_0^1L_g(\mathcal{B}(\mathbf{x}_0, R/4); \mathbb{R}^N)$ *for some function* $\mathbf{v} \in W^1L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla \mathbf{v}|) \in L^r(\mathcal{B}(\mathbf{x}_0, R/2))$, *where* $r > 1$. *Then there is a constant* $C < \infty$ *and* $r_4 > 1$ *such that*

$$\left(\fint_{\mathcal{B}(\mathbf{x}_0,R/4)} g(\mathbf{x}, |\nabla \mathbf{w}|)^{r_4}\mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} \le C\left(\fint_{\mathcal{B}(\mathbf{x}_0,R/2)} \{1 + g(\mathbf{x}, |\nabla \mathbf{v}|)^r\}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{r}}.$$

Here, the constants C *and* r_4 *only depend on* n, N, p, q, M, α_+, ω, $\left|\mathbf{G}_0^{-1}\right|$, r, *and* $\int_\Omega (1 + g(\mathbf{x}, |\nabla \mathbf{u}|))\mathrm{d}\mathbf{x}$.

We also have the following theorem that gives Morrey regularity for the function itself if the gradient possesses Morrey regularity. The proof of this theorem uses some ideas from Lemma 2.2 and Proposition 2.3 in [18].

Theorem 7. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded, and has no external cusps. Suppose that $\alpha : \Omega \to [1, \infty)$ satisfies* (3) *and* (4)*, and that $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (A$_1$) *and* (A$_2$)*. Suppose also that $\mathbf{u} \in W^1L_g(\Omega; \mathbb{R}^N)$ and $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$ for some $0 \leq \lambda < n$. Then, if r_2 is as in Remark 3 and $1 \leq s < r_2$, we have that $g(\cdot, |\mathbf{u}|)^s \in L^{1,\kappa}(\Omega)$ for every $0 \leq \kappa < \min\{n + s(p + \lambda - n), n\}$.*

Proof. Put $\mu := \min\{n + s(p + \lambda - n), n\}$, fix $\kappa < \mu$, and let $\gamma \in (\kappa, \mu)$. Since $\gamma < \mu \leq n + s(p + \lambda - n)$, there is some $s' \in (s, r_2)$ such that $n + s'(p + \lambda - n) \geq \gamma$. With R_0 as in Theorem 3, find $R_1 \in (0, R_0]$ so that $\omega(2R_1) \leq \frac{s'}{s} - 1$. Since Ω has no external cusps, there is some constant $A < \infty$ such that

$$\frac{1}{A} |\Omega(\mathbf{x}_0, a)| \left(\frac{b}{a}\right)^n \leq |\Omega(\mathbf{x}_0, b)| \leq A |\Omega(\mathbf{x}_0, a)| \left(\frac{b}{a}\right)^n \tag{12}$$

for any a, $b \in (0, \operatorname{diam}(\Omega))$ and $\mathbf{x}_0 \in \Omega$, and hence to show that $g(\cdot, |\mathbf{u}|) \in L^{1,\kappa}(\Omega)$, it suffices to show that

$$\sup_{\substack{\mathbf{x}_0 \in \Omega \\ 0 < \rho < R_1}} |\Omega(\mathbf{x}_0, \rho)|^{-\frac{\kappa}{n}} \int_{\Omega(\mathbf{x}_0, \rho)} g(\mathbf{x}, |\mathbf{u}|) \mathrm{d}\mathbf{x} < \infty.$$

With $\mathbf{x}_0 \in \Omega$ and $\rho \in (0, R_1)$ fixed for the remainder of the proof, we denote by Ω_r the set $\Omega(\mathbf{x}_0, r)$, by α_1 and α_2 the quantities $\inf_{\mathbf{x} \in \Omega_{R_1}} \alpha(\mathbf{x})$ and $\sup_{\mathbf{x} \in \Omega_{R_1}} \alpha(\mathbf{x})$, respectively, and by $\mathbf{u}_t$ the average $(\mathbf{u})_{\Omega_t}$. We observe that our choice of R_1 ensures that $\alpha_2 s \leq \alpha_1 s'$, since $\alpha_2/\alpha_1 \leq \omega(2R_1)/\alpha_1 + 1 \leq s'/s$. In what follows, we write C for any constant that does not depend on either $\mathbf{x}_0$ or ρ; in particular, we allow C to depend on $\mathbf{u}$, κ, and A. We have

$$\begin{aligned} |\Omega_\rho|^{-\frac{\kappa}{n}} \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}|)^s \mathrm{d}\mathbf{x} &\leq C |\Omega_\rho|^{-\frac{\kappa}{n}} \left\{ \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u} - \mathbf{u}_\rho|)^s \mathrm{d}\mathbf{x} \right. \\ &\qquad \left. + \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}_{R_0} - \mathbf{u}_\rho|)^s \mathrm{d}\mathbf{x} + \int_{\Omega_\rho} g(\mathbf{x}, |\mathbf{u}_{R_0}|)^s \mathrm{d}\mathbf{x} \right\} \\ &= C |\Omega_\rho|^{-\frac{\kappa}{n}} \{I_1 + I_2 + I_3\}, \end{aligned} \tag{13}$$

where I_1, I_2, and I_3 are defined naturally. We first note that for any $R < R_0$, by Theorem 3 and the previously noted inequality $\alpha_2 s \leq \alpha_1 s'$, it holds that

$$\begin{aligned} \int_{\Omega_R} h(|\mathbf{u} - \mathbf{u}_R|)^{\alpha_2 s} \mathrm{d}\mathbf{x} &\leq C \int_{\Omega_R} \left\{ 1 + h\left(|\mathbf{u} - \mathbf{u}_R|\right)^{\alpha(\mathbf{x}) s'} \right\} \mathrm{d}\mathbf{x} \\ &\leq C \left(|\Omega_R| + |\Omega_R|^{\frac{n + ps'}{n}} \fint_{\Omega_R} \left\{ h\left(\frac{|\mathbf{u} - \mathbf{u}_R|}{R}\right)^{\alpha(\mathbf{x}) s'} \right\} \mathrm{d}\mathbf{x} \right) \\ &\leq C \left(|\Omega_R| + |\Omega_R|^{\frac{n + ps'}{n}} \left(\fint_{\Omega_R} \left\{ 1 + h(|\nabla \mathbf{u}|)^{\alpha(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} \right)^{s'} \right) \\ &\leq C \left(|\Omega_R| + |\Omega_R|^{\frac{n + ps'}{n}} + |\Omega_R|^{\frac{n + s'(p + \lambda - n)}{n}} \|g(\cdot, |\nabla \mathbf{u}|)\|_{L^{1,\lambda}}^{s'} \right). \end{aligned}$$

But $\gamma < n$, and clearly $n + ps' \geq n$; also, by the selection of s', we have that $n + s'(p + \lambda - n) \geq \gamma$, so defining the finite constant G by $G := 1 + \|g(\cdot, |\nabla \mathbf{u}|)\|_{L^{1,\lambda}}^{s'}$, the above string of inequalities yields

$$|\Omega_R|^{-\frac{\gamma}{n}}\int_{\Omega_R} h(|\mathbf{u}-\mathbf{u}_R|)^{\alpha_2 s}\mathrm{d}\mathbf{x}\le CG, \tag{14}$$

for $0<R<R_1$. With this work, we quickly observe

$$I_1\le C\int_{\Omega_\rho}\{1+h(|\mathbf{u}-\mathbf{u}_\rho|)^{\alpha_2 s}\}\,\mathrm{d}\mathbf{x}\le CG\,|\Omega_\rho|^{\frac{\gamma}{n}}.$$

To estimate I_2, we first let $0<a<b\le R_1$ be given, and define $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ to be the inverse of $h^{\alpha_2 s}$. Then by Jensen's inequality and (14), we find

$$\begin{aligned}h(|\mathbf{u}_b-\mathbf{u}_a|)^{\alpha_2 s}&\le h\left(|\Omega_a|^{-1}\int_{\Omega_a}|\mathbf{u}-\mathbf{u}_b|\,\mathrm{d}\mathbf{x}\right)^{\alpha_2 s}\le|\Omega_a|^{-1}\int_{\Omega_a}h(|\mathbf{u}-\mathbf{u}_b|)^{\alpha_2 s}\mathrm{d}\mathbf{x}\\&\le|\Omega_a|^{-1}\,|\Omega_b|^{\frac{\gamma}{n}}\,G.\end{aligned}$$

For each $i=0,1,\ldots,$, put $r_i=2^{-i}R_1$. Putting $a=r_i$ and $b=r_{i-1}$ in the above inequality and using that φ is the inverse of $h^{\alpha_2 s}$, we have

$$\left|\mathbf{u}_{r_{i-1}}-\mathbf{u}_{r_i}\right|\le\varphi\left(|\Omega_{r_i}|\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}G\right). \tag{15}$$

Now, if a nonnegative integer k is selected so that $2^{-k-1}R_1\le\rho\le 2^{-k}R_1$, then it follows that $k\le\frac{1}{\log(2)}\log\left(\frac{R_1}{\rho}\right)$, and so, using the triangle inequality and (15), we obtain

$$\begin{aligned}|\mathbf{u}_{R_0}-\mathbf{u}_\rho|&\le\sum_{i=1}^{k}\varphi\left(|\Omega_{r_i}|\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}G\right)+\varphi\left(|\Omega_\rho|^{-1}\left|\Omega_{r_k}\right|^{\frac{\gamma}{n}}G\right)\\&\le\sum_{i=1}^{k+1}\varphi\left(|\Omega_{r_i}|\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}G\right).\end{aligned}$$

Hence, employing Lemma 1 and Jensen's inequality, along with the two preceding inequalities, we have

$$\begin{aligned}h(|\mathbf{u}_{R_0}-\mathbf{u}_\rho|)^{\alpha_2 s}&\le(k+1)^{qs\alpha_+}h\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\varphi\left(|\Omega_{r_i}|\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}G\right)\right)^{\alpha_2 s}\\&\le(k+1)^{qs\alpha_+-1}\sum_{i=1}^{k+1}|\Omega_{r_i}|\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}G\\&\le\left(\frac{1}{\log(2)}\log\left(\frac{R_1}{\rho}\right)+1\right)^{qs\alpha_+-1}G\sum_{i=1}^{k+1}|\Omega_{r_i}|^{-1}\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}.\end{aligned}$$

But (12) gives that

$$|\Omega_{r_i}|^{-1}\left|\Omega_{r_{i-1}}\right|^{\frac{\gamma}{n}}\le C\left(2^{(n-\gamma)(i-k)}\right)\left|\Omega_{r_{k+1}}\right|^{\frac{\gamma-n}{n}}\le C\left(2^{(n-\gamma)(i-k)}\right)|\Omega_\rho|^{\frac{\gamma-n}{n}},$$

and hence

$$\begin{aligned} h(|\mathbf{u}_{R_1}-\mathbf{u}_\rho|)^{\alpha_2 s} &\le \left(\frac{1}{\log(2)}\log\left(\frac{R_1}{\rho}\right)+1\right)^{qs\alpha_+-1} G\,|\Omega_\rho|^{\frac{\gamma-n}{n}}\sum_{i=1}^{k+1}\left(2^{n-\gamma}\right)^{i-k} \\ &\le \left(\frac{1}{\log(2)}\log\left(\frac{R_1}{\rho}\right)+1\right)^{qs\alpha_+-1} G\,|\Omega_\rho|^{\frac{\gamma-n}{n}}\sum_{i=-1}^{k-1}\left(2^{n-\gamma}\right)^{-i} \\ &\le \left(\frac{1}{\log(2)}\log\left(\frac{R_1}{\rho}\right)+1\right)^{qs\alpha_+-1} G\,|\Omega_\rho|^{\frac{\gamma-n}{n}}. \end{aligned}$$

Using this estimate and that $\gamma > \kappa$, we have that

$$\begin{aligned} I_2 &\le C\int_{\Omega_\rho}\{1+h(|\mathbf{u}_{R_1}-\mathbf{u}_\rho|)^{\alpha_2}\}\,\mathrm{d}\mathbf{x} \le C\left(\frac{1}{\log(2)}\log\left(\frac{R_1}{\rho}\right)+1\right)^{qs\alpha_+-1} G\,|\Omega_\rho|^{\frac{\gamma}{n}} \\ &\le CG\,|\Omega_\rho|^{\frac{\kappa}{n}}. \end{aligned}$$

Turning now to I_3, we have by Jensen's inequality and the inequality $\alpha_2 s \le \alpha_1 s' \le \alpha_1 r_2$ that

$$\begin{aligned} I_3 &\le |\Omega_\rho|\left(1+h(|\mathbf{u}_{R_1}|)^{\alpha_2 s}\right) \le |\Omega_\rho|\left(1+|\Omega_{R_1}|^{-1}\int_{\Omega_{R_1}} h(|\mathbf{u}|)^{\alpha_2 s}\mathrm{d}\mathbf{x}\right) \\ &\le |\Omega_\rho|\left(1+|\Omega_{R_1}|^{-1}\int_{\Omega}\{g(\mathbf{x},|\mathbf{u}|)^{r_2}\mathrm{d}\mathbf{x}\right). \end{aligned}$$

By Remark 3, we have that $g(\cdot,|\mathbf{u}|)^{r_2} \in L^1(\Omega)$, and so we have

$$I_3 \le C\left(1+\|g(\cdot,|\mathbf{u}|)^{r_2}\|_{L^1}\right)|\Omega_\rho|.$$

Putting our estimates for I_1, I_2, and I_3 into (13) and using that $\gamma > \kappa$, we have

$$|\Omega_\rho|^{-\frac{\kappa}{n}}\int_{\Omega_\rho} g(\mathbf{x},|\mathbf{u}|)\mathrm{d}\mathbf{x} \le C\left(1+G+|\Omega_{R_1}|^{-1}\|g(\cdot,|\mathbf{u}|)^{r_2}\|_{L^1(\Omega)}\right)$$

for all $0 < \rho \le R_1$, which gives that $g(\cdot,|\mathbf{u}|) \in L^{1,\kappa}(\Omega)$, as desired. □

The next lemma is essentially a restatement of Lemma 1 in [13], and is proved there.

Lemma 3. *Let* $\varphi : (0,\infty) \to \mathbb{R}$ *be nondecreasing, and suppose that there exist* $A \ge 1$, $B \ge 0$, $R_0 > 0$, *and* $\alpha > \beta \ge 0$ *such that for some* $0 \le \varepsilon \le \left(\frac{1}{2A}\right)^{\frac{2\alpha}{\alpha-\beta}}$, *the inequality*

$$\varphi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^{\alpha}+\varepsilon\right]\varphi(R)+BR^{\beta}$$

holds for each $0 < \rho \le R \le R_0$. *Then there is some finite constant* $C = C(A,\alpha,\beta)$ *such that*

$$\varphi(\rho) \le C\left(\frac{\rho}{R}\right)^{\beta}(\varphi(R)+B)$$

for all $0 < \rho \le R \le R_0$.

4. Results for Convex Functionals

We recall the following local Lipschitz regularity result from [12], which is also proved in [10].

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^N)$ be a minimizer of the functional*

$$\mathbf{u} \mapsto \int\limits_{\Omega} h(|\nabla \mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x},$$

where h is a function with (p,q)-structure and $\mathbf{G}_0$ is an invertible $n \times n$ constant matrix. Then there is a constant $C = C\left(n,p,q,\left|\mathbf{G}_0^{-1}\right|,|\mathbf{G}_0|\right)$ such that

$$\|h(|\nabla \mathbf{v}|)\|_{L^\infty(\mathcal{B}_{\mathbf{x}_0,\rho})} \leq \frac{C}{(R-\rho)^n} \int\limits_{\mathcal{B}_{\mathbf{x}_0,R}} h(|\nabla \mathbf{v}|)\mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$ and $0<\rho<R$.

The proof of the following theorem uses many of the ideas from the proof for Proposition 3.1 in [1].

Theorem 9. *Suppose that $\alpha : \Omega \to [1,\infty)$ satisfies (3)-(5), and that $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (A_1)-(A_3). Let $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ be an invertible matrix, and suppose $\mathbf{v}$ is a minimizer for the functional J defined by*

$$\mathbf{w} \mapsto \int_{\Omega} g(\mathbf{x},|\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x},$$

among all mappings $\mathbf{w} \in \mathbf{v} + W_0^1 L_g(\Omega;\mathbb{R}^N)$. Then for every $0 \leq \kappa < n$, there are constants C_κ and R_κ, which, in addition to κ, also depend on n, N, p, q, α_+, $|\mathbf{G}_0|$, $\left|\mathbf{G}_0^{-1}\right|$, M, ω, δ, κ, and $L := \int_\Omega (1+g(\mathbf{x},|\nabla \mathbf{w}\mathbf{G}_0|))\mathrm{d}\mathbf{x}$, such that

$$\int_{\mathcal{B}(\mathbf{x}_0,\rho)} (1+g(\mathbf{x},|\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x} \leq C_\kappa \left(\frac{\rho}{R}\right)^\kappa \int_{\mathcal{B}(\mathbf{x}_0,R)} (1+g(\mathbf{x},|\nabla \mathbf{v}|))\,\mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}(\mathbf{x}_0,R) \subset \Omega$ and $0 < \rho \leq R \leq R_\kappa$.

Proof. Fix $\kappa \in [0,n)$. Throughout the proof, C will denote a constant that may depend only on the parameters listed in the statement of the theorem, and its value may change from line to line. We define the functions $\alpha_1, \alpha_2 : \Omega \times (0,\infty) \to [1,\infty)$ by

$$\alpha_1(\mathbf{x},r) := \min\left\{\alpha(\mathbf{y}) : \mathbf{y} \in \overline{\Omega(\mathbf{x},r)}\right\},$$
$$\alpha_2(\mathbf{x},r) := \max\left\{\alpha(\mathbf{y}) : \mathbf{y} \in \overline{\Omega(\mathbf{x},r)}\right\}.$$

By Theorem 5 and Hölder's inequality, we have that there are constants $C>0$ and $r_3>1$ such that

$$\left(\fint_{\mathcal{B}(\mathbf{x},r)} \{1+g(\mathbf{x},|\nabla \mathbf{v}|)^s\}\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{s}} \leq C\fint_{\mathcal{B}(\mathbf{x},2r)} \{1+g(\mathbf{x},|\nabla \mathbf{v}|)\}\,\mathrm{d}\mathbf{x} \tag{16}$$

whenever $\mathcal{B}(\mathbf{x},2r) \subset \Omega$ and $s \in [1,r_3]$. Fix $r' \in (1,\min\{2,r_3\})$, and select $R_0 \in (0,1/2)$ so that $\omega(2R_0) < \frac{r_3}{r'}-1$. Now we suppose that $0 < 8\rho < R \leq R_0$ and $\mathbf{x}_0 \in \Omega$ are such that $\mathcal{B}(\mathbf{x}_0,R) \subset \Omega$. With $\mathbf{x}_0$ and R fixed, we will use for convenience α_1 and α_2 to denote $\alpha_1(\mathbf{x}_0,R)$ and $\alpha_2(\mathbf{x}_0,R)$, respectively, and $\mathcal{B}_r := \mathcal{B}(\mathbf{x}_0,r)$ for

$r > 0$. Note that $\alpha_2 \leq \alpha_1 + \omega(2R)$, and hence the choices for R_0 and r' above imply that

$$\alpha_2 r' \leq \alpha(\mathbf{x}) r'(1+\omega(2R)) \leq \alpha(\mathbf{x}) r_3 \tag{17}$$

for all $\mathbf{x} \in \mathcal{B}_R$.

We can select $\mathbf{x}_2 \in \overline{\mathcal{B}_R}$ so that $\alpha(\mathbf{x}_2) = \alpha_2$. Let $\mathbf{w} \in W^1 L_{h^{\alpha_2}}(\mathcal{B}_{R/4}; \mathbb{R}^N)$ be the minimizer for the functional $J_{\mathbf{x}_2} : W^1 L_{h^{\alpha_2}}(\mathcal{B}_{R/4}; \mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_{\mathbf{x}_2}(\mathbf{u}) := \int_{\mathcal{B}_{R/4}} g(\mathbf{x}_2, |\nabla \mathbf{u} \mathbf{G}_0|) \mathrm{d}\mathbf{x},$$

satisfying $\mathbf{w} = \mathbf{v}$ on $\partial \mathcal{B}_{R/4}$. (Note that by Theorem 5 and (17), we have that $\mathbf{v} \in W^1 L_{h^{\alpha_2}}(\mathcal{B}_{R/4}; \mathbb{R}^N)$.) We clearly have

$$\begin{aligned}
\int\limits_{\mathcal{B}_\rho} g(\mathbf{x}_2, |\nabla \mathbf{v} \mathbf{G}_0|) \mathrm{d}\mathbf{x} = \int\limits_{\mathcal{B}_\rho} g(\mathbf{x}_2, |\nabla \mathbf{w} g_0|) \mathrm{d}\mathbf{x} + \int\limits_{\mathcal{B}_\rho} \Big\{ & g(\mathbf{x}_2, |\nabla \mathbf{v} \mathbf{G}_0|) - g(\mathbf{x}_2, |\nabla \mathbf{w} \mathbf{G}_0|) \\
& - g_t(\mathbf{x}_2, |\nabla \mathbf{w} \mathbf{G}_0|) \frac{\nabla \mathbf{w} \mathbf{G}_0}{|\nabla \mathbf{w} \mathbf{G}_0|} \cdot (\nabla \mathbf{v} - \nabla \mathbf{w}) \mathbf{G}_0 \Big\} \mathrm{d}\mathbf{x} \\
& + \int\limits_{\mathcal{B}_\rho} g_t(\mathbf{x}_2, |\nabla \mathbf{w} \mathbf{G}_0|) \frac{\nabla \mathbf{w} \mathbf{G}_0}{|\nabla \mathbf{w} \mathbf{G}_0|} \cdot (\nabla \mathbf{v} - \nabla \mathbf{w}) \mathbf{G}_0 \mathrm{d}\mathbf{x} \\
= I_1 + I_2 + I_3, &
\end{aligned} \tag{18}$$

where I_1, I_2, and I_3 are defined to be the first, second, and third integrals, respectively. By part (vii) of Lemma 1, there is a constant C depending only on n, p, and $q\alpha_+$ such that $I_3 \leq C(I_1 + I_2)$, so that it remains only to estimate I_1 and I_2.

For I_1, we have from Theorem 8 and the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_2}$ that

$$I_1 \leq C \left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}_{R/4}} g(\mathbf{x}_2, |\nabla \mathbf{w}|) \mathrm{d}\mathbf{x} \leq C \left(\frac{\rho}{R}\right)^n \int_{\mathcal{B}_{R/4}} g(\mathbf{x}_2, |\nabla \mathbf{v}|) \mathrm{d}\mathbf{x}.$$

To estimate I_2, we note that the integrand is nonnegative because of the convexity of $g(\mathbf{x}_2, \cdot)$, so we can expand the domain of integration to $\mathcal{B}_{R/4}$. Then using the Euler-Lagrange equations for $\mathbf{w}$ (Lemma 2), the minimality of $\mathbf{v}$ for J, and (A$_3$), we have

$$\begin{aligned}
I_2 \leq\ & C \int\limits_{\mathcal{B}_{R/4}} \{ g(\mathbf{x}_2, |\nabla \mathbf{v} \mathbf{G}_0|) - g(\mathbf{x}, |\nabla \mathbf{v} \mathbf{G}_0|) \} \mathrm{d}\mathbf{x} \\
& + C \int\limits_{\mathcal{B}_{R/4}} \{ g(\mathbf{x}, |\nabla \mathbf{w} \mathbf{G}_0|) - g(\mathbf{x}_2 |\nabla \mathbf{w} \mathbf{G}_0|) \} \mathrm{d}\mathbf{x} \\
\leq\ & C \int\limits_{\mathcal{B}_{R/4}} \{ \omega(R) \, (1 + h(|\nabla \mathbf{v}|)^{\alpha_2}) \log{(e + h(|\nabla \mathbf{v}|))} + \delta(R) \, (1 + h(|\nabla \mathbf{v}|)^{\alpha_2}) \} \mathrm{d}\mathbf{x} \\
& + C \int\limits_{\mathcal{B}_{R/4}} \{ \omega(R) \, (1 + h(|\nabla \mathbf{w}|)^{\alpha_2}) \log(e + h(|\nabla \mathbf{w}|)) + \delta(R) (1 + h(|\nabla \mathbf{w}|)^{\alpha_2}) \} \mathrm{d}\mathbf{x}
\end{aligned}$$

Now we use the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_2}$ in the above inequality to conclude that

$$
\begin{aligned}
I_2 &\leq C\omega(R) \int\limits_{\mathcal{B}_{R/4}} \left(1 + h(|\nabla \mathbf{v}|)^{\alpha_2}\right) \log\left(e + h(|\nabla \mathbf{v}|)\right) \mathrm{d}\mathbf{x} \\
&\quad + C\omega(R) \int\limits_{\mathcal{B}_{R/4}} \left(1 + h(|\nabla \mathbf{w}|)^{\alpha_2}\right) \log\left(e + h(|\nabla \mathbf{w}|)\right) \mathrm{d}\mathbf{x} \\
&\quad + C\delta(R) \int\limits_{\mathcal{B}_{R/4}} \left(1 + h(|\nabla \mathbf{v}|)^{\alpha_2}\right) \mathrm{d}\mathbf{x} \\
&= I_{2,1} + I_{2,2} + I_{2,3}.
\end{aligned} \tag{19}
$$

Before we estimate $I_{2,1}$, $I_{2,2}$, and $I_{2,3}$, we introduce some notation and ideas that we will use in these estimates. As in [1], for each $s \in [1, \infty)$, we use the norm $\|\cdot\|_s$ on $L^s(\mathcal{B}_{R/4})$ by

$$
\left\|\hat{h}\right\|_s := \left(\fint_{\mathcal{B}_{R/4}} \left|\hat{h}\right|^s \right)^{\frac{1}{s}}.
$$

We recall the following from [1], which follows from a result in [19]: for each $s > 1$, there is a constant $c(s)$, which does not depend on R or the mapping $\hat{h}$, such that

$$
\fint_{\mathcal{B}_{R/4}} \left|\hat{h}\right| \log\left(e + \frac{\left|\hat{h}\right|}{\left\|\hat{h}\right\|_1} \right) \mathrm{d}\mathbf{x} \leq c(s) \left\|\hat{h}\right\|_s. \tag{20}
$$

Using the inequality $\log(e + ab) \leq \log(e + a) + \log(e + b)$, which is valid for all $a, b \geq 0$, we have that

$$
\begin{aligned}
I_{2,1} &\leq C\omega(R) \int_{\mathcal{B}_{R/4} \cap \{h(|\nabla \mathbf{v}|) \geq e\}} h(|\nabla \mathbf{v}|)^{\alpha_2} \log(h(|\nabla \mathbf{v}|)^{\alpha_2}) \mathrm{d}\mathbf{x} + C\omega(R) R^n \\
&\leq C\omega(R) R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \log\left(e + \frac{h(|\nabla \mathbf{v}|)^{\alpha_2}}{\|h(|\nabla \mathbf{v}|)^{\alpha_2}\|_1} \right) \mathrm{d}\mathbf{x} \\
&\quad + C\omega(R) R^n \log\left(e + \|h(|\nabla \mathbf{v}|)^{\alpha_2}\|_1\right) \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \mathrm{d}\mathbf{x} + C\omega(R) R^n.
\end{aligned} \tag{21}
$$

By Hölder's inequality and (17), we find that

$$
\begin{aligned}
\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \mathrm{d}\mathbf{x} &\leq \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2 r'} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} \\
&\leq \left(\fint_{\mathcal{B}_{R/4}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x}) r' (1 + \omega(2R))} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}}.
\end{aligned} \tag{22}
$$

Employing (17) and (16), we have that

$$
\log\left(e + \|h(|\nabla \mathbf{v}|)^{\alpha_2}\|_1\right) \leq C \log\left(\frac{1}{R}\right). \tag{23}
$$

Putting (22) and (23) into (21), and using (20) in the first integral of the right side of (21) yields

$$\begin{aligned} I_{2,1} \leq{}& C\omega(R)R^n \left(\fint_{\mathcal{B}_{R/4}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})r'(1+\omega(2R)} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} \\ &+ C\omega(R)\log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R/4}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})r'(1+\omega(2R))} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}} \\ &+ C\omega(R)R^n. \end{aligned}$$

Using (16) in the above inequality with $s = r'(1+\omega(2R))$, we obtain

$$\begin{aligned} I_{2,1} &\leq C\omega(R)\log\left(\frac{1}{R}\right)R^n \left(\fint_{\mathcal{B}_{R/2}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} \right)^{1+\omega(2R)} + C\omega(R)R^n \\ &\leq C\omega(R)\log\left(\frac{1}{R}\right) R^{-n\omega(2R)} \int_{\mathcal{B}_{R/2}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} + C\omega(R)R^n \\ &\leq C\omega(R)\log\left(\frac{1}{R}\right) \int_{\mathcal{B}_{R/2}} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x}. \end{aligned} \tag{24}$$

We note here that we used (4) to conclude that $R^{-n\omega(2R)} \leq C$ in the last line.

Now we turn to the task of providing an estimate for $I_{2,2}$. In the same way that we obtained (21), we find that

$$\begin{aligned} I_{2,2} \leq{}& C\omega(R)R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2} \log\left(e + \frac{h(|\nabla \mathbf{w}|)^{\alpha_2}}{\left\|h(|\nabla \mathbf{w}|)^{\alpha_2}\right\|_1} \right) \mathrm{d}\mathbf{x} \\ &+ C\omega(R)R^n \log\left(e + \left\|h(|\nabla \mathbf{w}|)^{\alpha_2}\right\|_1\right) \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2}\mathrm{d}\mathbf{x} + C\omega(R)R^n. \end{aligned}$$

Using the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_2}$, we can bound $\log(e + \left\|h(|\nabla \mathbf{w}|)^{\alpha_2}\right\|_1)$ in the same way that we bounded the analogous term for $\mathbf{v}$, so that the above inequality gives

$$\begin{aligned} I_{2,2} \leq{}& C\omega(R)R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2} \log\left(e + \frac{h(|\nabla \mathbf{w}|)^{\alpha_2}}{\left\|h(|\nabla \mathbf{v}|)^{\alpha_2}\right\|_1} \right) \mathrm{d}\mathbf{x} \\ &+ C\omega(R)\log\left(\frac{1}{R}\right) R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2}\mathrm{d}\mathbf{x} + C\omega(R)R^n. \end{aligned} \tag{25}$$

Now, by Theorem 6, we have that there is some $r_4 > 1$ such that

$$\left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4}\mathrm{d}\mathbf{x} \right)^{\frac{1}{r_4}} \leq C \left(\fint_{\mathcal{B}_{R/2}} \left\{ 1 + h(|\nabla \mathbf{v}|)^{\alpha_2 r'} \right\} \mathrm{d}\mathbf{x} \right)^{\frac{1}{r'}}. \tag{26}$$

Utilizing (20) in the first term of (25) and Hölder's inequality in the second, we find that

$$
\begin{aligned}
I_{2,2} &\leq C\omega(R)R^n \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} \\
&\quad + C\omega(R)\log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} + C\omega(R)R^n \\
&\leq C\omega(R)\log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{w}|)^{\alpha_2 r_4} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r_4}} + C\omega(R)R^n.
\end{aligned}
\tag{27}
$$

We use (26) and (17) in (27) to arrive at the inequality

$$
I_{2,2} \leq C\omega(R)\log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R/2}} \left\{1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})r'(1+\omega(2R))}\right\} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r'}} + C\omega(R)R^n.
$$

Now we employ (16) and conclude as in (24) that

$$
\begin{aligned}
I_{2,2} &\leq C\omega(R)\log\left(\frac{1}{R}\right) R^n \left(\fint_{\mathcal{B}_{R}} \left\{1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})}\right\} \mathrm{d}\mathbf{x}\right)^{1+\omega(2R)} + C\omega(R)R^n \\
&\leq C\omega(R)\log\left(\frac{1}{R}\right) R^{-n\omega(2R)} \int_{\mathcal{B}_{R}} \left\{1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})}\right\} \mathrm{d}\mathbf{x} + C\omega(R)R^n \\
&\leq C\omega(R)\log\left(\frac{1}{R}\right) \int_{\mathcal{B}_{R}} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|)) \,\mathrm{d}\mathbf{x}.
\end{aligned}
\tag{28}
$$

The estimate for $I_{2,3}$ is easier. By (22) and (16), we have

$$
\begin{aligned}
I_{2,3} &\leq C\delta(R)R^n \fint_{\mathcal{B}_{R/4}} h(|\nabla \mathbf{v}|)^{\alpha_2} \mathrm{d}\mathbf{x} + C\delta(R)R^n \\
&\leq C\delta(R)R^n \left(\fint_{\mathcal{B}_{R/4}} \left\{1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})r'(1+\omega(2R))}\right\} \mathrm{d}\mathbf{x}\right)^{\frac{1}{r'}} + C\delta(R)R^n \\
&\leq C\delta(R)R^n \left(\fint_{\mathcal{B}_{R/2}} \left\{1 + h(|\nabla \mathbf{v}|)^{\alpha(\mathbf{x})}\right\} \mathrm{d}\mathbf{x}\right)^{1+\omega(2R)} + C\delta(R)R^n.
\end{aligned}
$$

Similarly to (24), we can conclude

$$
I_{2,3} \leq C\delta(R) \int_{\mathcal{B}_{R/2}} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|)) \,\mathrm{d}\mathbf{x}. \tag{29}
$$

Putting (24), (28), and (29) into (19) yields

$$
I_2 \leq C\left(\omega(R)\log\left(\frac{1}{R}\right) + \delta(R)\right) \int_{\mathcal{B}_{R/2}} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|)) \,\mathrm{d}\mathbf{x}. \tag{30}
$$

We have already noted that $I_3 \leq C(I_1 + I_2)$, so putting our estimates for I_1 and I_2 into (18) gives

$$
\int_{\mathcal{B}_\rho} (1+g(\mathbf{x}, |\nabla \mathbf{v}|)) \,\mathrm{d}\mathbf{x} \leq C\left(\left(\frac{\rho}{R}\right)^n + \omega(R)\log\left(\frac{1}{R}\right) + \delta(R)\right) \int_{\mathcal{B}_R} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|)) \,\mathrm{d}\mathbf{x} \tag{31}
$$

for all $0 < 8\rho < R \leq R_0$. But by enlarging C if necessary, we clearly have that (31) holds for $0 < \rho \leq R \leq 8\rho \leq R_0$ as well, so that, in fact, the inequality in (31) holds for all $0 < \rho \leq R \leq R_0$. By (5), for each $\kappa \in [0, n)$, we can find $R_\kappa \in (0, R_0)$ so that

$$\omega(R) \log\left(\frac{1}{R}\right) + \delta(R) \leq \varepsilon_0 := \left(\frac{1}{2C}\right)^{\frac{n}{n-\kappa}}$$

for all $0 < R < R_\kappa$. Then by Lemma 3, we conclude that

$$\int_{\mathcal{B}_\rho} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x} \leq C_\kappa \left(\frac{\rho}{R}\right)^\kappa \int_{\mathcal{B}_R} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x}$$

for all $0 < \rho \leq R \leq R_\kappa$, which concludes the proof. □

Using a reflection argument and Theorem 9, we can show the following version of the result for the half-ball.

Theorem 10. *Suppose that* $\alpha : \mathcal{B}^+ \to [1, \infty)$ *satisfies* (3)-(5), *and that* $g : \mathcal{B}^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A$_1$)-(A$_3$). *Let* $\mathbf{G}_0 \in \mathbb{R}^{n\times n}$ *be an invertible matrix, and suppose that* $\mathbf{v}$ *is a minimizer for the functional* $J : W^{1,1}(\mathcal{B}^+; \mathbb{R}^N) \to \mathbb{R}^*$ *defined by*

$$J(\mathbf{w}) := \int_{\mathcal{B}^+} g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x},$$

satisfying $\mathbf{v} = 0$ *on* $\mathcal{B} \cap \partial\mathcal{H}^+$ *in the sense of trace. Then for every* $0 \leq \kappa < n$, *there are constants* C_κ *and* R_κ, *which, in addition to* κ, *also depend on* n, N, p, q, α_+, $|\mathbf{G}_0|$, $\left|\mathbf{G}_0^{-1}\right|$, M, ω, δ, κ, *and* $L := \int_{\mathcal{B}^+}(1 + g(\mathbf{x}, |\nabla \mathbf{w}\mathbf{G}_0|))\mathrm{d}\mathbf{x}$, *such that*

$$\int_{\mathcal{B}(\mathbf{x}_0,\rho)^+} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x} \leq C \left(\frac{\rho}{R}\right)^\kappa \int_{\mathcal{B}(\mathbf{x}_0,R)^+} (1 + g(\mathbf{x}, |\nabla \mathbf{v}|))\mathrm{d}\mathbf{x}$$

whenever $\mathcal{B}(\mathbf{x}_0, R)^+ \subset \mathcal{B}^+$ *and* $0 < \rho \leq R$.

Now we can prove the following lemma.

Lemma 4. *Suppose that* $\alpha : \mathcal{B}^+ \times [1, \infty)$ *satisfies* (3)-(5) *and that* $g : \mathcal{B}^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A$_1$)-(A$_3$). *Assume* $0 \leq \lambda < n$. *Let*

$$\mathcal{A} := \left\{\mathbf{u} \in W^{1,1}(\mathcal{B}^+; \mathbb{R}^N) : \mathbf{u} = 0 \text{ on } \mathcal{B} \cap \partial\mathcal{H}^+ \text{in the sense of trace}\right\},$$

and define the functional $K : \mathcal{A} \to \mathbb{R}^*$ *by*

$$K(\mathbf{w}) := \int_{\mathcal{B}^+} g(\mathbf{x}, |[\nabla \mathbf{w} + \mathbf{A}]\mathbf{G}|)\mathrm{d}\mathbf{x},$$

where $g(\cdot, |\mathbf{A}|) \in L^{1,\lambda}(\mathcal{B}^+)$ *and* $\mathbf{G} \in \mathcal{C}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$ *has continuous matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}(\overline{\mathcal{B}^+}; \mathbb{R}^{n\times n})$. *If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\mathcal{B}^+)$ *and* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *satisfying* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$, *then* $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}_{\text{loc}}(\mathcal{B} \cap \overline{\mathcal{H}^+})$.

Proof. Fix $\mathbf{x}_0 \in \mathcal{B}^+$ and $R > 0$ so that $\mathcal{B}^+(\mathbf{x}_0, R) \subset \mathcal{B}^+$. For ease of notation, let $\mathbf{G}_0 = \mathbf{G}(\mathbf{x}_0)$. Also, for each $r > 0$, define

$$\mu(r) := \sup_{|\mathbf{x}-\mathbf{y}|\leq r} |\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})|,$$

and set $\mathcal{B}_r^+ := \mathcal{B}(\mathbf{x}_0, r)^+$. Fix $R > 0$ such that $\mathcal{B}_R^+ \subset \mathcal{B}^+$ and $R \leq R_{(n+\lambda)/2}$, where $R_{(n+\lambda)/2}$ is the value of R_κ given by Theorem 10 for $\kappa = (n + \lambda)/2$ and

$L = \int_{\mathcal{B}^+}(1 + g(\mathbf{x}, |\nabla\mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x}$. Let $\mathbf{v} \in W^1L_g(\mathcal{B}_R^+; \mathbb{R}^N)$ be the minimizer of the functional $J : W^1L_g(\mathcal{B}_R^+; \mathbb{R}^N) \to \mathbb{R}$ defined by

$$J(\mathbf{w}) := \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla\mathbf{w}\mathbf{G}_0|)\mathrm{d}\mathbf{x},$$

satisfying $\mathbf{v} = \mathbf{u}$ on $\partial\mathcal{B}_R^+$ in the sense of trace. Then for $0 < \rho < R/2$, we have

$$\begin{aligned} \int\limits_{\mathcal{B}_\rho^+} g(\mathbf{x}, |\nabla\mathbf{u}\mathbf{G}_0|)\mathrm{d}\mathbf{x} = \int\limits_{\mathcal{B}_\rho^+} g(\mathbf{x}, |\nabla\mathbf{v}\mathbf{G}_0|)\mathrm{d}\mathbf{x} + \int\limits_{\mathcal{B}_\rho^+} \Big\{ & g(\mathbf{x}, |\nabla\mathbf{u}\mathbf{G}_0|) - g(\mathbf{x}, |\nabla\mathbf{v}\mathbf{G}_0|) \\ & - g_t(\mathbf{x}, |\nabla\mathbf{v}\mathbf{G}_0|)\frac{\nabla\mathbf{v}\mathbf{G}_0}{|\nabla\mathbf{v}\mathbf{G}_0|} \cdot [\nabla\mathbf{u} - \nabla\mathbf{v}]\mathbf{G}_0 \Big\}\mathrm{d}\mathbf{x} \\ & + \int\limits_{\mathcal{B}_\rho^+} g_t(\mathbf{x}, |\nabla\mathbf{v}\mathbf{G}_0|)\frac{\nabla\mathbf{v}\mathbf{G}_0}{|\nabla\mathbf{v}\mathbf{G}_0|} \cdot [\nabla\mathbf{u} - \nabla\mathbf{v}]\mathbf{G}_0\mathrm{d}\mathbf{x} \\ & = I_1 + I_2 + I_3, \end{aligned} \tag{32}$$

where I_1, I_2, and I_3 are defined to be the respective integrals. By Theorem 10 and the minimality of $\mathbf{v}$, we have that

$$I_1 \le C\left(\frac{\rho}{R}\right)^{(\lambda+n)/2} \int_{\mathcal{B}_R^+} (1 + g(\mathbf{x}, |\nabla\mathbf{u}|))\,\mathrm{d}\mathbf{x}.$$

We now consider I_2. By the convexity of g, the integrand in I_2 is nonnegative, so we can expand the domain of integration to $\mathcal{B}_R^+$ and use Lemma 2 to arrive at

$$I_2 \le \int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |\nabla\mathbf{u}\mathbf{G}_0|) - g(\mathbf{x}, |\nabla\mathbf{v}\mathbf{G}_0|)\}\,\mathrm{d}\mathbf{x}.$$

Since $\mathbf{u}$ is a $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer, we have

$$\begin{aligned} I_2 \le & \int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |\nabla\mathbf{u}\mathbf{G}_0|) - g(\mathbf{x}, |[\nabla\mathbf{u} + \mathbf{A}]\mathbf{G}|)\}\mathrm{d}\mathbf{x} \\ & + \int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |[\nabla\mathbf{v} + \mathbf{A}]\mathbf{G}|) - g(\mathbf{x}, |\nabla\mathbf{v}\mathbf{G}_0|)\}\mathrm{d}\mathbf{x} \\ & + (\gamma_\varepsilon(R) + \varepsilon)\int_{\mathcal{B}_R^+} \{g(\mathbf{x}, |\nabla\mathbf{u}|) + g(\mathbf{x}, |\nabla\mathbf{v}|)\}\mathrm{d}\mathbf{x} + \int_{\mathcal{B}_R^+} |\nu_\varepsilon|\,\mathrm{d}\mathbf{x} \\ \le & \; I_{2,1} + I_{2,2} + I_{2,3} + R^\lambda \left\|\nu_\varepsilon\right\|_{L^{1,\lambda}}. \end{aligned} \tag{33}$$

First we estimate $I_{2,1}$ using part (vi) of Lemma 1.

$$\begin{aligned} I_{2,1} = & \int\limits_{\mathcal{B}_R^+}\int\limits_0^1 \frac{\partial}{\partial\mathbf{F}} g(\mathbf{x}, \nabla\mathbf{u}\mathbf{G}_0 + t([\nabla\mathbf{u} + \mathbf{A}]\mathbf{G} - \nabla\mathbf{u}\mathbf{G}_0)) \cdot (\nabla\mathbf{u}[\mathbf{G} - \mathbf{G}_0] + \mathbf{A}\mathbf{G})\mathrm{d}t\mathrm{d}\mathbf{x} \\ \le & \; C\mu(R)\int_{\mathcal{B}_R^+} g_t(\mathbf{x}, |\nabla\mathbf{u}| + |\mathbf{A}|)\,|\nabla\mathbf{u}|\,\mathrm{d}\mathbf{x} + C\varepsilon\int_{\mathcal{B}_R^+} g_t(\mathbf{x}, |\nabla\mathbf{u}| + |\mathbf{A}|)\frac{|\mathbf{A}|}{\varepsilon}\mathrm{d}\mathbf{x} \\ \le & \; C(\varepsilon + \mu(R))\int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla\mathbf{u}|) + C_\varepsilon\int_{\mathcal{B}_R^+} g(\mathbf{x}, |\mathbf{A}|)\mathrm{d}\mathbf{x} \\ \le & \; C(\varepsilon + \mu(R))\int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla\mathbf{u}|) + C_\varepsilon R^\lambda \left\|g(\cdot, |\mathbf{A}|)\right\|_{L^{1,\lambda}}. \end{aligned}$$

Because $\mathbf{v}$ is a minimizer for J, a similar computation yields

$$I_{2,2} \leq C(\varepsilon + \mu(R)) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + C_\varepsilon R^\lambda \left\| g(\cdot, |\mathbf{A}|) \right\|_{L^{1,\lambda}}.$$

For $I_{2,3}$, we again use the fact that $\mathbf{v}$ is a minimizer for J to conclude that

$$I_{2,3} \leq C(\gamma_\varepsilon(R) + \varepsilon) \int_{\mathcal{B}^+(\mathbf{x}_0,R)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x}.$$

Collecting our estimates for $I_{2,1}$, $I_{2,2}$, and $I_{2,3}$, by (33) we have

$$I_2 \leq C(\varepsilon + \mu(R) + \gamma_\varepsilon(R)) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + \left(C_\varepsilon \left\| g(\cdot, |\mathbf{A}|) \right\|_{L^{1,\lambda}} + \left\| \nu_\varepsilon \right\|_{L^{1,\lambda}}\right) R^\lambda.$$

Using (vii) from Lemma 1 as we did in the proof of Theorem 9, we see that $I_3 \leq C(I_1 + I_2)$, and so our estimates for I_1 and I_2, along with (32), give

$$\begin{aligned} &\int_{\mathcal{B}^+(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \\ &\qquad \leq C\left(\left(\frac{\rho}{R}\right)^{\frac{n+\lambda}{2}} + \varepsilon + \mu(R) + \gamma_\varepsilon(R) \right) \int_{\mathcal{B}^+(\mathbf{x}_0,R)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \\ &\qquad\quad + \left(C_\varepsilon \left\| g(\cdot, |\mathbf{A}|) \right\|_{L^{1,\lambda}} + \left\| \nu_\varepsilon \right\|_{L^{1,\lambda}} + 1\right) R^\lambda \end{aligned} \tag{34}$$

Let $\varepsilon_0 = (2C)^{-\frac{2(n+\lambda)}{n-\lambda}}/2$. Find $0 < R^* < 1$ such that $\mu(R^*) < \varepsilon_0/4$ and $\gamma_{\varepsilon_0/2}(R^*) < \varepsilon_0/4$. Let $R_0 = \min\{R^*, 1 - |\mathbf{x}_0|, R_{(n+\lambda)/2}\}$. Then for $0 < \rho \leq R \leq R_0$, putting $\varepsilon = \varepsilon_0/2$ in (34) we have

$$\begin{aligned} &\int_{\mathcal{B}_\rho^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \\ &\qquad \leq C\left(\left(\frac{\rho}{R}\right)^{\frac{n+\lambda}{2}} + \varepsilon_0 \right) \int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \\ &\qquad\quad + (C_{\varepsilon_0/2} \left\| g(\cdot, |\mathbf{A}| \right\|_{L^{1,\lambda}} + \left\| \nu_{\varepsilon_0/2} \right\|_{L^{1,\lambda}} + 1) R^\lambda \end{aligned} \tag{35}$$

By Lemma 3, we have that

$$\begin{aligned} \int_{\mathcal{B}_\rho^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} &\leq C\left(\frac{\rho}{R}\right)^\lambda \left[\int_{\mathcal{B}_R^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + 1 \right] \\ &\leq C\left(\frac{\rho}{R}\right)^\lambda \left[\int_{\mathcal{B}^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + 1 \right] \end{aligned} \tag{36}$$

whenever $0 < \rho \leq R \leq R_0$. Now if $\mathcal{U} \subset\subset \mathcal{B} \cap \overline{\mathcal{H}^+}$, letting $d = \operatorname{dist}(\mathcal{U}; \partial\mathcal{B})$ and $c(\mathcal{U}) = C/\min\{R_0^\lambda, d^\lambda\}$, we have from (36) that

$$\rho^{-\lambda} \int_{\mathcal{B}^+(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} \leq c(\mathcal{U}) \left[\int_{\mathcal{B}^+} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} + 1 \right]$$

for all $\rho < \min\{R_0, d\}$ and $\mathbf{x}_0 \in \mathcal{U}$, and hence $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}_{\mathrm{loc}}(\mathcal{B}^+ \cap \overline{\mathcal{H}^+}; \mathbb{R}^{N\times n})$, which completes the proof. □

Using Theorem 9 instead of Theorem 10, we can demonstrate the following lemma in the same way that we proved Lemma 4.

Lemma 5. *Suppose that* $\alpha : \mathcal{B} \to [1, \infty)$ *satisfies* (3)-(5) *and that* $g : \mathcal{B} \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A$_1$)-(A$_3$). *Assume* $0 \leq \lambda < n$. *Let* $\mathcal{A} := W^{1,1}(\mathcal{B}; \mathbb{R}^N)$, *and define the functional* $K : \mathcal{A} \to \mathbb{R}^*$ *by*

$$K(\mathbf{w}) := \int_{\mathcal{B}} g(\mathbf{x}, |[\nabla \mathbf{w} + \mathbf{A}]\mathbf{G}|)\mathrm{d}\mathbf{x},$$

where $g(\cdot, |\mathbf{A}|) \in L^{1,\lambda}(\mathcal{B})$ *and* $\mathbf{G} \in \mathcal{C}(\overline{\mathcal{B}}; \mathbb{R}^{n \times n})$ *has continuous matrix inverse* $\mathbf{G}^{-1} \in \mathcal{C}(\overline{\mathcal{B}^+}; \mathbb{R}^{n \times n})$. *If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon > 0} \subset L^{1,\lambda}(\mathcal{B}^+)$ *and* $\{\gamma_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{C}(\mathbb{R}_+)$ *satisfying* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$, *then* $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}_{\mathrm{loc}}(\mathcal{B})$.

Using Lemmas 4 and 5, we prove the following result.

Theorem 11. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded with* $\mathcal{C}^1$ *boundary, and that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (3)-(5) *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A$_1$)-(A$_3$). *Let* $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$ *be given, and define*

$$\mathcal{A} := \left\{ \mathbf{w} \in W^{1,1}(\Omega; \mathbb{R}^N) : \mathbf{w} - \overline{\mathbf{u}} \in W^{1,1}_0(\Omega; \mathbb{R}^N) \right\}.$$

Define the functional $J : \mathcal{A} \to \mathbb{R}^*$ *by*

$$J(\mathbf{w}) := \int_{\Omega} g(\mathbf{x}, |\nabla \mathbf{w}(\mathbf{x})|)\mathrm{d}\mathbf{x}.$$

Let $\mathbf{u} \in \mathcal{A}$ *be given. Then there are functions* $\{\nu_\varepsilon\}_{\varepsilon > 0} \subset L^{1,\lambda}(\Omega)$ *and nondecreasing functions* $\{\gamma_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{C}(\mathbb{R}_+)$ *with* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(J, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$ *if and only if* $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

Remark 4. We emphasize here that this result is a biconditional statement. That is, the condition that $\mathbf{u}$ is an almost minimizer for J is not only sufficient to have that $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$, but also necessary.

Proof. First, if $\mathbf{u} \in \mathcal{A}$ is a $(J, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer over $\mathcal{A}$ for some $\{\nu_\varepsilon\}_{\varepsilon > 0} \subset L^{1,\lambda}(\Omega)$ and nondecreasing functions $\{\gamma_\varepsilon\}_{\varepsilon > 0}$ with $\gamma_\varepsilon(0) = 0$, we use a standard argument to incorporate the boundary values into the functional and straighten out the boundary, and then use a covering argument along with Lemmas 4 and 5 to obtain that $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

On the other hand, if $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$, then $J(\mathbf{u}) < \infty$. Let $\mathbf{v} \in \mathcal{A}$ with $\mathbf{u} - \mathbf{v} \in W^{1,1}_0(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$ be given. With $\nu_\varepsilon := \varepsilon^{-1} g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$, we have

$$J(\mathbf{u}) \leq J(\mathbf{v}) + \int_{\Omega(\mathbf{x}_0, \rho)} g(\mathbf{x}, |\nabla \mathbf{u}|)\mathrm{d}\mathbf{x} = J(\mathbf{v}) + \varepsilon \int_{\Omega(\mathbf{x}_0, \rho)} \nu_\varepsilon(\mathbf{x})\mathrm{d}\mathbf{x}.$$

Hence $\mathbf{u}$ is a $(J, \{0\}, \{\nu_\varepsilon\})$-minimizer over $\mathcal{A}$, and the proof is complete. □

5. Results for Asymptotically Convex Functionals

As mentioned in the introduction, this section is devoted to extending the results of the previous section to almost minimizers of functionals of the form

$$K(\mathbf{w}) := \int_{\Omega} f(\mathbf{x}, \mathbf{w}, \nabla \mathbf{w}), \tag{37}$$

where for each $\mathbf{x}$ and $\mathbf{u}$, the function $\mathbf{F} \mapsto f(\mathbf{x}, \mathbf{u}, \mathbf{F})$ looks like $\mathbf{F} \mapsto g(\mathbf{x}, |\mathbf{F}|)$ when $|\mathbf{F}|$ is large. Define the functional J by

$$J(\mathbf{w}) := \int_{\Omega} g(\mathbf{x}, |\nabla \mathbf{w}|)\mathrm{d}\mathbf{x}. \tag{38}$$

The following lemma establishes that almost minimizers for K will also be almost minimizers for J. In this lemma and in the sequel, we denote by p^* the Sobolev-conjugate of p; i.e., if $p < n$, then we put $p^* = np/(n-p)$, and if $p \geq n$, we set $p^* = +\infty$.

Lemma 6. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and suppose also that $\alpha : \Omega \to [1, \infty)$ satisfies* (3)-(5) *and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (A$_1$)-(A$_3$)*. Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$, and assume the following hypotheses hold for some $0 \leq \lambda < n$ and $1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, where r_2 is as in Remark 3.*

- (i) *For every $\varepsilon > 0$, there is a function $\sigma_\varepsilon \in L^{1,\lambda}(\Omega)$ and a constant $\Sigma_\varepsilon < \infty$ such that*
$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F}) - g(\mathbf{x}, |\mathbf{F}|)| < \varepsilon g(\mathbf{x}, |\mathbf{F}|)$$
for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ satisfying $g(\mathbf{x}, |\mathbf{F}|) \geq \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s$.
- (ii) *There is some $\beta \in L^{1,\lambda}(\Omega)$ such that*
$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F})| \leq C(\beta(\mathbf{x}) + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{F}|))$$
for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

For a fixed $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ with $g(\cdot, |\nabla\overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$, define the admissible class
$$\mathcal{A} := \left\{\mathbf{u} \in W^1L_g(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^1L_g(\Omega; \mathbb{R}^N)\right\}.$$
Let the functionals J and K, each mapping $\mathcal{A}$ into $\mathbb{R}$, be defined by (38) *and* (37)*, respectively. Let $\mathbf{u} \in \mathcal{A}$, and suppose that there are functions $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and nondecreasing functions $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ satisfying $\gamma_\varepsilon(0) = 0$, along with constants $\{T_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$ such that*

$$\begin{aligned} K(\mathbf{u}) \leq K(\mathbf{v}) &+ (\gamma_\varepsilon(\rho) + \varepsilon)\int_{\Omega(\mathbf{x}_0,\rho)} \{\nu_\varepsilon(\mathbf{x}) + g(\mathbf{x}, |\nabla\mathbf{u}|) + g(\mathbf{x}, |\nabla\mathbf{v}|)\}\,\mathrm{d}\mathbf{x} \\ &+ T_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{v}|)^s\}\,\mathrm{d}\mathbf{x} \end{aligned} \tag{39}$$

for all $\mathbf{v} \in \mathcal{A}$ with $\mathbf{u} - \mathbf{v} \in W_0^{1,1}(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. Then there are functions $\{\tilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and nondecreasing functions $\{\tilde{\gamma}_\varepsilon\}_{\varepsilon>0} \subset C(\mathbb{R}_+)$ with $\tilde{\gamma}_\varepsilon(0) = 0$, as well as constants $\{\tilde{T}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, such that $\mathbf{u}$ is a $(J, \{\tilde{\gamma}_\varepsilon\}, \{\tilde{\nu}_\varepsilon + \tilde{T}_\varepsilon g(\cdot, |\mathbf{u}|)^s\})$-minimizer.

Proof. It suffices to show that

$$J(\mathbf{u}) \leq J(\mathbf{v}) + (\tilde{\gamma}_\varepsilon(\rho) + \varepsilon)\int_{\Omega(\mathbf{x}_0,\rho)} \left\{\tilde{\nu}_\varepsilon(\mathbf{x}) + \tilde{T}_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\nabla\mathbf{u}|)\right\}\mathrm{d}\mathbf{x} \tag{40}$$

for all $\mathbf{v} \in \mathcal{A}$ such that $\mathbf{u} - \mathbf{v} \in W_0^1L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. To this end, we let $\mathbf{w} \in W^1L_g(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$ be the minimizer of the functional $J_{\mathbf{x}_0,\rho}$ defined by

$$J_{\mathbf{x}_0,\rho}(\mathbf{v}) = \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x}, |\nabla\mathbf{v}|)\mathrm{d}\mathbf{x},$$

satisfying $\mathbf{w}-\mathbf{u} \in W_0^1 L_g(\Omega(\mathbf{x}_0,\rho);\mathbb{R}^N)$. Then for any $\mathbf{v} \in \mathcal{A}$ that satisfies $\mathbf{u}-\mathbf{v} \in W_0^1 L_g(\Omega(\mathbf{x}_0,\rho);\mathbb{R}^N)$, we have by the minimality of $\mathbf{w}$ that

$$\begin{aligned} J(\mathbf{u}) - J(\mathbf{v}) &\le J_{\mathbf{x}_0,\rho}(\mathbf{u}) - J_{\mathbf{x}_0,\rho}(\mathbf{w}) \\ &= \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x},|\nabla\mathbf{u}|) - f(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\}\,\mathrm{d}\mathbf{x} \\ &\qquad + \int_{\Omega(\mathbf{x}_0,\rho)} \{f(\mathbf{x},\mathbf{u},\nabla\mathbf{u}) - f(\mathbf{x},\mathbf{w},\nabla\mathbf{w})\}\,\mathrm{d}\mathbf{x} \\ &\qquad + \int_{\Omega(\mathbf{x}_0,\rho)} \{f(\mathbf{x},\mathbf{w},\nabla\mathbf{w}) - g(\mathbf{x},|\nabla\mathbf{w}|)\}\,\mathrm{d}\mathbf{x} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{41}$$

To estimate I_1, we partition $\Omega(\mathbf{x}_0,\rho)$ into the set on which $g(\mathbf{x},|\nabla\mathbf{u}|) \le \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon g(\mathbf{x},|\mathbf{u}|)^s$ (call this set $\mathcal{S}$), and the set on which the opposite inequality holds (call this set $\mathcal{T}$). By the growth conditions on f and g, we have

$$\begin{aligned} \int_{\mathcal{S}} \{g(\mathbf{x},|\nabla\mathbf{u}|) - f(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\}\,\mathrm{d}\mathbf{x} &\le C\int_{\mathcal{S}} \{1+\beta+g(\mathbf{x},|\mathbf{u}|)^s + g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x} \\ &\le C\int_{\Omega(\mathbf{x}_0,\rho)} \{1+\beta+\sigma_\varepsilon + (1+\Sigma_\varepsilon)g(\mathbf{x},|\mathbf{u}|)^s\}\,\mathrm{d}\mathbf{x}. \end{aligned}$$

To estimate the integral over $\mathcal{T}$, we use the assumption in (i) to conclude that

$$\int_{\mathcal{T}} \{g(\mathbf{x},|\nabla\mathbf{u}|) - f(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\}\,\mathrm{d}\mathbf{x} \le \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\nabla\mathbf{u}|)\mathrm{d}\mathbf{x}.$$

Combining the estimates for the integrals over $\mathcal{S}$ and $\mathcal{T}$ yields

$$I_1 \le C\int_{\Omega(\mathbf{x}_0,\rho)} \{1+\alpha+\sigma_\varepsilon + (1+\Sigma_\varepsilon)g(\mathbf{x},|\mathbf{u}|)^s + \varepsilon g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x}.$$

We estimate I_3 in a similar fashion, keeping in mind the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_0,\rho}$, to obtain

$$I_3 \le C\int_{\Omega(\mathbf{x}_0,\rho)} \{1+\alpha+\sigma_\varepsilon + (1+\Sigma_\varepsilon)g(\mathbf{x},|\mathbf{w}|)^s + \varepsilon g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x}. \tag{42}$$

We have

$$\begin{aligned} \int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{w}|)^s\mathrm{d}\mathbf{x} &\le C\rho^{n+ps} \fint_{\Omega(\mathbf{x}_0,\rho)} g\left(\mathbf{x}, \frac{|\mathbf{u}-\mathbf{w}|}{\rho}\right)^s \mathrm{d}\mathbf{x} \\ &\qquad + C\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{u}|)^s\mathrm{d}\mathbf{x} \end{aligned} \tag{43}$$

By Remark 3, the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_0,\rho}$, and (43), there is a constant C, which does not depend on $\mathbf{w}$, $\mathbf{x}_0$, or ρ, such that

$$\begin{aligned}\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{w}|)^s \mathrm{d}\mathbf{x} &\leq C\rho^{n+ps}\left(\fint_{\Omega(\mathbf{x}_0,\rho)}\{1+g\left(\mathbf{x},|\nabla\mathbf{u}-\nabla\mathbf{w}|\right)\}\,\mathrm{d}\mathbf{x}\right)^s \\ &\quad + C\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{u}|)^s \mathrm{d}\mathbf{x} \\ &\leq C\rho^{n+ps-ns}\left(\int_{\Omega(\mathbf{x}_0,\rho)}\{1+g\left(\mathbf{x},|\nabla\mathbf{u}|\right)\}\,\mathrm{d}\mathbf{x}\right)^s \\ &\quad + C\int_{\Omega(\mathbf{x}_0,\rho)} g(\mathbf{x},|\mathbf{u}|)^s \mathrm{d}\mathbf{x}.\end{aligned} \tag{44}$$

Define $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\Delta(r) := r^{n+ps-ns}\sup_{\mathbf{y}\in\Omega}\left(\int_{\Omega(\mathbf{y},r)}\{1+g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x}\right)^{s-1}.$$

Note that the exponent on r is positive, since we have assumed that $1 < s < p^*/p$, so we have that Δ is continuous with $\Delta(0) = 0$. With this notation in place and the estimates in (42) and (44), we now have that

$$I_3 \leq C(\varepsilon + \Sigma_\varepsilon\Delta(\rho))\int_{\Omega(\mathbf{x}_0,\rho)}\{1+g(\mathbf{x},|\nabla\mathbf{u}|)\}\,\mathrm{d}\mathbf{x} + C\int_{\Omega(\mathbf{x}_0,\rho)}\{1+\alpha+\sigma_\varepsilon\}\,\mathrm{d}\mathbf{x}.$$

Finally, to estimate I_2, we use the fact that $\mathbf{u}$ satisfies (39) to get

$$\begin{aligned}I_2 \leq (\varepsilon+\gamma_\varepsilon(\rho))&\int_{\Omega(\mathbf{x}_0,\rho)}\{\nu_\varepsilon + g(\mathbf{x},|\nabla\mathbf{u}|)+g(\mathbf{x},|\nabla\mathbf{w}|)\}\,\mathrm{d}\mathbf{x} \\ &+ T_\varepsilon\int_{\Omega(\mathbf{x}_0,\rho)}\{g(\mathbf{x},|\mathbf{u}|)^s + g(\mathbf{x},|\mathbf{w}|)^s\}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Using (44) and the definition of Δ, along with the minimality of $\mathbf{w}$ for $J_{\mathbf{x}_0,\rho}$, we have

$$I_2 \leq C\left(\varepsilon+\gamma_\varepsilon(\rho)+T_\varepsilon\Delta(\rho)\right)\int_{\Omega(\mathbf{x}_0,\rho)}\left\{\nu_\varepsilon(\mathbf{x})+1+\varepsilon^{-1}T_\varepsilon g(\mathbf{x},|\mathbf{u}|)^s + g(\mathbf{x},|\nabla\mathbf{u}|)\right\}\mathrm{d}\mathbf{x}.$$

Inserting our estimates for I_1, I_2, and I_3 into (41), we see that (40) holds with $\tilde{\nu}_\varepsilon$, $\tilde{\gamma}_\varepsilon$, and $\tilde{T}_\varepsilon$ defined by

$$\begin{aligned}\tilde{\nu}_\varepsilon &:= C(\varepsilon/C)^{-1}\left(1+\frac{\varepsilon}{C}+\alpha+\sigma_\varepsilon+\frac{\varepsilon}{C}\nu_{\varepsilon/C}\right), \\ \tilde{\gamma}_\varepsilon &:= C\left\{\gamma_{\varepsilon/C}+(T_{\varepsilon/C}+\Sigma_{\varepsilon/C}+1)\Delta\right\}, \\ \tilde{T}_\varepsilon &:= C(\varepsilon/C)^{-1}\left(1+\Sigma_{\varepsilon/C}+T_{\varepsilon/C}\right).\end{aligned}$$

Note that clearly $\{\tilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$, and $\{\tilde{\gamma}_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ satisfies $\tilde{\gamma}_\varepsilon = 0$ for each $\varepsilon > 0$. Furthermore, it is manifest that $\{\tilde{T}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, so the lemma is proved. □

We are now in a position to prove the main theorem.

Theorem 12. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and that $\alpha : \Omega \to [1,\infty)$ satisfies* (3) *and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (A_1)-(A_3). *Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy the following hypotheses for some $0 \leq \lambda < n$ and $1 < s < \min\{r_2, 1+pr_2/n, p^*/p\}$, where $r_2 > 1$ is as in Remark 3.*

(i) *For every* $\varepsilon > 0$*, there is a function* $\sigma_\varepsilon \in L^{1,\lambda}(\Omega)$ *and a constant* $\Sigma_\varepsilon < \infty$ *such that*
$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F}) - g(\mathbf{x}, |\mathbf{F}|)| < \varepsilon g(\mathbf{x}, |\mathbf{F}|)$$
for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ *satisfying* $g(\mathbf{x}, |\mathbf{F}|) \geq \sigma_\varepsilon(\mathbf{x})+\Sigma_\varepsilon g(\mathbf{x}, |\mathbf{u}|)^s$*;*

(ii) *There is some* $\beta \in L^{1,\lambda}(\Omega)$ *such that*
$$|f(\mathbf{x}, \mathbf{u}, \mathbf{F})| \leq C(\beta(\mathbf{x}) + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{F}|))$$
for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$*.*

For a fixed $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ *with* $g(\cdot, |\nabla\overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$*, define the admissible class*
$$\mathcal{A} := \left\{\mathbf{u} \in W^1L_g(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W^1_0 L_g(\Omega; \mathbb{R}^N)\right\}.$$
Let the functional $K : \mathcal{A} \to \mathbb{R}$ *be as defined in* (37)*. If* $\mathbf{u} \in \mathcal{A}$ *and there are functions* $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ *and nondecreasing functions* $\{\gamma_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R}_+)$ *with* $\gamma_\varepsilon(0) = 0$ *such that* $\mathbf{u}$ *is a* $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$*-minimizer over* $\mathcal{A}$*, then* $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda}(\Omega)$*.*

Before we prove this theorem, we make a few remarks for later convenience.

Remark 5. If $K(\mathbf{v}) = +\infty$ for any function $\mathbf{v} \in W^{1,1}(\Omega; \mathbb{R}^N)$ with $g(\cdot, |\nabla\mathbf{v}|) \notin L^1(\Omega; \mathbb{R}^{N\times n})$, then clearly we can enlarge the admissible class $\mathcal{A}$ to
$$\mathcal{A}' := \left\{\mathbf{w} \in W^{1,1}(\Omega; \mathbb{R}^N) : \mathbf{w} - \overline{\mathbf{u}} \in W^{1,1}_0(\Omega; \mathbb{R}^N)\right\},$$
and the same result holds.

Remark 6. Examining the proof of Theorem 11 and Lemma 6, we see that we do not actually need the inequality in (2) to hold for all $\varepsilon > 0$, but only for $\varepsilon \geq \varepsilon_0$, where $\varepsilon_0 > 0$ depends on n, N, p, q, α, $\overline{\mathbf{u}}$, Ω, and f.

Remark 7. By analyzing the proofs of Theorem 11 and Lemma 6, we see that the bound on the Morrey norm $\|g(\cdot, |\nabla\mathbf{u}|)\|_{L^{1,\lambda}}$ stays uniformly bounded if the quantity $L := \int_\Omega g(\mathbf{x}, |\nabla\mathbf{u}|)$ stays bounded. That is, if $\{\mathbf{u}_t\}_{t\in\Lambda}$ is a collection of $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizers such that $\int_\Omega g(\mathbf{x}, |\nabla\mathbf{u}_t|)\mathrm{d}\mathbf{x} \leq L$ for some $L < \infty$ and all $t \in \Lambda$, then there is a finite constant $\tilde{L}$ such that $\|g(\cdot, |\nabla\mathbf{u}_t|)\|_{L^{1,\lambda}} \leq \tilde{L}$ for all $t \in \Lambda$.

Proof. Since $\mathbf{u}$ is a $(K, \{\gamma_\varepsilon\}, \{\nu_\varepsilon\})$-minimizer over $\mathcal{A}$, by the growth conditions on f and Lemma 6, we have that there are functions $\{\tilde{\nu}_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and nondecreasing functions $\{\tilde{\gamma}_\varepsilon\}_{\varepsilon>0} \subset C(\mathbb{R}_+)$ with $\tilde{\gamma}_\varepsilon(0) = 0$, as well as constants $\{\tilde{T}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, such that $\mathbf{u}$ is a $(J, \{\tilde{\gamma}_\varepsilon\}, \{\tilde{\nu}_\varepsilon + \tilde{T}_\varepsilon g(\cdot, |\mathbf{u}|)^s\})$-minimizer. In view of Theorem 11, it therefore suffices to prove that $\mu_\varepsilon := \tilde{\nu}_\varepsilon + \tilde{T}_\varepsilon g(\cdot, |\mathbf{u}|)^s \in L^{1,\lambda}(\Omega)$. By hypothesis, we have that $\mathbf{u} \in W^1L_g(\Omega; \mathbb{R}^N)$. Therefore, $g(\cdot, |\mathbf{u}|)^s \in L^{r_2/s}(\Omega)$ by Remark 3, and hence we see by Hölder's inequality that $g(\cdot, |\mathbf{u}|)^s \in L^{1,n-ns/r_2}(\Omega)$. Therefore, letting $\lambda_1 := n(1 - s/r_2)$, we have that $\mu_\varepsilon \in \min\{\lambda_1, \lambda\}$, and hence Theorem 11 implies that $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\min\{\lambda,\lambda_1\}}(\Omega)$. If $\lambda_1 \geq \lambda$, the proof is therefore complete.

So suppose that $\lambda_1 < \lambda$. Since $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda_1}(\Omega)$, we can use Theorem 7 to conclude that $g(\cdot, |\mathbf{u}|)^s \in L^{1,\kappa}(\Omega)$ for every $\kappa < \min\{n + s(p + \lambda_1 - n), n\}$. Hence, if $n + s(p + \lambda_1 - n) > \lambda$, then $g(\cdot, |\mathbf{u}|)^s \in L^{1,\lambda}(\Omega)$, whence $\mu_\varepsilon \in L^{1,\lambda}(\Omega)$, and the proof is finished. If $n + s(p + \lambda_1 - n) \leq \lambda$, set $\lambda_2 = n + s(p + \lambda_1 - n)$. Arguing as before, we have that $\mu_\varepsilon \in L^{1,\kappa}(\Omega)$ for every $\kappa < \lambda_2$. Thus Theorem 11 implies that $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\kappa}$ for every $0 \leq \kappa < \lambda_2$. Recursively defining
$$\lambda_{j+1} := n + s(p + \lambda_j - n)$$

and continuing to bootstrap as above, we have that $\{\mu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$ for every $\kappa < \lambda_j$ if $\lambda_j \leq \lambda$, and $\{\mu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ if $\lambda_j > \lambda$. We claim that λ_j increases without bound. Indeed, inductively, one can easily show that

$$\lambda_j = n - \frac{ps}{s-1} + \left(\frac{p}{s-1} - \frac{n}{r_2}\right) s^j.$$

Since $1 < s < 1 + pr_2/n$, we have that the coefficient in front of s^j is positive, and since first two terms are constant in j, we see that indeed $\lim_{j\to\infty} \lambda_j = \infty$. Hence, if $j_0 \in \mathbb{N}$ is selected so that $\lambda_{j_0} > \lambda$, then we can use a bootstrap argument as above with j_0 iterations to get that $\{\mu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$. Applying Theorem 11 one last time gives $g(\cdot, |\nabla \mathbf{u}|) \in L^{1,\lambda}(\Omega)$, and the proof is complete. ☐

6. APPLICATIONS

In this section, we present some applications of these results to various problems.

6.1. Partial Differential Equations.

The first application we provide is to partial differential equations.

Theorem 13. *Suppose that* $\Omega \subset \mathbb{R}^n$ *is open and bounded with* $\mathcal{C}^1$ *boundary, and that* $\alpha : \Omega \to [1, \infty)$ *satisfies* (3)-(5) *and* $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* (A_1)-(A_3). *Suppose also that* $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ *satisfies* $g(\cdot, |\nabla u|) \in L^{1,\lambda}(\Omega)$ *for some* $0 \leq \lambda < n$. *Fix* $1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, *where* r_2 *is as in Remark 3, and suppose that the mappings* $\mathbf{A} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ *and* $\mathbf{b} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ *satisfy the following properties:*

(i) *For each* $\varepsilon > 0$, *there is a function* $\sigma_\varepsilon \in L_g(\Omega)$ *with* $g(\cdot, \sigma_\varepsilon(\cdot)) \in L^{1,\lambda}(\Omega)$ *and a constant* $\Sigma_\varepsilon < \infty$ *such that*

$$\left|\mathbf{A}(\mathbf{x}, \mathbf{u}, \mathbf{F}) - g_t(\mathbf{x}, |\mathbf{F}|)\frac{\mathbf{F}}{|\mathbf{F}|}\right| < \varepsilon g_t(\mathbf{x}, |\mathbf{F}|)$$

for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ *satisfying* $|\mathbf{F}| > \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon |\mathbf{u}|$.

(ii) *There is a constant* $M \geq 1$ *and a function* $\beta \in L_g(\Omega)$ *with* $g(\cdot, \beta(\cdot)) \in L^{1,\lambda}(\Omega)$ *such that*

$$|\mathbf{A}(\mathbf{x}, \mathbf{u}, \mathbf{F})| \leq M g_t(\mathbf{x}, \beta(\mathbf{x}) + |\mathbf{u}| + |\mathbf{F}|);$$
$$|\mathbf{b}(\mathbf{x}, \mathbf{u}, \mathbf{F})| \leq M g_t(\mathbf{x}, \beta(\mathbf{x}) + |\mathbf{u}| + |\mathbf{F}|);$$

for all $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$.

Suppose that $\mathbf{u} \in W^1L_g(\Omega; \mathbb{R}^N)$ *is a weak solution to the system*

$$\operatorname{div}[\mathbf{A}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))] = \mathbf{b}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})) \text{ in } \Omega,$$
$$\mathbf{u}(\mathbf{x}) = \overline{\mathbf{u}}(\mathbf{x}) \text{ on } \partial\Omega;$$

i.e. $\mathbf{u} - \overline{\mathbf{u}} \in W_0^1L_g(\Omega; \mathbb{R}^N)$, *and for each* $\boldsymbol{\varphi} \in W_0^1L_g(\Omega; \mathbb{R}^N)$,

$$\int_\Omega \{\mathbf{A}(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}) \cdot \nabla\boldsymbol{\varphi} + \mathbf{b}(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}) \cdot \boldsymbol{\varphi}\}\,\mathrm{d}\mathbf{x} = 0. \tag{45}$$

Then $g(\cdot, |\nabla\mathbf{u}|) \in L^{1,\lambda}(\Omega)$.

Proof. The proof given here is similar to the proofs given for the analogous theorems in [12, 15]; our overall strategy is to show that $\mathbf{u}$ is an almost minimizer for the functional $J : \mathcal{A} \to \mathbb{R}$ defined by

$$J(\mathbf{w}) := \int_\Omega g(\mathbf{x}, |\nabla \mathbf{w}(\mathbf{x})|)\mathrm{d}\mathbf{x},$$

where

$$\mathcal{A} := \left\{ \mathbf{w} \in W^1 L_g(\Omega; \mathbb{R}^N) : \mathbf{w} - \overline{\mathbf{u}} \in W_0^1 L_g(\Omega; \mathbb{R}^N) \right\}.$$

We will allow the constant C to depend on s, along with all the other usual parameters. Fix $\mathbf{x}_0 \in \Omega$, $0 < \rho < \operatorname{diam}(\Omega)$, and $\mathbf{v} \in \mathcal{A}$ with $\mathbf{v} - \mathbf{u} \in W^{1,1}(\Omega(\mathbf{x}_0, \rho); \mathbb{R}^N)$. Then using the convexity of $g(\mathbf{x}, \cdot)$ and (45), we have

$$\begin{aligned} K(\mathbf{u}) &\le K(\mathbf{v}) + \int_{\Omega(\mathbf{x}_0,\rho)} g_t(\mathbf{x}, \nabla \mathbf{u}) \frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \\ &\le K(\mathbf{v}) + \int_{\Omega(\mathbf{x}_0,\rho)} \left(g_t(\mathbf{x}, \nabla \mathbf{u}) \frac{\nabla \mathbf{u}}{|\nabla \mathbf{u}|} - \mathbf{A}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \right) \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}]\mathrm{d}\mathbf{x} \\ &\qquad - \int_{\Omega(\mathbf{x}_0,\rho)} \mathbf{b}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \cdot (\mathbf{u} - \mathbf{v})\mathrm{d}\mathbf{x}. \\ &= K(\mathbf{v}) + I_1 + I_2. \end{aligned} \tag{46}$$

To estimate I_1, for $0 < \varepsilon < 1$ we split $\Omega(\mathbf{x}_0, \rho)$ into the set on which $|\nabla \mathbf{u}(\mathbf{x})| \le \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon \, |\mathbf{u}(\mathbf{x})|$ (call this set $\mathcal{S}$), and the set on which the reverse inequality holds (call this set $\mathcal{T}$); using the growth conditions on f and $\mathbf{A}$, followed by part (iv) of Lemma 1, gives

$$\begin{aligned} I_1 &\le C \int_{\mathcal{S}} g_t(\mathbf{x}, \sigma_\varepsilon + \beta + (1 + \Sigma_\varepsilon) \, |\mathbf{u}|) \, |\nabla \mathbf{u} - \nabla \mathbf{v}| \, \mathrm{d}\mathbf{x} + \varepsilon \int_{\mathcal{T}} g_t(\mathbf{x}, |\nabla \mathbf{u}|) \, |\nabla \mathbf{u} - \nabla \mathbf{v}| \, \mathrm{d}\mathbf{x} \\ &\le C\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} g_t\left(\mathbf{x}, \varepsilon^{-1/(p\alpha(\mathbf{x})-1)} (\sigma_\varepsilon + \beta + (1 + \Sigma_\varepsilon) \, |\mathbf{u}|) \right) |\nabla \mathbf{u} - \nabla \mathbf{v}| \, \mathrm{d}\mathbf{x} \\ &\qquad + \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} g_t(\mathbf{x}, |\nabla \mathbf{u}|) \, |\nabla \mathbf{u} - \nabla \mathbf{v}| \, \mathrm{d}\mathbf{x}. \end{aligned}$$

Now utilizing (vi) in Lemma 1 yields

$$\begin{aligned} I_1 &\le C\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{ g(\mathbf{x}, |\nabla \mathbf{u}|) + g(\mathbf{x}, |\nabla \mathbf{v}|) \} \, \mathrm{d}\mathbf{x} \\ &\qquad + C_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{ g(\mathbf{x}, \beta) + g(\mathbf{x}, \sigma_\varepsilon) + (1 + \Sigma_\varepsilon) g(\mathbf{x}, |\mathbf{u}|) \} \, \mathrm{d}\mathbf{x}. \end{aligned}$$

To estimate I_2, we use the growth constraints on $\mathbf{b}$ and again employ (vi) in Lemma 1 to obtain

$$\begin{aligned} I_2 &\le \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{ g(\mathbf{x}, \beta) + g(\mathbf{x}, |\mathbf{u}|) + g(\mathbf{x}, |\nabla \mathbf{u}|) \} \, \mathrm{d}\mathbf{x} \\ &\qquad + C_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{ g(\mathbf{x}, |\mathbf{u}|) + g(\mathbf{x}, |\mathbf{v}|) \} \, \mathrm{d}\mathbf{x}. \end{aligned}$$

Defining

$$\begin{aligned} \nu_\varepsilon(\mathbf{x}) &:= C_\varepsilon (\varepsilon / C_\varepsilon)^{-1} \left\{ g(\mathbf{x}, \beta(\mathbf{x})) + g(\mathbf{x}, \sigma_{\varepsilon/C_\varepsilon}(\mathbf{x})) \right\} + g(\mathbf{x}, \beta(\mathbf{x})) \\ T_\varepsilon &:= C_\varepsilon (1 + \varepsilon + \Sigma_\varepsilon), \end{aligned}$$

we have shown that

$$I_1 + I_2 \leq \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{\nu_\varepsilon + g(\mathbf{x}, |\nabla \mathbf{u}|) + g(\mathbf{x}, |\nabla \mathbf{v}|)\} \, \mathrm{d}\mathbf{x} \\ + T_\varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} \{g(\mathbf{x}, |\mathbf{u}|) + g(\mathbf{x}, |\mathbf{v}|)\} \, \mathrm{d}\mathbf{x}.$$

Note that $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\lambda}(\Omega)$ and $\{T_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}_+$, so putting this estimate into (46) and employing Lemma 6 and Theorem 11 then bootstrapping as we did in Theorem 12 gives the desired result. ☐

6.2. **Regularity for minimizing sequences.** To prove the existence of Morrey regular minimizing sequences, we will use the following version of Ekeland's variational principle. For the proof of this result, see, for example, [18].

Theorem 14. *Let $(\mathcal{V}, d)$ be a complete metric space, and let $J : \mathcal{V} \to \mathbb{R}^*$ be a lower semicontinuous functional that is finite at some point in $\mathcal{V}$. Assume that for some $\mathbf{v} \in \mathcal{V}$ and some $\varepsilon > 0$, we have*

$$J(\mathbf{v}) \leq \inf_{\mathbf{w}\in\mathcal{V}} J(\mathbf{w}) + \varepsilon.$$

Then there exists a point $\mathbf{u} \in \mathcal{V}$ such that

$$J(\mathbf{u}) \leq J(\mathbf{v}) \text{ and } J(\mathbf{u}) \leq J(\mathbf{w}) + \varepsilon d(\mathbf{u}, \mathbf{w}) \text{ for all } \mathbf{w} \in \mathcal{V}.$$

We use Theorem 14 to prove the following, which supplies uniform regularity for minimizing sequences.

Theorem 15. *Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded with $\mathcal{C}^1$ boundary, and that $\alpha : \Omega \to [1, \infty)$ satisfies (3)-(5) and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (A_1)-(A_3). Suppose that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the following hypotheses for some $0 \leq \lambda < n$, $r < 1 < s < \min\{r_2, 1 + pr_2/n, p^*/p\}$, $\beta \in L^1(\Omega)$, and $\gamma \in L^{1,\lambda}(\Omega)$:*

$$\frac{1}{M} g(\mathbf{x}, |\mathbf{F}|) - M g(\mathbf{x}, |\mathbf{u}|)^r - \beta(\mathbf{x}) \leq f(\mathbf{x}, \mathbf{u}, \mathbf{F}) \leq M(\gamma(\mathbf{x}) + g(\mathbf{x}, |\mathbf{u}|)^s + g(\mathbf{x}, |\mathbf{F}|).$$

Here, r_2 is as in Remark 3. Suppose further that $\overline{\mathbf{u}} \in W^1L_g(\Omega; \mathbb{R}^N)$ with $g(\cdot, |\nabla \overline{\mathbf{u}}|) \in L^{1,\lambda}(\Omega)$ is given and define the admissible class by

$$\mathcal{A} := \left\{ \mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,1}(\Omega; \mathbb{R}^N) \right\}.$$

If the functional $K : \mathcal{A} \to \mathbb{R}^*$ is defined by (37), then there is a minimizing sequence $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathcal{A}$ for K such that the sequence $\{g(\cdot, |\nabla \mathbf{u}_k|)\}_{k=1}^\infty$ is uniformly bounded in $L^{1,\lambda}(\Omega)$.

Proof. By the growth condition imposed on f, we have that $K(\overline{\mathbf{u}}) < \infty$, so that K is finite at some point in $\mathcal{A}$. Let $\{\mathbf{v}_k\}_{k=1}^\infty \subset \mathcal{A}$ be a minimizing sequence for K, and let ε_k be defined by

$$\varepsilon_k := K(\mathbf{v}_k) - \inf_{\mathbf{w}\in\mathcal{V}} K(\mathbf{w}).$$

Without loss of generality, we assume that $\varepsilon_k \leq \varepsilon_0$, where ε_0 is as in Remark 6. By the coercivity condition on g, we have that K is bounded from below and we have that the sequence $\{g(\cdot, |\nabla \mathbf{v}_k|)\}_{k=1}^\infty$ is bounded in $L^1(\Omega)$. Notice that $\mathcal{A}$ equipped with the metric

$$d(\mathbf{u}, \mathbf{v}) := \|\nabla \mathbf{u} - \nabla \mathbf{v}\|_{L^1}$$

is a complete metric space, so by Theorem 14, we have that there is a sequence $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathcal{A}$ such that $K(\mathbf{u}_k) \leq K(\mathbf{v}_k)$ and $K(\mathbf{u}_k) \leq K(\mathbf{w}) + \varepsilon_k \|\nabla \mathbf{u}_k - \nabla \mathbf{w}\|_{L^1}$ for every $\mathbf{w} \in \mathcal{A}$. Since $K(\mathbf{u}_k)$ is dominated by $K(\mathbf{v}_k)$, it is clear that $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is a minimizing sequence for K. Also, if $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega(\mathbf{x}_0,\rho);\mathbb{R}^N)$, then from the above inequality, we have that

$$\begin{aligned} K(\mathbf{u}_k) &\leq K(\mathbf{u}_k + \boldsymbol{\varphi}) + \varepsilon_k \int_{\Omega(\mathbf{x}_0,\rho)} |\nabla \boldsymbol{\varphi}| \,\mathrm{d}\mathbf{x} \\ &\leq K(\mathbf{u}_k + \boldsymbol{\varphi}) + \varepsilon_k \int_{\Omega(\mathbf{x}_0,\rho)} C\left(1 + g(\mathbf{x}, |\nabla \boldsymbol{\varphi}|)\right) \mathrm{d}\mathbf{x}. \end{aligned}$$

Recall that $\varepsilon_k \leq \varepsilon_0$, so that we have

$$K(\mathbf{u}_k) \leq K(\mathbf{u}_k + \boldsymbol{\varphi}) + \varepsilon \int_{\Omega(\mathbf{x}_0,\rho)} C\left(1 + g(\mathbf{x}, |\nabla \boldsymbol{\varphi}|)\right)$$

for all $\varepsilon \geq \varepsilon_0$. Therefore, by Theorem 12 and Remarks 5 and 6, we have that $\{g(\cdot, |\nabla \mathbf{u}_k|)\}_{k=1}^{\infty} \subset L^{1,\lambda}(\Omega)$. In fact, since $\{\mathbf{u}_k\}_{k=1}^{\infty}$ is a minimizing sequence for K, the coercivity assumption on f implies that the quantities $\int_\Omega g(\mathbf{x}, |\nabla \mathbf{u}_k|)\mathrm{d}\mathbf{x}$ are uniformly bounded, so by Remark 7, the Morrey norms $\|g(\cdot, |\nabla \mathbf{u}_k|)\|_{L^{1,\lambda}}$ are uniformly bounded also, as desired. □

References

[1] Emilio Acerbi and Giuseppe Mingione. Regularity results for a class of functionals with non-standard growth. *Arch. Ration. Mech. Anal.*, 156(2):121–140, 2001.

[2] Emilio Acerbi and Giuseppe Mingione. Regularity results for a class of quasiconvex functionals with nonstandard growth. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 30(2):311–339, 2001.

[3] Emilio Acerbi and Giuseppe Mingione. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal.*, 164(3):213–259, 2002.

[4] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[5] Yunmei Chen, Stacey Levine, and Murali Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406 (electronic), 2006.

[6] Valeria Chiadò Piat and Alessandra Coscia. Hölder continuity of minimizers of functionals with variable growth exponent. *Manuscripta Math.*, 93(3):283–299, 1997.

[7] Giovanni Cupini, Nicola Fusco, and Raffaella Petti. Hölder continuity of local minimizers. *J. Math. Anal. Appl.*, 235(2):578–597, 1999.

[8] A. Dall'Aglio, E. Mascolo, and G. Papi. Local boundedness for minima of functionals with nonstandard growth conditions. *Rend. Mat. Appl. (7)*, 18(2):305–326, 1998.

[9] Lars Diening and Frank Ettwein. Fractional estimates for non-differentiable elliptic systems with general growth. *Forum Math.*, 20(3):523–556, 2008.

[10] Lars Diening, Bianca Stroffolini, and Anna Verde. Everywhere regularity of functionals with ϕ-growth. *Manuscripta Math.*, 129(4):449–481, 2009.

[11] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[12] Kyle Fey and Mikil Foss. Morrey regularity results for asymptotically convex variational problems with (p,q) growth. *J. Differential Equations*, 246(12):4519–4551, 2009.

[13] Mikil Foss. Global regularity for almost minimizers of nonconvex variational problems. *Ann. Mat. Pura Appl. (4)*, 187(2):263–321, 2008.

[14] Mikil Foss, Antonia Passarelli di Napoli, and Anna Verde. Global Morrey regularity results for asymptotically convex variational problems. *Forum Math.*, 20(5):921–953, 2008.

[15] Mikil Foss, Antonia Passarelli di Napoli, and Anna Verde. Morrey regularity and continuity results for almost minimizers of asymptotically convex integrals. *Appl. Math. (Warsaw)*, 35(3):335–353, 2008.

[16] M. Giaquinta and G. Modica. Regularity results for some classes of higher order nonlinear elliptic systems. *J. Reine Angew. Math.*, 311/312:145–169, 1979.

[17] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[18] Enrico Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[19] Tadeusz Iwaniec and Anne Verde. On the operator $\mathcal{L}(f) = f \log |f|$. *J. Funct. Anal.*, 169(2):391–420, 1999.

[20] Jan Malý and William P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[21] Paolo Marcellini and Gloria Papi. Nonlinear elliptic systems with general growth. *J. Differential Equations*, 221(2):412–443, 2006.

[22] Giuseppe Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.*, 51(4):355–426, 2006.

[23] Julian Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[24] Vasiliĭ V. Zhikov. On some variational problems. *Russian J. Math. Phys.*, 5(1):105–116 (1998), 1997.