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History

1632: Galileo Galilei publishes *Dialogue concerning the two chief world systems*

1687: Isaac Newton releases *Philosophiae Naturalis Principia Mathematica*.

1865: James Clerk Maxwell unifies the theories of electricity and magnetism.

\[ \nabla \cdot \mathbf{D} = 4\pi \rho \\
\nabla \cdot \mathbf{B} = 0 \\
\n\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \\
\n\n\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]
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**Postulate 1**

There is no absolute standard of rest; only relative motion is observable.
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**Postulate 1**
There is no absolute standard of rest; only relative motion is observable.

**Postulate 2**
The velocity of light $c$ is independent of the motion of the source.
Definitions

Definition

The four-dimensional set of axes is known as *spacetime*.
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Definition
In classical mechanics, motion is described in a *frame of reference*.
Definition

A reference frame is **inertial** if every test particle initially at rest remains at rest and every particle in motion remains in motion without changing speed or direction.

Definition

An **event** is a point \((t, x, y, z)\) in spacetime.

Definition

Any line which joins different events associated with a given object will be called a **world-line**.

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An **observer** is a series of clocks in an inertial reference frame which measures the time.
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Any line which joins different events associated with a given object will be called a *world-line*.

Definition

An *observer* is a series of clocks in an inertial reference frame which measures the time.
Theorem

Given two observers, $O$ and $O'$, events in the inertial frames of reference set up by the observers are related by:

$$
\begin{align*}
    t' &= t \\
    x' &= x + vt \\
    y' &= y \\
    z' &= z
\end{align*}
$$

where $v$ is the relative velocity which $O'$ is moving with respect to $O$. 

This theorem is incorrect (physically). We have to reexamine our ideas of time.
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The *proper time* $\tau$ along the worldline of a particle in constant motion is the time measured in an inertial coordinate system in which the particle is at rest.
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Definition

The *proper time* $\tau$ along the worldline of a particle in constant motion is the time measured in an inertial coordinate system in which the particle is at rest.

If we assume Postulate 2 by Einstein (that $c$ is a constant), then $t = kt'$. But if $O'$ is at rest relative to $O$, $t' = kt$. This $k$ is known as Bondi’s $k$-factor.
According to $O$, $t = kt'$, where $k$ is a constant. According to $O'$, $t' = kt$.

$d_B = \frac{1}{2}c(k^2 - 1)t$ and $t_B = \frac{1}{2}(k^2 + 1)t$. 
Bondi’s $k$-factor

According to $O$, $t = kt'$, where $k$ is a constant. According to $O'$, $t' = kt$.

d_E = \frac{1}{2}c(k^2 - 1)t_E$ and $t_B = \frac{1}{2}(k^2 + 1)t_B$. 
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According to $O$, $t = kt'$, where $k$ is a constant. According to $O'$, $t' = kt$.

$t_B$ and $d_B$ can be expressed as:

\[ d_B = \frac{1}{2}c(k^2 - 1)t \quad \text{and} \quad t_B = \frac{1}{2}(k^2 + 1)t. \]
The Gamma Factor

\[ v = \frac{d_B}{t_B} = \frac{1}{2} c (k^2 - 1) t = \frac{1}{2} \left( \frac{k^2}{k^2 + 1} \right) \frac{c(k^2 - 1)}{(k^2 + 1)}. \]
The Gamma Factor

\[ v = \frac{d_B}{t_B} = \frac{1}{2} c (k^2 - 1)t = \frac{c(k^2 - 1)}{(k^2 + 1)} . \]

\[ v k^2 + v = c k^2 - c \]

\[ c + v = c k^2 - v k^2 \]

\[ c + v = k^2 (c - v) \]

\[ k = \sqrt{\frac{c + v}{c - v}} > 1. \]
The Gamma Factor

\[ v = \frac{d_B}{t_B} = \frac{1}{2} c(k^2 - 1)t = \frac{c(k^2 - 1)}{(k^2 + 1)}. \]

\[ vk^2 + v = ck^2 - c \]
\[ c + v = ck^2 - vk^2 \]
\[ c + v = k^2(c - v) \]
\[ k = \sqrt{\frac{c + v}{c - v}} > 1. \]

\[ \frac{\text{time } E \text{ to } B \text{ measured by } O}{\text{time } E \text{ to } B \text{ measured by } O'} = \frac{t_B}{kt} = \frac{(k^2 + 1)t}{2kt} = \gamma(v) \]
The Gamma Factor

\[ v = \frac{d_B}{t_B} = \frac{\frac{1}{2} c (k^2 - 1) t}{\frac{1}{2} (k^2 + 1) t} = \frac{c (k^2 - 1)}{(k^2 + 1)}. \]

\[ \gamma(v) = \frac{(k^2 + 1) t}{2 k t} = \frac{1}{\sqrt{1 - v^2/c^2}}. \]
The Lorentz Transformation

The inertial coordinate systems set up by two observers are related by:

\[
\begin{align*}
    t &= t' + \left( \frac{v_x'}{c^2} \right) \sqrt{1 - \left( \frac{v}{c} \right)^2} \\
    x &= x' + v t' \sqrt{1 - \left( \frac{v}{c} \right)^2}
\end{align*}
\]

We can write this more concisely as

\[
(\gamma v) (cv') = \gamma \left( \frac{v}{c} \right) (cv')
\]

where \( v \) is the relative velocity.
The Lorentz Transformation

Theorem

The inertial coordinate systems set up by two observers are related by:

\[
\begin{align*}
  t &= t' + \frac{vx'/c^2}{\sqrt{1 - (v/c)^2}} \\
  x &= \frac{x' + vt'}{\sqrt{1 - (v/c)^2}}
\end{align*}
\]

We can write this more concisely as

\[
\begin{pmatrix}
  ct \\
  x
\end{pmatrix} = \gamma(v) \begin{pmatrix}
  1 & v/c \\
  v/c & 1
\end{pmatrix} \begin{pmatrix}
  ct' \\
  x'
\end{pmatrix}
\]

where \( v \) is the relative velocity.
Proof

First consider two observers moving at constant speeds. We express an event in terms of both coordinate systems. We substitute in for Bondi’s \( k \)-factor:

\[
\frac{ct}{x} = \frac{1}{2} \left( k + \frac{1}{k} - 1 \right) \left( \frac{ct'}{x'} \right)
\]

After some algebra and simplification, we have our result (2).
Proof

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Proof

- First consider two observers moving at constant speeds.
- We express an event in terms of both coordinate systems.
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\[
\begin{pmatrix}
  c t \\
  x
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
  k + k^{-1} & k - k^{-1} \\
  k - k^{-1} & k + k^{-1}
\end{pmatrix} \begin{pmatrix}
  c t' \\
  x'
\end{pmatrix}
\]

- After some algebra and simplification, we have our result $\square$. 
Proof

- First consider two observers moving at constant speeds.
- We express an event in terms of both coordinate systems.
- We substitute in for Bondi’s $k$-factor

$$
\begin{pmatrix}
  ct \\
  x
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
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  k - k^{-1} & k + k^{-1}
\end{pmatrix}\begin{pmatrix}
  ct' \\
  x'
\end{pmatrix}
$$

- After some algebra and simplification, we have our result $\square$. 
In four dimensions,

\[
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
\gamma & \gamma v/c & 0 & 0 \\
\gamma v/c & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix}
\] (2)

where \(v\) is the relative velocity and \(\gamma = \gamma(v)\).
In four dimensions,

\[
\begin{pmatrix}
ct \\
x \\
y \\
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= \begin{pmatrix}
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\gamma v/c & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix}
\]

(2)

where \( v \) is the relative velocity and \( \gamma = \gamma(v) \).

**Definition**

The 4 \times 4 matrix in (2) is known as the *boost*, denoted \( L_v \).

With the properties of the boost, we can look at how an observer would judge a moving particle’s velocity.
Theorem

A particle cannot travel faster than $c$, the velocity of light.
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A particle cannot travel faster than $c$, the velocity of light.

Proof:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & -u/c \\ -u/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ -vt + b \end{pmatrix}.$$
Theorem

A particle cannot travel faster than $c$, the velocity of light.

Proof:

\[
\begin{pmatrix}
  ct' \\
  x'
\end{pmatrix} = \gamma(u) \begin{pmatrix}
  1 & -u/c \\
  -u/c & 1
\end{pmatrix} \begin{pmatrix}
  ct \\
  -\nu t + b
\end{pmatrix}.
\]

\[
w = -\frac{dx'}{dt'} = \frac{v + u}{1 + uv/c^2}.
\]
\[ w = \frac{v + u}{1 + uv/c^2}. \]

Letting \(|u| < c\) and \(|v| < c\), we see that \(|w| < c\) since
\[ w = \frac{v + u}{1 + uv/c^2}. \]

Letting \(|u| < c\) and \(|v| < c\), we see that \(|w| < c\) since

\[
(c - u)(c - v) > 0 \iff -(u + v)c > -c^2 - uv
\]

\[
u + v < c \left(1 + \frac{uv}{c^2}\right)
\]

\[w < c.\]
\[ w = \frac{v + u}{1 + uv/c^2}. \]

Letting \(|u| < c\) and \(|v| < c\), we see that \(|w| < c\) since

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\]

\[
u + v < c \left(1 + \frac{uv}{c^2}\right)
\]

\[ w < c. \]

\[
(c + u)(c + v) > 0 \iff (u + v)c > -c^2 - uv
\]

\[
u + v > -c \left(1 + \frac{uv}{c^2}\right)
\]

\[ w > -c. \quad \square\]
In four-dimensions, we can say that

\[
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix} = \mathbf{L}
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix}
\tag{3}
\]

where \( \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \), and \( \mathbf{H}, \mathbf{K} \) are 3\times3 orthogonal matrices.

Definition
The matrix \( \mathbf{L} \) in (3) is a Lorentz transformation if

\[ \mathbf{L}^{-1} = \mathbf{gL}^T \mathbf{g}, \]

where \( \mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \).

\( \mathbf{L} \) is orthochronous if \( l_{1,1} \), \( l_{2,2} \) > 0, where \( l_{1,1} \) is the first entry of the first row of \( \mathbf{L} \).
In four-dimensions, we can say that

\[
\begin{pmatrix}
ct \\
 x \\
y \\
z
\end{pmatrix} = L
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix}
\]

where

\[L = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} L_v \begin{pmatrix} 1 & 0 \\ 0 & K^T \end{pmatrix}\]

and \(H, K\) are 3 \times 3 orthogonal matrices.
In four-dimensions, we can say that

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\begin{pmatrix}
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x \\
y \\
z
\end{pmatrix}
= L
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x' \\
y' \\
z'
\end{pmatrix}
\tag{3}
\]

where

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L = \begin{pmatrix}
1 & 0 \\
0 & H
\end{pmatrix}
L_v
\begin{pmatrix}
1 & 0 \\
0 & K^T
\end{pmatrix}
\]

and \(H, K\) are 3 \(\times\) 3 orthogonal matrices.

**Definition**

The matrix \(L\) in (3) is a *Lorentz transformation* if
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y \\
z
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\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix}
$$

(3)

where $L = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} L_v \begin{pmatrix} 1 & 0 \\ 0 & K^T \end{pmatrix}$ and $H, K$ are $3 \times 3$ orthogonal matrices.

**Definition**

The matrix $L$ in (3) is a *Lorentz transformation* if $L^{-1} = g L^T g$, where

$$
g = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
$$
In four-dimensions, we can say that

\[
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix} = L
\begin{pmatrix}
c t' \\
x' \\
y' \\
z'
\end{pmatrix}
\]

where

\[
L = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} L_v \begin{pmatrix} 1 & 0 \\ 0 & K^T \end{pmatrix}
\]

and \(H, K\) are 3 \(\times\) 3 orthogonal matrices.

**Definition**

The matrix \(L\) in (3) is a *Lorentz transformation* if \(L^{-1} = gL^T g\), where

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g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

\(L\) is *orthochronous* if \(l_{1,1} > 0\), where \(l_{1,1}\) is the first entry of the first row of \(L\).
Definition

In special relativity, the space we live in is called *Minkowski Space*, denoted by \( M = \mathbb{R} \times \mathbb{R}^3 \) where \( \mathbb{R} = \mathbb{R} \times 0 \) is the time axes, and \( \mathbb{R}^3 = 0 \times \mathbb{R}^3 \) the space axes.
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The distance $D$ from a point to the origin is $D = \sqrt{x^2 + y^2 + z^2}$. 
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The distance $D$ from a point to the origin is $D = \sqrt{x^2 + y^2 + z^2}$.

- Emit a light pulse when $t = x = y = z = 0$. 

Jared Ruiz
Advised by Dr. Steven Kent
The Mathematics of Special Relativity
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- Emit a light pulse when $t = x = y = z = 0$.
- It arrives at $(ct, x, y, z)$ if $ct = \sqrt{x^2 + y^2 + z^2} = D$. 
Definition

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- $c^2 t^2 = D^2$, and $c^2 t^2 - x^2 - y^2 - z^2 = 0$. 
Definition

In special relativity, the space we live in is called *Minkowski Space*, denoted by $\mathbb{M} = \mathbb{R} \times \mathbb{R}^3$ where $\mathbb{R} = \mathbb{R} \times 0$ is the time axes, and $\mathbb{R}^3 = 0 \times \mathbb{R}^3$ the space axes.

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- $c^2 t^2 = D^2$, and $c^2 t^2 - x^2 - y^2 - z^2 = 0$.

Definition

In Minkowski space, the *interval* between any two events $x = (t_1, x_1, y_1, z_1)$ and $y = (t_2, x_2, y_2, z_2)$ is defined to be

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2.$$
Invariance of the Interval

If

\[ c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \]

for an observer \( O \),
Invariance of the Interval

If

\[ c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \]

for an observer \( O \), then

\[ c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2 = 0 \]

for an observer \( O' \), moving with constant velocity relative to \( O \).
Invariance of the Interval

If

\[ c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \]

for an observer \( O \), then

\[ c^2(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 - (y'_2 - y'_1)^2 - (z'_2 - z'_1)^2 = 0 \]

for an observer \( O' \), moving with constant velocity relative to \( O \). Because of this, we say the interval is \textit{invariant}.
The Inner Product

**Definition**

In $\mathbb{M}$, two objects $X = (X_0, X_1, X_2, X_3)$ and $X' = (X'_0, X'_1, X'_2, X'_3)$ are called *four-vectors* if

$$X = LX'$$

where $L$ is the general Lorentz transformation.

$\text{Definition}$
The Inner Product

Definition

In $\mathbb{M}$, two objects $X = (X_0, X_1, X_2, X_3)$ and $X' = (X'_0, X'_1, X'_2, X'_3)$ are called *four-vectors* if

$$X = LX'$$

where $L$ is the general Lorentz transformation.

For $x = (ct_1, x_1, y_1, z_1), y = (ct_2, x_2, y_2, z_2) \in \mathbb{M}$, the displacement four-vector

$$X = y - x = c(t_2 - t_1) + (x_2 - x_1) + (y_2 - y_1) + (z_2 - z_1).$$
The Inner Product

Definition

In $\mathbb{M}$, two objects $X = (X_0, X_1, X_2, X_3)$ and $X' = (X'_0, X'_1, X'_2, X'_3)$ are called four-vectors if

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For $x = (ct_1, x_1, y_1, z_1)$, $y = (ct_2, x_2, y_2, z_2) \in \mathbb{M}$, the displacement four-vector

$$X = y - x = c(t_2 - t_1) + (x_2 - x_1) + (y_2 - y_1) + (z_2 - z_1).$$

Definition

The inner product between two four-vectors $X, Y \in \mathbb{M}$ is:

$$g(X, Y) = X_0 Y_0 - X_1 Y_1 - X_2 Y_2 - X_3 Y_3.$$
Predictions

Definition

The *four-velocity* of a particle, \((V_0, V_1, V_2, V_3)\), is given by:

\[
V_0 = c \frac{dt}{d\tau}, \quad V_1 = \frac{dx}{d\tau}, \quad V_2 = \frac{dy}{d\tau}, \quad V_3 = \frac{dz}{d\tau}.
\]
The **four-velocity** of a particle, \((V_0, V_1, V_2, V_3)\), is given by:

\[
V_0 = c \frac{dt}{d\tau}, \quad V_1 = \frac{dx}{d\tau}, \quad V_2 = \frac{dy}{d\tau}, \quad V_3 = \frac{dz}{d\tau}.
\]

**Theorem**

Let an observer \(O\) be moving with constant velocity \(V\). Then \(O\) sees two events \(A\) and \(B\) as being simultaneous if and only if the displacement \(g(X, V) = 0\), where \(X = B - A\).
### Definition

The *four-velocity* of a particle, \((V_0, V_1, V_2, V_3)\), is given by:

\[
V_0 = c \frac{dt}{d\tau}, \quad V_1 = \frac{dx}{d\tau}, \quad V_2 = \frac{dy}{d\tau}, \quad V_3 = \frac{dz}{d\tau}.
\]

### Theorem

Let an observer \(O\) be moving with constant velocity \(V\). Then \(O\) sees two events \(A\) and \(B\) as being simultaneous if and only if the displacement \(g(X, V) = 0\), where \(X = B - A\).

### Corollary

Two events which are simultaneous for one observer may not be simultaneous for another observer moving at constant velocity with the respect to the first observer.
Theorem

If a rod has length $R_0$ in a rest frame, then in an inertial coordinate system oriented in the direction of the unit vector $e$ and moving with respect to the rod with velocity $v$, the length of the rod is:
The Lorentz Contraction

Theorem

If a rod has length $R_0$ in a rest frame, then in an inertial coordinate system oriented in the direction of the unit vector $\mathbf{e}$ and moving with respect to the rod with velocity $\mathbf{v}$, the length of the rod is:

$$R = \frac{R_0 \sqrt{c^2 - v^2}}{\sqrt{c^2 - v^2 \sin^2(\theta)}}$$

where $\theta$ is the angle between $\mathbf{e}$ and $\mathbf{v}$. 
**Definition**

The *rest mass* $m_0 = m(0)$ of a body is the mass of a body measured in an inertial coordinate system in which the body is at rest.
Definition

The \textit{rest mass} $m_0 = m(0)$ of a body is the mass of a body measured in an inertial coordinate system in which the body is at rest.

Theorem

A body’s inertial mass $m$ is a function of $v$, and can be rewritten as $m = m(v) = \gamma(v)m_0$. 

$E = mc^2$. 

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**Definition**

The *rest mass* \( m_0 = m(0) \) of a body is the mass of a body measured in an inertial coordinate system in which the body is at rest.

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A body’s inertial mass \( m \) is a function of \( v \), and can be rewritten as \( m = m(v) = \gamma(v)m_0 \).

**Theorem**

\[ E = mc^2. \]
Recall:

\[ g(X,Y) = X_0 Y_0 - X_1 Y_1 - X_2 Y_2 - X_3 Y_3. \]

For any three four-vectors \( X, Y, Z \in M \), and \( \alpha \in \mathbb{R} \):

\[ g(X,Y) = g(Y,X). \]

\[ g(\alpha X + \beta Y, Z) = \alpha g(X,Z) + \beta g(Y,Z). \]

But \( g(X,Y) < 0 \) if \( X_0 Y_0 < X_1 Y_1 + X_2 Y_2 + X_3 Y_3 \).

So \( g \) is an indefinite inner product.

Thus \( g(X,X) \) (or the interval) can be used to split Minkowski space into cones.
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So \( g \) is an *indefinite* inner product. Thus \( g(X, X) \) (or the interval) can be used to split Minkowski space into cones.
Definition

For an event \( x \in M \), we have:

- Light-Cone \( C_L(x) = \{ y : g(X, X) = 0 \} \)

- Time-Cone \( C_T(x) = \{ y : g(X, X) > 0 \} \)

- Space-Cone \( C_S(x) = \{ y : g(X, X) < 0 \} \)

where \( y \in M \), and \( X \) is the displacement vector from \( x \) to \( y \).
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where $y \in \mathbb{M}$, and $X$ is the displacement vector from $x$ to $y$. 
For events $x, y \in \mathbb{M}$ we define a partial ordering $<$ on $\mathbb{M}$ by $x < y$ if the displacement vector $X$ from $x$ to $y$ lies in the future light-cone.
Definition

For events $x, y \in \mathbb{M}$ we define a partial ordering $<$ on $\mathbb{M}$ by $x < y$ if the displacement vector $X$ from $x$ to $y$ lies in the future light-cone.

That is, $x < y$ if $t_y > t_x$, and $g(X, X) > 0$. 
The matrix $L$ in (3) is a Lorentz transformation if $L^{-1} = gL^T g$, where $g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$. 
**Definition**

The matrix $L$ in (3) is a **Lorentz transformation** if $L^{-1} = gL^Tg$, where 

$$g = \begin{pmatrix} 
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}.$$ 

**Definition**

Define the **Lorentz group**

$$\mathcal{L} = \{ \lambda : \mathbb{M} \to \mathbb{M} : \forall X, Y \in \mathbb{M}, g(\lambda X, \lambda Y) = g(X, Y) \}.$$
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$g(X', Y')$
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Definition

Define the *Lorentz group* $\mathcal{L} = \{ \lambda : M \rightarrow M : \forall X, Y \in M, g(\lambda X, \lambda Y) = g(X, Y) \}$

$g(X', Y') = g(LX, LY)$
Definition

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Definition

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The matrix \( L \) in (3) is a *Lorentz transformation* if \( L^{-1} = gL^T g \), where \( g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \). \( L \) is *orthochronous* if \( l_{1,1} > 0 \), where \( l_{1,1} \) is the first entry of the first row of \( L \).

Define the *Lorentz group* \( \mathcal{L} = \{ \lambda : M \to M : \forall X, Y \in M, g(\lambda X, \lambda Y) = g(X, Y) \} \)

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The matrix $L$ in (3) is a *Lorentz transformation* if $L^{-1} = gL^T g$,
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1 & 0 & 0 & 0 \\
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Definition

Define the *Lorentz group* $L = \{ \lambda : \mathbb{M} \to \mathbb{M} : \forall X, Y \in \mathbb{M}, g(\lambda X, \lambda Y) = g(X, Y) \}$

$$g(X', Y') = g(LX, LY) = g(\lambda X, \lambda Y) = g(X, Y).$$

Definition

The *orthochronous Lorentz group* $L_+$ is the subgroup of $L$ whose elements preserve the partial ordering $<$ on $\mathbb{M}$.
Usually we think of the topology on $\mathbb{R}^4$ as the standard Euclidean topology $\mathcal{T}$. 
Usually we think of the topology on $\mathbb{R}^4$ as the standard Euclidean topology $T$.
Physically, this topology is not useful because:

1. The 4-dimensional Euclidean topology is locally homogeneous, yet $\mathbb{M}$ is not (the light cone separates timelike and spacelike events).
Usually we think of the topology on $\mathbb{R}^4$ as the standard Euclidean topology $\mathcal{T}$. Physically, this topology is not useful because:

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Two properties which $\mathcal{T}^F$ will satisfy are:

1*. $\mathcal{T}^F$ is not locally homogenous, and the light cone through any point can be deduced from $\mathcal{T}^F$. 

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Usually we think of the topology on $\mathbb{R}^4$ as the standard Euclidean topology $\mathcal{T}$.

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Two properties which $\mathcal{T}^F$ will satisfy are:

1*. $\mathcal{T}^F$ is not locally homogenous, and the light cone through any point can be deduced from $\mathcal{T}^F$.

2*. The group of all homeomorphisms of the fine topology is generated by the inhomogeneous Lorentz group and dilatations.
Definition

Define the group $G$ to be the group consisting of:

1. The Lorentz group $\mathcal{L}$.
2. Translations ($X' = X + K$ where $K \in \mathbb{M}$ is constant).
3. Multiplication by a scalar, or dilatations ($X' = \alpha X$ for $\alpha \in \mathbb{R}$).
Definition

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Definition

Define the group $G_0$ to be the group consisting of:
1. The orthochronous Lorentz group $\mathcal{L}_+$. 
2. Translations. 
3. Dilatations.
Definition

The fine topology $\mathcal{T}^F$ of $\mathbb{M}$ induces the 1-dimensional Euclidean topology on $g^\mathbb{R}$ and the 3-dimensional topology on $g^\mathbb{R}^3$, where $g \in G$. 
Definition

The fine topology $\mathcal{T}^F$ of $\mathbb{M}$ induces the 1-dimensional Euclidean topology on $g\mathbb{R}$ and the 3-dimensional topology on $g\mathbb{R}^3$, where $g \in G$. A set $U \in \mathbb{M}$ is open in $\mathcal{T}^F$ $\iff$ $U \cap g\mathbb{R}$ is open in $g\mathbb{R}$, and $U \cap g\mathbb{R}^3$ is open in $g\mathbb{R}^3$. 

$\epsilon$-neighborhoods in $\mathcal{T}^F$ are $N_{\mathcal{F}} \epsilon(x) = N_{\mathcal{E}} \epsilon(x) \cap (C_{\mathcal{T}}(x) \cup C_{\mathcal{S}}(x))$ 
where $x \in \mathbb{M}$. 

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Definition

The fine topology $T^F$ of $\mathbb{M}$ induces the 1-dimensional Euclidean topology on $g^\mathbb{R}$ and the 3-dimensional topology on $g^\mathbb{R}^3$, where $g \in G$. A set $U \in \mathbb{M}$ is open in $T^F$ $\iff$ $U \cap g^\mathbb{R}$ is open in $g^\mathbb{R}$, and $U \cap g^\mathbb{R}^3$ is open in $g^\mathbb{R}^3$.

The $\epsilon$-neighborhoods in $T$ are $N^E_\epsilon(x) = \{y : d(x, y) < \epsilon\}$.
Definition

The fine topology $\mathcal{T}^F$ of $\mathbb{M}$ induces the 1-dimensional Euclidean topology on $g\mathbb{R}$ and the 3-dimensional topology on $g\mathbb{R}^3$, where $g \in G$. A set $U \in \mathbb{M}$ is open in $\mathcal{T}^F \iff U \cap g\mathbb{R}$ is open in $g\mathbb{R}$, and $U \cap g\mathbb{R}^3$ is open in $g\mathbb{R}^3$.

The $\epsilon$-neighborhoods in $\mathcal{T}$ are $N^E_\epsilon(x) = \{y : d(x, y) < \epsilon\}$.

Definition

We define the $\epsilon$-neighborhoods in $\mathcal{T}^F$ as

$$N^F_\epsilon(x) = N^E_\epsilon(x) \cap \left( C^T(x) \cup C^S(x) \right)$$

where $x \in \mathbb{M}$.
Theorem

\( N_\epsilon^F (x) \) is open in \( T^F \).
Theorem

$N^F_\epsilon(x)$ is open in $\mathcal{T}^F$.

Proof: Let $A$ be either the time axes or space axes. Then

$$N^F_\epsilon(x) \cap A = \begin{cases} & \text{if } x \in A \\ & \text{if } x \notin A \end{cases}$$
Theorem

$N^F_\epsilon(x)$ is open in $T^F$.

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**Theorem**

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N^E_\epsilon(x) \cap A & \text{if } x \in A \\
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\end{cases}$$
The group of all homeomorphisms of $T^F$ is $G$.

Proof:

$T^F$ is defined invariantly under $G$, so every $g \in G$ is a homeomorphism.

Now we have to show every homeomorphism $h : (M, T^F) \to (M, T^F)$ is in $G$.

Now let $g \in G$ be the element corresponding to time reflection.

Lemma

Let $h : (M, T^F) \to (M, T^F)$ be a homeomorphism. Then $h$ either preserves the partial ordering or reverses it.

So either $h$ or $gh$ preserves the partial ordering, and is an element of $G_0$.

The group of all homeomorphisms is thus generated by $g$ and $G_0$, which is $G$.2
First Result

Theorem

The group of all homeomorphisms of $T^F$ is $G$. 

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The Mathematics of Special Relativity
First Result

Theorem

The group of all homeomorphisms of $\mathcal{T}^F$ is $G$.

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The group of all homeomorphisms of $\mathcal{T}^F$ is $G$.

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The group of all homeomorphisms is thus generated by $g$ and $G_0$, which is $G$. 

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First Result

**Theorem**

The group of all homeomorphisms of \( T^F \) is \( G \).

**Proof:** \( T^F \) is defined invariantly under \( G \), so every \( g \in G \) is a homeomorphism. Now we have to show every homeomorphism \( h : (\mathcal{M}, T^F) \rightarrow (\mathcal{M}, T^F) \) is in \( G \).

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**Lemma**

Let \( h : (\mathcal{M}, T^F) \rightarrow (\mathcal{M}, T^F) \) be a homeomorphism. Then \( h \) either preserves the partial ordering or reverses it.
Theorem

The group of all homeomorphisms of $\mathcal{T}^F$ is $G$.

**Proof:** $\mathcal{T}^F$ is defined invariantly under $G$, so every $g \in G$ is a homeomorphism. Now we have to show every homeomorphism $h : (\mathbb{M}, \mathcal{T}^F) \rightarrow (\mathbb{M}, \mathcal{T}^F)$ is in $G$.

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Theorem

The group of all homeomorphisms of $\mathcal{T}^F$ is $G$.

Proof: $\mathcal{T}^F$ is defined invariantly under $G$, so every $g \in G$ is a homeomorphism. Now we have to show every homeomorphism $h : (\mathbb{M}, \mathcal{T}^F) \rightarrow (\mathbb{M}, \mathcal{T}^F)$ is in $G$.

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So either $h$ or $gh$ preserves the partial ordering, and is an element of $G_0$.

The group of all homeomorphisms is thus generated by $g$ and $G_0$, which is $G$. □
Second Result

Corollary

The light, time, and space cones through a point can be deduced from the topology.
The light, time, and space cones through a point can be deduced from the topology.

**Proof:** For $x \in M$, let $G_x$ be the group of homeomorphisms which fix $x$. 

**Corollary**
The light, time, and space cones through a point can be deduced from the topology.

**Proof:** For $x \in M$, let $G_x$ be the group of homeomorphisms which fix $x$. By the theorem, $G_x$ is generated by the Lorentz group and dilatations.
Corollary

The light, time, and space cones through a point can be deduced from the topology.

Proof: For $x \in M$, let $G_x$ be the group of homeomorphisms which fix $x$. By the theorem, $G_x$ is generated by the Lorentz group and dilatations. Therefore there are exactly four orbits under $G_x$: $C^L(x) \setminus x$, $C^T(x) \setminus x$, $C^S(x) \setminus x$, the point $x$. □
Assumed Einstein's Two Postulates.

Bondi's $k$-factor.

Gamma factor $\gamma(v) = \sqrt{1 - \frac{v^2}{c^2}}$.

Lorentz Boost $L$ and Lorentz Transformation $L$.

Invariance of the Interval.

Inner Product $g$.

Predictions (simultaneity, Lorentz Contraction, mass, etc.).

Cones.

Defined Lorentz group $L$ and $L^+$.

Defined fine topology $T_F$ on $M$.

All mappings considered are in the Lorentz group.
Assumed Einstein’s Two Postulates.
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- Assumed Einstein’s Two Postulates.
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- Gamma factor $\gamma(v) = \frac{1}{\sqrt{1 - (v/c)^2}}$. 

\[ L^\nu_{\lambda} \quad \text{and} \quad L^\nu_{\lambda} + \] 

\[ \text{Invariance of the Interval.} \]

\[ \text{Inner Product} \] 

\[ g. \]

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The Mathematics of Special Relativity

Jared Ruiz Advised by Dr. Steven Kent
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References