Let's recall some facts from trigonometry. We have the two functions $\sin(x)$ and $\cos(x)$. We usually introduce these as giving the ratios of the sides in a right triangle (remember “SohCahToa”), but then later learn that sin and cos are defined for every real number, and give values between -1 and 1. Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$, and its domain is every number such that $\cos(x) \neq 0$. Recall $\cos(x) = 0$ if and only if $x = \pi/2 + k\pi$, for some integer $k$. The graph of the tangent function is:

These all have natural inverse functions: $\arcsin$, $\arccos$, and $\arctan$. To define these, we restrict the domain of the original functions. For our purposes, we are only interested in the $\arctan$ function. Here, the domain of $\arctan$ is any real number, and its range is $(-\pi/2, \pi/2)$. The graph of $\arctan$ is:
Today we are going to prove the following “well-known” theorem:

**Theorem:** \( \arctan 1 + \arctan 2 + \arctan 3 = \pi. \)

We will do this without any calculators too! Go back to the graph of \( \arctan \), and graphically sketch what this theorem is really saying.

To warm-up, let’s do a few quick calculations. Let’s draw our two favorite triangles, the \( 30° - 60° - 90° \) triangle, and \( 45° - 45° - 90° \) triangle:

Then....

- \( \tan(\pi/4) = \ldots \), so \( \arctan(\ldots) = \ldots \)
- \( \tan(\pi/3) = \ldots \), so \( \arctan(\ldots) = \ldots \)
- \( \tan(\pi/6) = \ldots \), so \( \arctan(\ldots) = \ldots \)

The following fact will also come in handy:

**Fact 1.** For all \( a \in \mathbb{R} \),

\[ \arctan(-a) = -\arctan a \]

Notice that all this says is that \( \arctan \) is an odd function. You can verify that this is true by looking at the graph of the arctangent function.
To prove the main result (again, without calculators!), there are two lemmas that we will need to prove first.

**Lemma 1.** For a real number \( x \) such that \( x \neq 0 \),

\[
\arctan x + \arctan(1/x) = \begin{cases} 
\frac{\pi}{2}; & x > 0 \\
-\frac{\pi}{2}; & x < 0 
\end{cases}
\]

**Proof:** Define a function \( f(x) = \arctan x + \arctan(1/x) \). Notice, this function is defined for \( x \neq 0 \); so \( f(x) \) is continuous for \( x > 0 \) and for \( x < 0 \). We are trying to prove \( f(x) = \pi/2 \) if \( x > 0 \), and \( f(x) = -\pi/2 \) if \( x < 0 \). So we are trying to prove \( f(x) \) (after restricting the domain to either all positive or all negatives) is constant. What’s the easiest way to show a function is constant?

\[
f'(x) = \frac{1}{1 + x^2} + \frac{1}{1 + (1/x)^2} \left( -\frac{1}{x^2} \right)
\]

So now that we know \( f(x) \) is constant for all positive values and all negative values, what can we do? If it’s constant, we can plug in ANY value we want, and the output will always be the same. So, what should we plug in?

Notice, this is NOT a proof by example. Can anyone tell us why?
Lemma 2. For $a, b \in \mathbb{R}$, $ab \neq 1$

$$\arctan a + \arctan b = \begin{cases} 
\arctan \left( \frac{a+b}{1-ab} \right); & ab < 1 \\
\arctan \left( \frac{a+b}{1-ab} \right) + \pi; & ab > 1, a, b > 0 \\
\arctan \left( \frac{a+b}{1-ab} \right) - \pi; & ab > 1, a, b < 0
\end{cases}.$$ 

Proof: We really only care about the second line, so that’s what we’ll prove. But to do that, we need a few things. Recall the sum formulas for sin and cos:

$$\sin(x+y) = \sin(x) \cos(y) + \sin(y) \cos(x) \quad \text{and} \quad \cos(x+y) = \cos(x) \cos(y) - \sin(x) \sin(y).$$

Using these, prove that $\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x) \tan(y)}$.

Now we have to be careful. The graph of the tangent function shows us that there are lots of vertical asymptotes, ie. places where tan is undefined. So this formula is only defined if $x + y$ is not a multiple of $\pi/2$. In particular, its true for $x = \arctan(a)$, and $y = \arctan(b)$, if $a$ and $b$ are chosen correctly (we’ll show this on the next page). If we plug these values in for $x$ and $y$, what happens?
But wait, we aren’t done yet! Notice that we can’t have $ab = 1$, so we need to consider what happens when $ab < 1$ and $ab > 1$.

- First suppose $ab < 1$, and that both $a$ and $b$ are positive. If $b > 1$, then $a < 1/b$, so

$$\arctan(a) + \arctan(b) < \arctan\left(\frac{1}{b}\right) + \arctan(b) \quad \text{(since arctan is an increasing function)}$$

$$= \frac{\pi}{2} \quad \text{(from Lemma 1)}.$$ 

We can do something similar if $a, b$ are both negative. So, as long as $ab < 1$ then $\arctan a + \arctan b < \pi/2$, which means everything is defined, and we’ve proved the first line of the lemma:

$$\arctan a + \arctan b = \arctan\left(\frac{a + b}{1 - ab}\right).$$

- Now what if $ab > 1$? Suppose that $a, b > 0$. If we take reciprocals, we have

$$0 < \frac{1}{ab} = \frac{1}{a} \frac{1}{b} < 1.$$ 

So...

$$\arctan a + \arctan b = \arctan\left(\frac{a + b}{1 - ab}\right). \quad \text{(use Lemma 1)}$$

$$= \arctan\left(\frac{a + b}{1 - ab}\right) \quad \text{(simplify)}$$

$$= \arctan\left(\frac{a + b}{1 - ab}\right) \quad \text{(use the fact that } 0 < \frac{1}{ab} < 1\text{)}$$

From here, just simplify this last term. What do you get?
Now we can prove the original result:

\[
\arctan 1 + \arctan 2 + \arctan 3 = \pi
\]

How do you do it?

Proof:

In fact, something much stronger can be proved:

**Theorem 1.** For every positive integer \( n \), there exists an increasing sequence of positive integers \( k_1, k_2, \ldots, k_s \) such that

\[
\sum_{j=1}^{s} t_j \arctan k_j = n\pi
\]

for suitable \( t_j \in \{-1, 1\} \).

For instance,

\[
- \arctan 1 + \arctan 2 + \arctan 4 + \arctan 11 - \arctan 12 + \arctan 13 + \arctan 158 + \arctan 21351 = 2\pi.
\]