Morrey Regularity for Almost Minimizers of Nonconvex Functionals with $p(\mathbf{x})$ Growth

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Setup

A Prototype Functional

$$J(\mathbf{u}) := \int_\Omega \left\{ |\nabla \mathbf{u}(\mathbf{x})|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) \right\} \mathrm{d}\mathbf{x}, \quad \mathbf{u} \in \mathscr{A}$$

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We (initially) assume the following:

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Key point: When $|\nabla \mathbf{u}|$ is large, the integrand “looks like” $|\nabla \mathbf{u}|^{p(\mathbf{x})}$.

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1. Electro-rheological fluids
2. Thermistors

An Example Result

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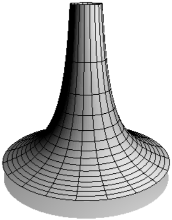
Note: This regularity result is global.

Morrey Spaces

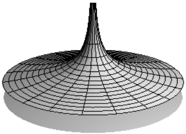
For each $p \in [1,+\infty)$ and $\lambda \in [0,n]$, define

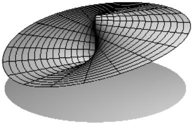
Morrey Space

$$L^{p,\lambda}(\Omega;\mathbb{R}^N) := \left\{ \mathbf{u} \in L^p(\Omega;\mathbb{R}^N) : \sup_{\substack{\mathbf{x}_0 \in \Omega \\ \rho > 0}} \rho^{-\lambda} \int_{\Omega \cap B_{\mathbf{x}_0,\rho}} |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}x < \infty \right\}.$$



$L^{p,0} \cong L^p$





$L^{p,n} \cong L^\infty$

Fact: If $\lambda < n$, then $L^{p,\lambda} \not\subseteq L^q$ for any $q > p$.

Sobolev-Morrey Spaces

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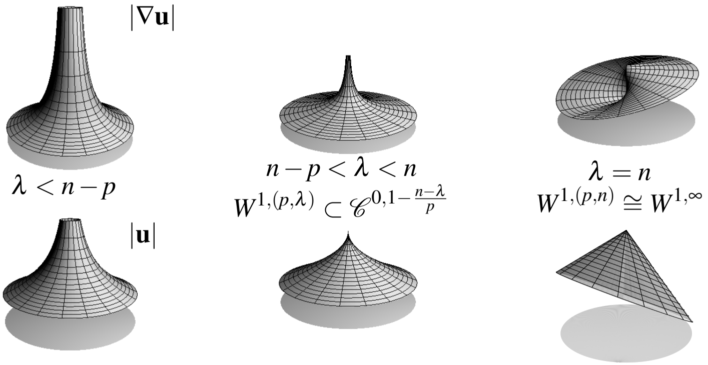
$$W^{1,(p,\lambda)}(\Omega;\mathbb{R}^N) := \left\{ \mathbf{u} \in L^{p,\lambda}(\Omega;\mathbb{R}^N) \,\middle|\, \nabla \mathbf{u} \in L^{p,\lambda}(\Omega;\mathbb{R}^{N\times n}) \right\}.$$



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Remark: If $p^- > n - \lambda$, then $W^{1,(p^-,\lambda)} \subset \mathscr{C}^{0,1-\frac{n-\lambda}{p^-}}$.

Strategy of Proof

Recall $\mathbf{u}$ minimizes

$$J(\mathbf{w}) := \int_\Omega \left\{ |\nabla \mathbf{w}(\mathbf{x})|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{w}(\mathbf{x})) \right\} \mathrm{d}\mathbf{x}.$$

Main Idea: Compare $\mathbf{u}$ to various minimizers for

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Interior Regularity, Nice Functional - Acerbi & Mingione (1999)

If $\mathbf{v}$ minimizes K, then $\forall \gamma \in [0,n)$, $\exists C_\gamma < \infty$ such that

$$\int_{B(\mathbf{x}_0,\rho)} |\nabla \mathbf{v}|^{p(\mathbf{x})} d\mathbf{x} \leq C_\gamma \left(\frac{\rho}{R}\right)^\gamma \int_{B(\mathbf{x}_0,R)} |\nabla \mathbf{v}|^{p(\mathbf{x})} d\mathbf{x}$$

whenever $B(\mathbf{x}_0,R) \subset \Omega$.

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This implies that $|\nabla \mathbf{v}|^{p(\mathbf{x})} \in L^{1,\gamma}_{\text{loc}}(\Omega)$ for every $\gamma < n$.

Proof Sketch - Interior Regularity

Let $B_\rho \subset B_R \subset \Omega$.

Let $\mathbf{v}$ be minimizer for K satisfying $\mathbf{v} = \mathbf{u}$ on ∂B_R.

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$$\begin{aligned}\int_{B_\rho} |\nabla \mathbf{u}|^{p(\mathbf{x})}\, d\mathbf{x} = & \int_{B_\rho} |\nabla \mathbf{v}|^{p(\mathbf{x})}\, d\mathbf{x} \\ & + \int_{B_\rho} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}} [|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} d\mathbf{x} \\ & + \int_{B_\rho} \frac{\partial}{\partial \mathbf{F}} [|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] d\mathbf{x}\end{aligned}$$

Proof Sketch - Interior Regularity

Let $B_\rho \subset B_R \subset \Omega$.

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$$\begin{aligned}\int_{B_\rho} |\nabla \mathbf{u}|^{p(\mathbf{x})}\,\mathrm{d}\mathbf{x} &= \int_{B_\rho} |\nabla \mathbf{v}|^{p(\mathbf{x})}\,\mathrm{d}\mathbf{x} \\ &\quad + \int_{B_\rho} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}[|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} \mathrm{d}\mathbf{x} \\ &\quad + \int_{B_\rho} \frac{\partial}{\partial \mathbf{F}}[|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}]\mathrm{d}\mathbf{x} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

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- Acerbi-Mingione $\implies I_1 \le C\left(\frac{\rho}{R}\right)^\gamma \int_{B_R} |\nabla \mathbf{u}|^{p(\mathbf{x})}$, where $\lambda < \gamma < n$.

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- Acerbi-Mingione $\implies I_1 \leq C \left(\frac{\rho}{R}\right)^\gamma \int_{B_R} |\nabla \mathbf{u}|^{p(\mathbf{x})}$, where $\lambda < \gamma < n$.
- Technical Lemma $\implies I_3 \leq C(I_1 + I_2)$.

Proof Sketch - Interior Regularity - Estimate for I_2

$$I_2 = \int_{B_\rho} \overbrace{\left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}} \left[|\nabla \mathbf{v}|^{p(\mathbf{x})} \right] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\}}^{\geq 0 \text{ by convexity of } \mathbf{F} \mapsto |\mathbf{F}|^{p(\mathbf{x})}} \mathrm{d}\mathbf{x}$$

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$$\leq \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}[|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} \mathrm{d}\mathbf{x}$$

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\begin{aligned}
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&\leq \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}\left[|\nabla \mathbf{v}|^{p(\mathbf{x})}\right] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} \mathrm{d}\mathbf{x} \\
&= \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} \quad \text{(by E-L eqns for } \mathbf{v}\text{)}
\end{aligned}
$$

Proof Sketch - Interior Regularity - Estimate for I_2

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$$\leq \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}\left[|\nabla \mathbf{v}|^{p(\mathbf{x})}\right] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} d\mathbf{x}$$

$$= \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} \right\} d\mathbf{x} \quad \text{(by E-L eqns for } \mathbf{v})$$

$$= \int_{B_R} \left\{ \left[|\nabla \mathbf{u}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{u}) \right] - \left[|\nabla \mathbf{v}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{v}) \right] \right\} d\mathbf{x} \quad (= J(\mathbf{u}) - J(\mathbf{v}))$$

$$+ \int_{B_R} \{ h(\mathbf{x}, \nabla \mathbf{v}) - h(\mathbf{x}, \nabla \mathbf{u}) \} d\mathbf{x}$$

Proof Sketch - Interior Regularity - Estimate for I_2

$$\begin{aligned}
I_2 &= \int_{B_\rho} \overbrace{\left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}[|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\}}^{\geq 0 \text{ by convexity of } \mathbf{F} \mapsto |\mathbf{F}|^{p(\mathbf{x})}} d\mathbf{x} \\
&\leq \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}[|\nabla \mathbf{v}|^{p(\mathbf{x})}] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} d\mathbf{x} \\
&= \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} \right\} d\mathbf{x} \quad \text{(by E-L eqns for } \mathbf{v}) \\
&= \int_{B_R} \left\{ \left[|\nabla \mathbf{u}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{u}) \right] - \left[|\nabla \mathbf{v}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{v}) \right] \right\} d\mathbf{x} \quad (= J(\mathbf{u}) - J(\mathbf{v})) \\
&\quad + \int_{B_R} \{ h(\mathbf{x}, \nabla \mathbf{v}) - h(\mathbf{x}, \nabla \mathbf{u}) \} d\mathbf{x} \\
&\leq \int_{B_R} \{ h(\mathbf{x}, \nabla \mathbf{v}) - h(\mathbf{x}, \nabla \mathbf{u}) \} d\mathbf{x} \quad \text{(since } \mathbf{u} \text{ minimizes } J)
\end{aligned}$$

Proof Sketch - Interior Regularity - Estimate for I_2

$$
\begin{aligned}
I_2 &= \int_{B_\rho} \overbrace{\left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}\left[|\nabla \mathbf{v}|^{p(\mathbf{x})}\right] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\}}^{\geq 0 \text{ by convexity of } \mathbf{F} \mapsto |\mathbf{F}|^{p(\mathbf{x})}} \mathrm{d}\mathbf{x} \\
&\leq \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} - \frac{\partial}{\partial \mathbf{F}}\left[|\nabla \mathbf{v}|^{p(\mathbf{x})}\right] \cdot [\nabla \mathbf{u} - \nabla \mathbf{v}] \right\} \mathrm{d}\mathbf{x} \\
&= \int_{B_R} \left\{ |\nabla \mathbf{u}|^{p(\mathbf{x})} - |\nabla \mathbf{v}|^{p(\mathbf{x})} \right\} \mathrm{d}\mathbf{x} \quad \text{(by E-L eqns for } \mathbf{v}\text{)} \\
&= \int_{B_R} \left\{ \left[|\nabla \mathbf{u}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{u}) \right] - \left[|\nabla \mathbf{v}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{v}) \right] \right\} \mathrm{d}\mathbf{x} \quad (= J(\mathbf{u}) - J(\mathbf{v})) \\
&\quad + \int_{B_R} \{ h(\mathbf{x}, \nabla \mathbf{v}) - h(\mathbf{x}, \nabla \mathbf{u}) \} \mathrm{d}\mathbf{x} \\
&\leq \int_{B_R} \{ h(\mathbf{x}, \nabla \mathbf{v}) - h(\mathbf{x}, \nabla \mathbf{u}) \} \mathrm{d}\mathbf{x} \quad \text{(since } \mathbf{u} \text{ minimizes } J\text{)} \\
&\leq \varepsilon \int_{B_R} |\nabla \mathbf{u}|^{p(\mathbf{x})} \mathrm{d}\mathbf{x} + C_\varepsilon R^\lambda \quad \text{(by hypotheses on } h\text{)}
\end{aligned}
$$

Proof Sketch - Interior Regularity

Combining estimates for I_1, I_2, I_3 yields

$$\int_{B_\rho} |\nabla \mathbf{u}|^{p(\mathbf{x})}\, \mathrm{d}\mathbf{x} \le C\left(\left(\frac{\rho}{R}\right)^{\gamma} + \varepsilon\right)\int_{B_R} |\nabla \mathbf{u}|^{p(\mathbf{x})}\, \mathrm{d}\mathbf{x} + C_\varepsilon R^{\lambda}$$

for a fixed $\gamma \in (\lambda, n)$ and any $\varepsilon > 0$.

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for a fixed $\gamma \in (\lambda, n)$ and any $\varepsilon > 0$.

Choosing $\varepsilon > 0$ small enough and iterating this estimate, we obtain

$$\int_{B_\rho} |\nabla \mathbf{u}|^{p(\mathbf{x})}\,\mathrm{d}\mathbf{x} \leq C\left(\frac{\rho}{R}\right)^{\lambda}\left(\int_{B_R} |\nabla \mathbf{u}|^{p(\mathbf{x})}\,\mathrm{d}\mathbf{x} + 1\right)$$

Combining estimates for I_1, I_2, I_3 yields

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for a fixed $\gamma \in (\lambda, n)$ and any $\varepsilon > 0$.

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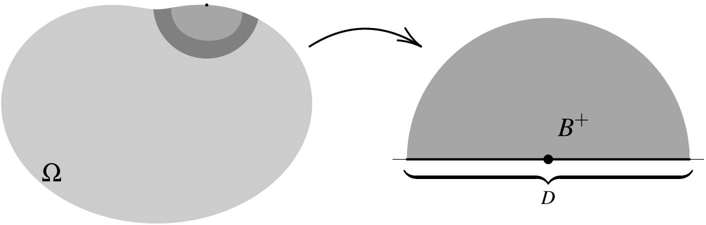
We conclude $|\nabla \mathbf{u}|^{p(\mathbf{x})} \in L^{1,\lambda}_{\text{loc}}(\Omega)$.

Proof Sketch - Boundary

Using standard methods to straighten out the boundary, we can work on half-ball.



Recall

Interior Regularity, Nice Functional - Acerbi & Mingione (1999)

If $\mathbf{v}$ minimizes K, then $\forall \gamma \in [0,n)$, $\exists C_\gamma < \infty$ such that

$$\int_{B(\mathbf{x}_0,\rho)} |\nabla \mathbf{v}|^{p(\mathbf{x})}\, \mathrm{d}\mathbf{x} \le C_\gamma \left(\frac{\rho}{R}\right)^{\gamma} \int_{B(\mathbf{x}_0,R)} |\nabla \mathbf{v}|^{p(\mathbf{x})}\, \mathrm{d}\mathbf{x}$$

whenever $B(\mathbf{x}_0, R) \subset \Omega$.

Reflection over D

Recall

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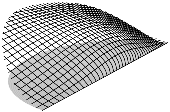
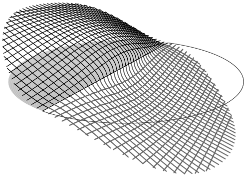
whenever $B(\mathbf{x}_0,R) \subset \Omega$.

If $\mathbf{v} = 0$ on D, we can use a reflection argument to obtain a boundary version.





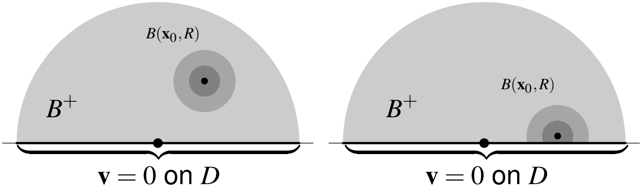
Boundary Version of Acerbi-Mingione

Theorem - Boundary Version of Acerbi-Mingione, Nice Functional

If $\mathbf{v}$ is a minimizer for $\mathbf{w} \mapsto \int_{B^+} |\nabla \mathbf{w}|^{p(\mathbf{x})}\, d\mathbf{x}$ satisfying $\mathbf{v} = 0$ on D, then $\forall \gamma \in [0, n)$, $\exists C_\gamma < \infty$ such that

$$\int_{B^+ \cap B(\mathbf{x}_0,\rho)} |\nabla \mathbf{v}|^{p(\mathbf{x})}\, d\mathbf{x} \le C_\gamma \left(\frac{\rho}{R}\right)^\gamma \int_{B^+ \cap B(\mathbf{x}_0,R)} |\nabla \mathbf{v}|^{p(\mathbf{x})}\, d\mathbf{x}$$

whenever $B^+ \cap B(\mathbf{x}_0, R) \subset B^+$. The constant C_γ does not depend on $\mathrm{dist}(\mathbf{x}_0, D)$.



Boundary Regularity

Since the boundary values $\overline{\mathbf{u}} \in W^{1,(p^+,\lambda)}(\Omega;\mathbb{R}^N)$, it suffices to prove regularity for $\tilde{\mathbf{u}} := \mathbf{u} - \overline{\mathbf{u}}$.

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Since $\mathbf{u}$ is a minimizer for $J(\mathbf{w}) = \int_\Omega \left\{ |\nabla \mathbf{w}|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{w}(\mathbf{x})) \right\} \mathrm{d}\mathbf{x}$ satisfying $\mathbf{u} = \overline{\mathbf{u}}$ on $\partial\Omega$, we have that $\tilde{\mathbf{u}}$ is a minimizer for

$$\tilde{J}(\mathbf{w}) := \int_\Omega \left\{ |\nabla(\mathbf{w} + \overline{\mathbf{u}})|^{p(\mathbf{x})} + h(\mathbf{x}, \nabla(\mathbf{w} + \overline{\mathbf{u}})) \right\} \mathrm{d}\mathbf{x}$$

satisfying $\tilde{\mathbf{u}} = 0$ on $\partial\Omega$,

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$$= \int_\Omega \left\{ |\nabla \mathbf{w}|^{p(\mathbf{x})} + H(\mathbf{x}, \nabla \mathbf{w}) \right\}$$

satisfying $\tilde{\mathbf{u}} = 0$ on $\partial\Omega$, where

$$H(\mathbf{x}, \mathbf{F}) := |\mathbf{F} + \nabla \bar{\mathbf{u}}(\mathbf{x})|^{p(\mathbf{x})} - |\mathbf{F}|^{p(\mathbf{x})} + h(\mathbf{x}, \mathbf{F} + \nabla \bar{\mathbf{u}}(\mathbf{x})).$$

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Now repeat essentially the same argument as before, using boundary version of Acerbi-Mingione instead of the interior version, to get the boundary regularity for $\tilde{\mathbf{u}}$, and hence for $\mathbf{u}$.

Functionals with More General Growth

The result actually holds for functionals with more general growth:

A More General Prototype Functional

$$J(\mathbf{u}) := \int_\Omega \left\{ g(|\nabla \mathbf{u}(\mathbf{x})|)^{\alpha(\mathbf{x})} + h(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) \right\} \mathrm{d}\mathbf{x}, \quad \mathbf{u} \in \mathscr{A}$$

Here, g is a Young function with $tg''(t) \sim g'(t)$.

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3. $g(t) := \begin{cases} t^p & \text{if } 0 \leq t \leq t_0 \\ t^{\frac{p+q}{2} + \frac{p-q}{2} \sin\log\log\log t} & \text{if } t \geq t_0 \end{cases} \qquad 1 < p \leq q.$

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Proof is similar, but need to establish analogues of the various ingredients in the more general setting.