1 Hilbert Spaces

Definition (1): A vector space (over $\mathbb{F}$) is an inner product space if there exists a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$ satisfying the following properties:

1. $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$,
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, for all $x, y \in \mathbb{C}^n$,
3. $\langle x + \alpha y, z \rangle = \langle x, z \rangle + \alpha \langle y, z \rangle$, for $\alpha \in \mathbb{C}^n$ and $x, y, z \in \mathbb{C}^n$,
4. $\langle x, x \rangle \geq 0$, for all $x \in \mathbb{C}^n$,
5. If $\langle x, x \rangle = 0$, then $x \equiv 0$.

If $\langle \cdot, \cdot \rangle$ only satisfies (a)-(d), we say X is a semi-inner product space.

Example (2):

1. The Euclidean inner product space is $\mathbb{F}^n$ with the inner product
$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$
2. Let $\lambda_1, ..., \lambda_n$ be positive real numbers. Then,
$$\langle x, y \rangle := \sum_{i=1}^{n} \lambda_i x_i \overline{y_i}$$
is also an inner product on $\mathbb{F}^n$. If we only require $\lambda_1, ..., \lambda_n$ to be nonnegative, then the object defined above is a semi-inner product.

Theorem (3): (Cauchy-Schwarz Inequality) Let X be an inner product space. Then for every $x, y \in X$,
$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2}.$$
Moreover, equality holds if and only if $\{x, y\}$ are linearly dependent.

Corollary (4): (Cauchy-Schwarz) For any semi-inner product space X, let $\|x\| = \langle x, x \rangle$.

1. Then for all $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$.
2. For all $x \in X$ and $\alpha \in \mathbb{F}$, $\|\alpha x\| = |\alpha| \|x\|$.

Remark:

1. If X is an inner product space, then $\|x\| = 0$ if and only if $x = 0$.
2. For any inner product space define $d(x, y) := \|x - y\|$ for all $x, y \in X$. Then, d is a metric on X.

Definition (5): A Hilbert Space is an inner product space (over $\mathbb{F}$) which is complete in the metric arising from the inner product.

Definition (6): A norm on a vector space X is a function $\|\cdot\| : X \to [0, \infty)$ such that the following hold:

1. $\|x\| = 0$ if and only if $x = 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$, for all $\alpha \in \mathbb{F}$ and $x \in X$
3. $\|x + y\| = \|x\| + \|y\|$

Notation: $\mathcal{H}$ is a Hilbert space over $\mathbb{F}$. For $f, g \in \mathcal{H}$, $f \perp g$ if and only if $\langle f, g \rangle = 0$.

Proposition (7): (Pythagorean Theorem) If $f_1, ..., f_n \in \mathcal{H}$ are pairwise orthogonal, then

$$\sum_{j=1}^{n} \|f_j\| = \left\| \sum_{j=1}^{n} f_j \right\|.$$

Proposition (8): (Parallelogram Law) If $f, g \in \mathcal{H}$, then

$$\|f + g\|^2 + \|f - g\|^2 = 2 \left(\|f\|^2 + \|g\|^2 \right).$$

Useful Fact: Suppose that X is a vector space over $\mathbb{F}$ equipped with a norm and the norm satisfies the Parallelogram Law, then X is an inner product space with the inner product

$$\langle x, y \rangle = \frac{1}{4} \sum_{n=0}^{3} \|x + i^n y\|^2 \, i^n. \qquad \text{(Polarization Identity)}$$

Definition (9): A set in a vector space X over $\mathbb{F}$ is convex if whenever $x, y \in S$ and $t \in \mathbb{R}$ such that $0 \leq t \leq 1$, $tx + (1 - t)y \in S$.

Theorem (10): Let $\mathcal{H}$ be a Hilbert space and suppose $K \neq \emptyset$ is a closed, convex subset of $\mathcal{H}$. Given $h \in \mathcal{H}$, there is a unique $k_0 \in K$ such that

$$\|h - k_0\| = \operatorname{dist}(h, K) \equiv \inf\{\|h - k\| : k \in K\}.$$

Definition (11): Let $\mathcal{H}$ be a Hilbert space and $M \subseteq \mathcal{H}$. Then for $x \in \mathcal{H}$, we say $x \perp M$ if $\langle x, y \rangle = 0$ for all $y \in M$.

Theorem (11): Let $\mathcal{H}$ be a Hilbert space and $M \subseteq \mathcal{H}$ a closed subspace. For $h \in \mathcal{H}$ let $x \in M$ be the unique point of M such that $\|h - x\| = \text{dist}(h, M)$. Then $(h - x) \perp M$. Conversely, if $x \in M$ and $(h - x) \perp M$, then $\text{dist}(h, M) = \|h - x\|$.

Theorem (12): (Müntz-Szats) If $0 \leq \lambda_1 \leq \lambda_2 \leq ....$ Then,

1. $\sum \frac{1}{\lambda_j} = \infty$ implies $\overline{\text{span}} \left\{ t^{\lambda_j} \right\} = C_{\mathbb{R}}[a, b]$.
2. $\sum \frac{1}{\lambda_j} < \infty$ implies $t^{\lambda} \notin \overline{\text{span}}\{t^{\lambda_j}\}$ if $\lambda = 0$ and $\lambda \notin \{\lambda_j\}$.

Notation: If $\mathcal{H}$ is a Hilbert space, $A \subseteq \mathcal{H}$, then

$$A^{\perp} := \{h \in \mathcal{H} : \langle a, h \rangle = 0, \text{ for all } a \in A\}.$$

Fact: $A^{\perp}$ is a closed, linear subspace of $\mathcal{H}$.

Definition (13): If M is subspace of $\mathcal{H}$ and $h \in \mathcal{H}$, write $\underline{P_M h}$ for the unique element of M closest to h. Thus, $P_M h \in M$, $\|h - P_m h\| = d(h, M)$, and $h - P_M h \in M^{\perp}$. P_M is called the $\underline{\text{projection of } h \text{ onto } M}$.

Facts:

1. P_M is a linear transformation $P_M : \mathcal{H} \to M \subseteq \mathcal{H}$.
2. For all $h \in \mathcal{H}$, $\|P_M h\| \leq \|h\|$.
3. $P_M(P_M h) = P_M h$.
4. $\ker P_M = M^{\perp}$, $R(P) = M$.

Remarks:

1. If $A \subseteq \mathcal{H}$, then $A \subseteq \left(A^{\perp}\right)^{\perp}$.
2. If $M \subseteq \mathcal{H}$ is a closed subspace, then $\left(M^{\perp}\right)^{\perp} = M$.
3. If $A, B \subseteq \mathcal{H}$ and $A \subseteq B$, then $B^{\perp} \subseteq A^{\perp}$.
4. If $W \subseteq \mathcal{H}$ is a sunspace, then $\overline{W} = \left(W^{\perp}\right)^{\perp}$.

Theorem (14): The following are equivalent:

1. L is continuous.
2. L is continuous at 0.

3. L is continuous at some $z \in X$.

4. There exists $c > 0$ such that $\|Lx\| \leq c\|x\|$, for all $x \in X$.

Definition (15): If L satisfies one of the above, we say L is a bounded linear transformation. When this holds, we write $\|L\| = \inf\{c > 0 : \|Lx\| \leq c\|x\|, \text{ for all } x \in X\}$.

Facts:

1. $\|Lx\| \leq \|L\|\,\|x\|$, for all $x \in X$.

2. $\|L\| = \sup\{\|Lx\| : x \in X, \|x\| \leq 1\}$.

3. $\|L\| = \sup\{\|Lx\| : x \in X, \|x\| = 1\}$.

4. $\|L\| = \sup\left\{\left\|L\frac{x}{\|x\|}\right\| : x \in X, x \neq 0\right\}$.

Theorem (17): (Riesz Representation Theorem) If $\mathcal{H}$ is a Hilbert space and $L : \mathcal{H} \to \mathbb{F}$ is a bounded linear functional, then there exists a unique $y \in \mathcal{H}$ such that $L(x) = \langle x, y\rangle$, for all $x \in \mathcal{H}$. Moreover, $\|L\| = \|y\|$.

Definition (18): A subset $\mathcal{E} \subset \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, is an orthonormal set if given $e_1, e_2\mathcal{E}$, we have

$$\langle e_1, e_2\rangle = \begin{cases} 0, & \text{if } e_1 \neq e_2, \\ 1, & \text{if } e_1 = e_2. \end{cases}$$

Definition (19): An orthonormal basis for $\mathcal{H}$, a Hilbert space, is a maximal orthonormal set.

Fact: Let $\mathcal{E}$ be an orthonormal set in the Hilbert space $\mathcal{H}$. Then there exists a basis B for $\mathcal{H}$ such that $\mathcal{E} \subseteq B$.

Lemma (20): Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{M}$ be a finite dimensional subspace with an orthonormal basis $\{e_1, ..., e_n\}$ for $\mathcal{M}$. Then $\mathcal{M}$ is a closed subspace and for all $x \in \mathcal{H}$,

$$P_{\mathcal{M}}x = \sum_{j=1}^{n} \langle x, e_j\rangle\, e_j.$$

Theorem (21): (Modern Bessel's Inequality) Let $\mathcal{H}$ be a Hilbert space and suppose $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal set. For $\mathfrak{h} \in \mathcal{H}$,

$$\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \|h\|^2.$$

Corollary (22): If $\mathcal{E}$ is any orthonormal set in the Hilbert space $\mathcal{H}$ and $h \in \mathcal{H}$, then $\mathcal{C} = \{e \in \mathcal{E} : \langle h, e \rangle \neq 0\}$ is at most countable.

Definition (23): Let $\mathcal{E}$ be any set such that $\mathcal{E} \ni e \mapsto a_e \in X$, where X is a complete normed vector space. We say $\underline{\sum_{e\in\mathcal{E}} a_e = b \in X}$ if given $\epsilon > 0$ there exists a finite set $\mathscr{F}_0 \subseteq \mathcal{E}$ such that whenever $\mathscr{F}$ is a finite subset of $\mathcal{E}$ with $\mathscr{F}_0 \subseteq \mathscr{F}$, we have

$$\left\| \sum_{e\in\mathcal{F}} a_e - b \right\| < \epsilon.$$

Definition (24): A normed vector space X with induced metric which is complete is called a <u>Banach space</u>.

Fact: Let $\mathcal{H}$ be a Hilbert space. Suppose $\mathcal{E} \subseteq \mathcal{H}$ is an orthonormal set. Then for all $h \in \mathcal{H}$, $\sum_{e\in\mathcal{E}} \langle h, e \rangle e$ converges in $\mathcal{H}$.

Theorem (25): (Main Theorem on Orthonormal Sets) Let $\mathcal{E}$ be an orthonormal set in $\mathcal{H}$, a Hilbert space. Then, the following are equivalent:

1. $\mathcal{E}$ is a basis.
2. If $h \in \mathcal{H}$ and $h \perp \mathcal{E}$, then $h = 0$.
3. $\overline{\text{span}}(\mathcal{E}) = \mathcal{H}$.
4. If $h \in \mathcal{H}$, then $h = \sum_{e\in\mathcal{E}} \langle h, e \rangle e$.
5. If $g, h \in \mathcal{H}$, then $\langle g, h \rangle = \sum_{e\in\mathcal{E}} \langle g, e \rangle \langle e, h \rangle$.
6. For all $h \in \mathcal{H}$, $\|h\|^2 = \sum_{e\in\mathcal{E}} |\langle h, e \rangle|^2$.

Proposition (26): Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{E}, \mathcal{F}$ are two bases for $\mathcal{H}$, then $\text{card}(\mathcal{E}) = \text{card}(\mathcal{F})$.

Definition (27): For a Hilbert space $\underline{\dim(\mathcal{H})}= \text{card}(\mathcal{E})$ where $\mathcal{E}$ is a basis of $\mathcal{H}$.

Proposition (28): $\mathcal{H}$ is a separable Hilbert space (countable dense subset) if and only if $\dim(\mathcal{H}) \leq \text{card}(\mathbb{N})$.

1.1 Isomorphisms

Definition (29): Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A linear map $U : \mathcal{H} \to \mathcal{K}$ is an isometry if $\|Uh\| = \|h\|$, for all $h \in \mathcal{H}$. (Preserves Distance)

Fact: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $U : \mathcal{H} \to \mathcal{K}$ is linear. Then U is an isometry if and only if for all $h \in \mathcal{H}$, $\langle Uh_1, Uh_2 \rangle_{\mathcal{K}} = \langle h_1, h_2 \rangle_{\mathcal{H}}$.

Definition (30): An <u>isomorphism</u> between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is an isometry which is onto. $\mathcal{H}$ and $\mathcal{K}$ are <u>isomorphic</u> if there exists such an isometry. U is often called a <u>unitary map</u>.
Theorem (31): Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then $\mathcal{H}$ is isomorphic to $\mathcal{K}$ if and only if $\dim(\mathcal{H}) = \dim(\mathcal{K})$.
Corollary (32): If $\mathcal{H}$ is infinite dimensional and seprarable and $\mathcal{K}$ is infinite dimensional and separable, then $\mathcal{H}$ is isomorphic to $\mathcal{K}$. In particular $L^2([-1,1])$, $L^2(\mathbb{R})$, $\ell^2(\mathbb{N})$, and $L^2_a(\mathcal{D})$ are all isomorphic to each other.

1.2 Direct Sum of Hilbert Spaces

Definition (33): If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $\mathcal{H} \oplus \mathcal{K} := \{h \oplus k : h \in \mathcal{H}, k \in \mathcal{K}\}$ and

$$\langle h_1 \oplus k_1, h_2 \oplus k_2 \rangle \equiv \langle h_1, h_2 \rangle + \langle k_1, k_2 \rangle$$

is the <u>Direct Sum of $\mathcal{H}$ and $\mathcal{K}$</u> and is a Hilbert Space.
Proposition (34): If $\mathcal{H}_1, \mathcal{H}_2 ...$ are Hilbert spaces, let

$$\mathcal{H} := \{(h_n)_{n=1}^{\infty} : h_n \in \mathcal{H}_n \text{ for all } n \text{ and } \sum_{n=1}^{\infty} \|h_n\|^2 < \infty\}.$$

For $h = (h_n)$ and $g = (g_n)$ in $\mathcal{H}$, define

$$\langle h, g \rangle = \sum_{n=1}^{\infty} \langle h_n, g_n \rangle .$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{H}$ and the norm relative to this inner product is $\|h\| = \left(\sum_{n=1}^{\infty} \|h_n\|^2\right)^{1/2}$. With this inner product $\mathcal{H}$ is a Hilbert space.
Definition (35): If $\mathcal{H}_1, \mathcal{H}_2, ...$ are Hilbert spaces, the space $\mathcal{H}$ of the above proposition is called the <u>direct sum of $\mathcal{H}_1, \mathcal{H}_2 ...$</u> and is denoted by $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots$.

1.3 Operators on Hilbert Spaces

Proposition (36):

1. If $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $A + B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and $\|A + B\| \leq \|A\| + \|B\|$.
2. If $\alpha \in \mathbb{F}$ and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $\alpha A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\|\alpha A\| = |\alpha| \, \|A\|$.
3. If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, then $BA \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ and $\|BA\| \leq \|B\| \, \|A\|$.

Theorem (37): Let (X, Ω, μ) be a σ-finite measure space and put $\mathcal{H} = L^2(X, \Omega, \mu)$. If $\phi \in L^\infty(\mu)$, define $M_\phi : L^2(\mu) \to L^2(\mu)$ by $M_\phi f = \phi f$. Then $M_\phi \in \mathcal{B}(L^2(\mu))$ and $\|M_\phi\| = \|\phi\|_{L^\infty}$.

Definition (38): If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, a function $u : \mathcal{H} \times \mathcal{K} \to \mathbb{F}$ is a sesquilinear form if for $h, g \in \mathcal{H}$ and $k, f \in \mathcal{K}$, and $\alpha, \beta \in \mathbb{F}$,

1. $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$,
2. $u(h, \alpha k + \beta f) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, f)$.

Theorem (39): If $u : \mathcal{H} \times \mathcal{K} \to \mathbb{F}$ is a bounded sesquilinear form with bound M, then there are unique operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$u(h, k) = \langle Ah, k \rangle = \langle h, Bk \rangle$$

for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$ and $\|A\|, \|B\| \leq M$.

Definition (40): If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the unique operator B is $\mathcal{B}(\mathcal{H}, \mathcal{K})$ satisfying the theorem above is called the adjoint of A and is denoted by $B = A^*$.

Proposition (41): If $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then U is an isomorphism if and only if U is invertible and $U^{-1} = U^*$.

Proposition (42): If $A, B \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{F}$, then

1. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$
2. $(AB)^* = B^* A^*$
3. $A^{**} = (A^*)^* = A$
4. If A is invertible in $\mathcal{B}(\mathcal{H})$ and A^{-1} is its inverse, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proposition (43): If $A \in \mathcal{B}(\mathcal{H})$, then $\|A\| = \|A^*\| = \|A^* A\|^{1/2}$.

Proposition (44): If $S : \ell^2 \to \ell^2$ is defined by $S(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...)$, then S is an isometry and $S^*(\alpha_1, \alpha_2 ...) = (\alpha_2, \alpha_3 ...)$.

Definition (45): Let $A \in \mathcal{B}(\mathcal{H})$.

1. A is called hermitian or self-adjoint if $A = A^*$.
2. A is called normal if $AA^* = A^* A$.

Example (46): $M_\varphi^* = M_{\overline{\varphi}}$. M_φ are normal operators. M_φ are self-adjoint operators if and only if $\varphi(x) = 0$ a.e. $x \in X$.

Definition (47): For $T \in \mathcal{B}(\mathcal{H})$, write

$$T = \frac{T+T^*}{2} + i\left(\frac{T+T^*}{2i}\right)$$

real and imaginary decomposition of T.

Proposition (48): An operator is normal if and only if $\mathrm{Re}(T)\mathrm{Im}(T) = \mathrm{Im}(T)\mathrm{Re}(T)$.

Proposition (49): Suppose $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$. For $A \in \mathcal{B}(\mathcal{H})$, $A = A^*$ is and only if $\langle Ax, x\rangle \in \mathbb{R}$, for all $x \in \mathcal{H}$.

Theorem (50): Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$ and $A \in \mathcal{B}(\mathcal{H})$. If $A = A^*$, then $\|A\| = \sup\{|\langle Ax, x\rangle| : \|x\| \leq 1\}$.

Corollary (51): If $A = A^*$, then $\langle Ah, h\rangle = 0$ for all $h \in \mathcal{H}$, implies $A = 0$. When $\mathcal{H}$ is a complex Hilbert space, then for $A \in \mathcal{B}(\mathcal{H})$, $\langle Ah, h\rangle = 0$ for all $h \in \mathcal{H}$ implies $A = 0$.

Theorem (52): For a Hilbert space $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, A is a normal operator if and only if $\|Ah\| = \|A^*h\|$, for all $h \in \mathcal{H}$.

Theorem (53): Let $A \in \mathcal{B}(\mathcal{H})$. The following are equivalent:

1. A is an isometry.
2. $A^*A = I_{\mathcal{H}}$.
3. $\langle Ah, Ay\rangle = \langle x, y\rangle$.

Theorem (54): Let $A \in \mathcal{B}(\mathcal{H})$. The following are equivalent:

1. $A^*A = AA^* = I_{\mathcal{H}}$
2. A is a unitary operator.
3. A is a normal isometry.

Theorem (55): For any $A \in \mathcal{B}(\mathcal{H})$, $\ker A = (\mathrm{Range} A^*)^{\perp}$.

Definition (56): An operator $E \in \mathcal{B}(\mathcal{H})$ is an idempotent if $E^2 = E$.

Proposition (57): Let $E \in \mathcal{B}(\mathcal{H})$.

1. E is an idempotent if and only if $I - E$ is an idempotent.
2. $\mathrm{Range}(E) = \ker(I-E), \mathrm{Range}(I-E) = \ker(E)$, and both $\mathrm{Range}(E)$ and $\ker(E)$ are both closed linear subspaces of $\mathcal{H}$.

Proposition (58): If E is a nonzero idempotent, the following are equivalent:

1. $\ker(E) = \text{Range}(E)^{\perp}$
2. $E = P_{\mathcal{M}}$, where $\mathcal{M} = \text{Range}(E)$
3. $E = E^*$
4. E is normal.
5. $\langle Ex, x\rangle \geq 0,$ for all $x \in \mathcal{H}$.
6. $\|E\| = 1$.

Notation: We showed that $\mathcal{M} \oplus \mathcal{M}^{\perp} \cong \mathcal{H}$ via $(m, n) \mapsto m + n$. More generally if $(\mathcal{M}_i)_{i\in I}$ is a family of closed, pairwise orthogonal subspaces, we write $\oplus_{i\in I}\mathcal{M}_i := \vee_{i\in I}\mathcal{M}_i$. Also, we define $\mathcal{M} \ominus \mathcal{N} := \mathcal{M} \cap \mathcal{N}^{\perp}$. Given $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ a closed subspace, we can form a "matrix" decomposition for T relative to the decomposition of $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$:

$$T = P_{\mathcal{M}}TP_{\mathcal{M}} + P_{\mathcal{M}}TP_{\mathcal{M}}^{\perp} + P_{\mathcal{M}}^{\perp}TP_{\mathcal{M}} + P_{\mathcal{M}}^{\perp}TP_{\mathcal{M}}^{\perp}$$

or

$$T = \begin{pmatrix} P_{\mathcal{M}}TP_{\mathcal{M}} & P_{\mathcal{M}}TP_{\mathcal{M}}^{\perp} \\ P_{\mathcal{M}}^{\perp}TP_{\mathcal{M}} & P_{\mathcal{M}}^{\perp}TP_{\mathcal{M}}^{\perp} \end{pmatrix}$$

Definition (59): Given $T \in \mathcal{B}(\mathcal{H})$. A closed subspace $\mathcal{M}$ is invariant for T if $T\mathcal{M} \subseteq \mathcal{M}$. $\mathcal{M}$ is reducing for T if $\mathcal{M}, \mathcal{M}^{\perp}$ are invariant for T.
Remark: In matrix notation: invariant ⇒ upper triangular. reducing ⇒ both corners=0.
Theorem (60): Let $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ a subspace of $\mathcal{H}$. The following are equivalent:

1. $\mathcal{M}$ is invariant for T.
2. $TP_{\mathcal{M}} = P_{\mathcal{M}}TP_{\mathcal{M}}$
3. $P_{\mathcal{M}}^{\perp}TP_{\mathcal{M}} = 0$.

Theorem (61): Let $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{M}$ a subspace of $\mathcal{H}$. The following are equivalent:

1. $\mathcal{M}$ is reducing on T.
2. $P_{\mathcal{M}}T = TP_{\mathcal{M}}$
3. $\mathcal{M}$ is invariant under T and T^*.

Definition (62): Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $T : \mathcal{H} \to \mathcal{K}$ be a linear operator. Call T a compact operator if $\{Tx : x \in \mathcal{H}, \|x\| \leq 1\}$ us precompact in $\mathcal{K}$.
Notation: $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.
Proposition (63): The following hold:

1. $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2) \subseteq \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$
2. $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ is a closed linear subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$
3. If $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_3)$, and $B \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)$ then $AT \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_3)$ and $TB \in \mathcal{K}(\mathcal{H}_0, \mathcal{H}_2)$.

Definition (64): An operator has finite rank if Range(T) is finite dimensional. (Note that finite rank operators are compact.)
Definition (65): The dimension of Range(T) is the rank of T.
Theorem (65): For $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, T is compact if and only if T is a norm limit of finite rank operators.
Remark: If T is compact, so is T^*.
Remark: If T is diagonal and $\mathcal{H} = \ell^2(\mathbb{N})$. T is compact if and only if $\lim_{n\to\infty} a_n = 0$.
Definition (66): For $T \in \mathcal{B}(\mathcal{H})$, $\lambda \in \mathbb{F}$ is an eigenvalue if $\mathbf{ker}(T - \lambda I) \neq (0)$. x is an eigenvector for T corresponding to λ if $x \neq 0$ and $x \in \ker(T - \lambda I)$.
Definition (67): For $T \in \mathcal{B}(\mathcal{H})$, $\sigma_{\text{pt}} = \{\lambda \in \mathbb{F} : \ker(T - \lambda I) \neq 0\}$. This is called the point spectrum and is the collection of eigenvalues. This is the set where $T - \lambda I$ is not 1-1.
Definition (68): For $T \in \mathcal{B}(\mathcal{H})$, the spectrum of T is $\sigma(T) = \{\lambda \in \mathbb{F} : T_\lambda I \text{ is not invertible}\}$.
Proposition (69): Suppose that $T \in \mathcal{K}(\mathcal{H})$ and $\lambda \neq 0$. Then $\mathbf{ker}(T - \lambda I)$ is finite dimensional.
Definition (70): Let $T \in \mathcal{B}(\mathcal{H})$, $\lambda \in \mathbb{F}$ is an approximate eigenvalue for T if there exists a sequence of unit vectors $x_n \in \mathcal{H}$ such that $T - \lambda I)x_n \to 0$.
Definition (71): T is bounded below if there exists $c > 0$ such that $\|Tx\| \geq c\,\|x\|$, for all $x \in \mathcal{H}$. (Notice that bounded below implies one-to-one. Also note if T is bounded below, then Range(T) is closed.
Note: The collection of approximate eigenvalues for $T \in \mathcal{B}(\mathcal{H})$ is $\{\lambda \in \mathbb{F} : T - \lambda I \text{ isn't bounded below}\}$.
Theorem (72): Suppose T is compact and $\lambda \neq 0$ is an approximate eigenvalue for T. Then λ is an eigenvalue for T.
Proposition (73): Suppose T is compact, $\lambda \neq 0$, $\lambda \notin \sigma_{\text{pt}}(T)$, $\overline{\lambda} \notin \sigma_{\text{pt}}(T^*)$. Then Range$(T - \lambda I) = \mathcal{H}$, $(T - \lambda I)^{-1}$ exists, and $(T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$.
Proposition (74): Suppose that $T \in \mathcal{B}(\mathcal{H})$ is normal. If $\lambda \in \mathbb{F}$, then $\ker(T - \lambda I) = \ker\left((T - \lambda I)^*\right)$ is a reducing subspace.

Proposition (75): If $T \in \mathcal{B}(\mathcal{H})$ is normal and $\lambda \neq \mu$ are distinct eigenvalues for T, then $\ker(T - \lambda I) \perp \ker(T - \mu I)$.

Proposition (76): Let $T = T^* \in \mathcal{B}(\mathcal{H})$. If $\lambda \in \sigma_{\text{pt}}(T)$, then $\lambda \in \mathbb{R}$.

Lemma (77): Suppose $T \in \mathcal{B}(\mathcal{H})$ is compact and $T = T^*$. Then $\sigma_{textpt}(T) \cap \{-\|T\|, \|T\|\} \neq \emptyset$.

Theorem (78): (Spectral Theorem for Compact Self-Adjoint Operators) Let $T \in \mathcal{B}(\mathcal{H})$ be compact and self-adjoint. Then T has at most countably many distinct eigenvalues, and if $\{\lambda_1, \lambda_2, ...\}$ are the non-zero eigenvalues for T and $P_n =$ Proj. onto $\ker(T - \lambda_n I)$, we have:

1. $P_n P_m = P_m P_n = 0$, if $n \neq m$
2. $\lambda_n \in \mathbb{R}$, for all $n \in \mathbb{N}$
3. $T = \sum_{\lambda \in \{\lambda_1, \lambda_2 ...\}} \lambda_n P_n$, where the sum converges in norm.

Theorem (79): Define $L : \mathscr{D} \to L^2([a,b])$ by $Lh = -h' + qh$. If L is 1-to-1, then

1. There exists a basis $e_1, e_2, ...$ for $L^2([a,b])$ and numbers $0 < |\lambda_1| \leq |\lambda_2| \leq ... \leq |\lambda| \to \infty$ such that $e_n \in \mathscr{D}$ and $Le_n = \lambda_n e_n$.
2. If $\lambda \in \mathbb{C}$ and $\lambda \neq \lambda_n$, for any n, then there exists a unique $h \in \mathscr{D}$ with $(L - \lambda)h = f$. (for any $f \in L^2([a,b])$)
3. If $\lambda = \lambda_n$ for some n and $f \in L^2([a,b])$, there exists $h \in \mathscr{D}$ with $Lh - \lambda h = f$ if and only if $\langle f, e_n \rangle = 0$. When this is the case, any two such h differ by a multiple of e_n $(h_1 - h_2 = ce_n)$.

Remark: Remember that $\mathscr{D}$ is simply the set of all functions in $L^2([a,b])$ that satisfy both boundary conditions.

2 Banach Spaces

Definition (80): Let X be a vector space over $\mathbb{F}$. Then, $\rho : X \to [0, \infty)$ is a <u>semi-norm</u> if

1. $\rho(\lambda x) = |\lambda| \rho(x), \lambda \in \mathbb{F}, x \in X$
2. $\rho(x + y) \leq \rho(x) + \rho(y), x, y \in X$.

Furthermore, ρ is a <u>norm</u> if in addtion,

3 $\rho(x) = 0$ implies $x = 0$.

We write $\|x\|$ instead of $\rho(x)$. $(X, \|\cdot\|)$ is a semi-norm space (or normal space). So we get a (psuedo)-metric: $d(x,y) = \|x-y\|$.

Definition (81): Given a normed space X, we say X is a Banach Space if X is complete with respect to d, given above.

Fact: For any normed space X, $+ : X \times X \to X$ and $\cdot : \mathbb{F} \times X \to X$ are continuous.

Definition (82): Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X. We say $\|\cdot\|_1 \sim \|\cdot\|_2$ if the open sets for d_1 and the open sets for d_2 are the same, i.e. they have the same topology.

Fact: $\|\cdot\|_1 \sim \|\cdot\|_2$ if and only if there exists $c, C > 0$ such that

$$c\,\|x\|_1 \leq \|x\|_2 \leq C\,\|x\|_1\,, \text{ for all } x \in X.$$

Fact: If X is a Banach space and $\mathcal{M} \subseteq X$ is a closed subspace, then $\mathcal{M}$ is a Banach space under the restriction of the norm on X to $\mathcal{M}$.

Example (83): Banach Spaces:

1. Hilbert Spaces
2. If $1 \leq p \leq \infty$ and (X, μ) is a measure space, then $L^p(X, \mu)$ is a Banach space by the Riesz-Fisher Theorem.
3. Let X be a Hausdorff Topological Space. Let $C_b(X)$ be the set of all $\mathbb{F}$-valued continuous, bounded functions on X. Let $\|f\| = \sup_{x \in X}\{|f(x)|\}$. Then, $(C_b(X), \|\cdot\|)$ is a Banach space.
4. Consider $C_b(X)$ as in example (3). Define $C_0(X)$ to be the set of all functions in $C_b(X)$ such that given $\epsilon > 0$ there exists a compact set $K \subseteq X$ such that for all $x \in X \backslash K$, $|f(x)| < \epsilon$. This set with the norm from example (3) is a Banach Space. (Use fact before this theorem to prove.)
5. If X, Y are Banach spaces (even X just normed), $\mathcal{B}(X, Y)$ is a Banach space under the operator norm.
6. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathbb{A}(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : f$ is analytic in $\mathbb{D}$ and f extends to a cnts. func in This is the Disk Algebra. Also define $\|f\| := \sup_{|z|=1} |f(z)|$. This is a Banach space.
7. Let $\mathcal{C}^n[a,b] = f : [a,b] \to \mathbb{F} : f^{(n)}$ exists and continuous on $[a,b]\}$. Define

$$\|f\| = \max\{\|f\|_\infty, \left\|f^(1)\right\|_\infty, ..., \left\|f^{(n)}\right\|_\infty\}.$$

Example (84):

1. Let $X = \mathcal{C}^1[a,b]$ and $Y = \mathcal{C}[a,b]$. Define $D : X \to Y$ by $Df = f'$. Then, $\|Df\| = \|f'\|_\infty \leq \|f\|$, by previous definition of $\|\cdot\|_\infty$. So $D \in \mathcal{B}(X,Y)$.
2. Let X and Y be topological spaces. Suppose $\tau : X \to Y$ is continuous. Define $T : \mathcal{C}_b(Y) \to \mathcal{C}_b(X)$ by $Tf = f \circ \tau$. Note that
$$\|Tf\|_{\mathcal{C}(X)} = \sup_{x \in X} |f(\tau(x))| \leq \sup_{y \in Y} |f(y)| = \|f\|_{\mathcal{C}(Y)} .$$
Therefore, $T \in \mathcal{B}(\mathcal{C}_b(Y), \mathcal{C}_b(X))$.
3. Let (X, Ω, μ) be a measure space. Fir $f \in L^\infty$, $1 \leq \leq \infty$, $M_f : L^p \to L^p$ defined by $(M_f g)(x) = f(x)g(x)$ is a bounded operator, with $\|M_f\| \leq \|f\|_\infty$. We have equality if X is semifinite.

Proposition (85): Let X be a finite dimensional vector space. Any two norms on X are equivalent.
Definition (86): Two Banach spaces are isomorphic (isometrically isomorphic) if there exists a bijective linear map $T : X \to Y$ such that for all $x \in X$, $\|Tx\| = \|x\|$.
Example (87): If X is a finite dimensional space, then $(X, \|\cdot\|_\infty)$ and $(\mathbb{F}, \|\cdot\|_\infty)$ are isometrically isomorphic.
Example (88): Let $c = \{x_n : x_n \in \mathbb{F}, \lim_{n\to\infty} x_n \text{exists}\}$. Let $\|(x_n)\| = \sup_n |x_n|$. Let $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Then c and $\mathcal{C}(X)$ are isometrically isomorphic via $\Phi(f)(n) = f\left(\frac{1}{n}\right)$, where $\Phi : \mathcal{C}(X) \to c$.
Corollary (89): Suppose C is a Banach space and M is a finite dimensional subspace of X. Then, M is closed.
Corollary (90): If M is finite dimensional, Y is normed, and $T : M \to Y$ is linear, then T is bounded.

2.1 Subspaces

Theorem (91): Let X be a normed space, $M \subseteq X$ be closed, and $\|x + M\| := \text{dist}\,(x, M)$. This gives a norm on X/M.
Theorem (91): Let X be a normed space and $M \subseteq X$ be closed subspace. Let $Q : X \to X/M$ be the quotient map. Then,

1. $\|Qx\| \leq \|x\|$, for all $x \in X$
2. If X is a Banach space, X/M is a Banach space.
3. $G \subseteq X/M$ is open if and only if $Q^{-1}(G)$ is open. (Is quotient topology.)

4. If $H \subseteq X$ is open set, so is $Q(H)$.

Corollary (92): Suppose X is a normed space, $M \subseteq X$ is a closed subspace, and $N \subseteq X$ is a finite dimensional subspace. Then $M + N$ is a closed subspace of X.

Proposition (93): Let f be a linear functional on the normed space X. Then f is continuous if and only if $\ker(f)$ is closed.

Fact: If X is normed and $f : X \to \mathbb{F}$ is a linear functional, then either $\ker(f)$ is closed or $\ker(f)$ is dense in X.

Definition (94): Let X^* be the set of all bounded linear functionals, $X^* = \mathcal{B}(X, \mathbb{F})$. X^* is called the dual of X.

Fact: X^* is a Banach space, even if X is just a normed space.

Example (95): Let $1 < p < \infty$ and (X, μ) is a measure space. (Can include $p = 1$ is (X, μ) is σ-finite. Then $L^p(X, \mu)^*$ can be identified as $L^q(X, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Example (96): Suppose X is a locally compact Hausdorff space. Let $M(X)$ be the set of all $\mathbb{F}$-valued regular Borel measures. For $\mu \in M(X)$, define $\|\mu\| = |\mu|(x)$. Then, $\mathcal{C}_0(X)^*$ is isometrically isomorphic to $M(X)$ via $\varphi_\mu(f) = \int_X f \, d\mu$, where $\mu \in M(X)$ and $f \in \mathcal{C}_0(X)$.

Fact: X can always be embedded in $X^* *$.

Definition (97): Given a real vector space X, a gauge is a functional $p : X \to \mathbb{R}$ satisfying:

1. $p(x + y) \leq p(x) + p(y), \text{ for all } x, y \in X,$
2. $p(tx) = tp(x), \text{ for all } x \in X, t \geq 0.$

Theorem (98): (**Hahn-Banach Theorem**) Let X be a real vector space, $M \subseteq X$ a subspace, and p a gauge. If $f : M \to \mathbb{R}$ is a linear functional such that $f(x) \leq p(x)$ for all $x \in M$, then there exists a linear functional $F : X \to \mathbb{R}$ such that $F(x) \leq p(x)$ for all $x \in X$, and $F|_M = f$.

Fact: There exists a one-to-one correspondence between linear functionals on $X^{\mathbb{C}}$ and linear functionals on $X^{\mathbb{R}}$. These formulas show that if X has a topology, then the correspondence preserves continuity, i.e. f is continuous if and only if Re f is continuous.

Corollary (99): Suppose X is a vector space over $\mathbb{F}$ and $p : X \to \mathbb{R}$ is a semi-norm. Let $M \subseteq X$ be a subspace and $f : M \to \mathbb{F}$ is linear such that $|f(x)| \leq p(x), \text{ for all } x \in M$. Then, there exists a linear functional $F : X \to \mathbb{R}$ such that $|F(x)| \leq p(x), \text{ for all } x \in X$ and $F|_M = f$.

Corollary (100): Suppose X is a normed space and $x_0 \in X$. Then there exists $f \in X^*$ such that $f(x_0) = \|x_0\|$ and $|f(y)| \leq \|y\|$, for all $y \in X$.

Corollary (101): Assume X is a normed space, $M \subseteq X$ is a subspace, and $f \in M^*$. Then there exists $F \in X^*$ such that $F|_M = f$ and $\|F\| = \|f\|$.

Corollary (102): Suppose X is a normed space and $\{x_1, ..., x_n\} \subseteq X$ is a linearly independent subset. Suppose $\{c_1, ..., c_n\} \subseteq \mathbb{F}$. Then there exists $f \in X^*$ such that $f(x_j) = c_j$, for $1 \leq j \leq n$.

Corollary (103): Suppose X is a normed space, $M \subseteq X$ a closed subspace, and $x_0 \in X/M$. Let $d = \text{dist}\,(x_o, M)$. Then there exists $f \in X^*$ such that $f(x_0) = 1$ and $f|_M \equiv 0$. Moreover, we can choose f such that $\|f\| = \frac{1}{d}$.

Theorem (104): There exists $\Lambda \in (\ell^\infty(\mathbb{R}))^*$ such that

1. $\Lambda(Lx) = \Lambda(x), \text{ for all } x \in \ell^\infty(\mathbb{R})$.
2. $\liminf x_n \leq \Lambda(x) \leq \limsup x_n$.
3. $\Lambda|_{C_0} = 0$
4. You can extend Λ to $\ell^\infty_{\mathbb{C}}$ by $\Lambda x := \Lambda(a) + i\Lambda(b)$.
5. Moreover if $x \in \ell^\infty_{\mathbb{C}}$ with $x_n \geq 0$ for all n, we have $\Lambda(x) \geq 0$. Also, $|\Lambda(x)| \leq \|x\|_\infty$. So $\|\Lambda\| = 1$.

Theorem (105): Let M be a subspace of a normed space X. Then

$$\overline{M} = \bigcap_{\substack{f \in X^* \\ M \subseteq \ker f}} \ker f$$

Corollary (106): If X is a normed space and $M \subseteq X$ is a subspace, then M is dense in X if and only if 0 is the only element of X^* which annihilates M.

2.2 Duals of Subspaces and Quotients

Definition (106): For X a normed vector space and $S \subseteq X$ with $S \neq \emptyset$, define

$$S^\perp = \{f \in X^* : f(s) = 0, \text{ for all } s \in S\}.$$

Fact: $S^\perp$ is a closed subspace of X^*.

Theorem (107): Let X be a normed space and $M \subseteq X$ be a closed subspace. Then

1. M^* is isometrically isomorphic to $X^*/M^\perp$ via the map $f + M^\perp \mapsto f|_M$, $f + M^\perp \in X^*/M^\perp$.
2. $(X/M)^*$ is isometrically isomorphic to $M^\perp$ via the map $g \in (X/M)^* \mapsto g \circ Q$, where Q is the natural map into the quotient space.

2.3 Reflexivity

Definition (108): Let J be the natural map from X into X^{**}, i.e. $(Jx)(\varphi) = \varphi(x)$. A Banach space is reflexive if J is onto, i.e. J is an isometric isometry. Note that X is not reflexive just because X is isomorphic to X^{**}. The natural map, J, must be an isomorphism.

Example (109):

1. If (X, Ω, μ) is σ-finite measure space. Then if $1 < p < \infty$, $L^p(X)$ is reflexive.
2. Hilbert Spaces are reflexive.
3. If X is a finite dimensional normed space. Then X is reflexive. (The proof is a dimension argument.)

Definition (110): If $T \in \mathcal{B}(X, Y)$, we define we define the <u>Banach space adjoint</u>, $T^* : Y^* \to X^*$, by $T^*\varphi = \varphi \circ T$.

Facts: The map $T \mapsto T^*$ (Banach space adjoint) has the following properties:

1. $(\lambda T)^* = \lambda T^*$
2. $(T + S)^* = T^* + S^*$
3. $(TS)^* = S^*T^*$

Theorem (111): (**Open Mapping Theorem**) Suppose X and Y are Banach spaces. Let $T : X \to Y$ be a bounded linear map. If T is onto, then T is an open map, i.e. if $G \subseteq X$ is an open set, $T(G)$ is open in Y.

Theorem (112): (**Baire Category Theorem**) If X is a locally compact Hausdorff space or X is a complete metric space, then X is 2^{nd} category.

Definition (113): A subset $V \subseteq X$ is <u>nowhere dense</u> if $(\overline{V})^0 = \emptyset$, where $(\overline{V})^0$ is the interior of the closure of V.

Definition (114): A set is 1^{st} category (Meager) if it is the countable union of nowhere dense sets. A set is 2^{nd} category if it is <u>not</u> 1^{st} category.

Theorem (115): If X is a normed space and X is locally compact, then X is finite dimensional.

Corollary (116): (Open Mapping Theorem) If $T : X \to Y$ is bounded, linear, and bijective, then $T^{-1} : Y \to X$ is bounded and linear.

Corollary (117): (Open Mapping Theorem) Let $T : X \to Y$ be a linear map. The graph of T is the set $\mathcal{G} := (T)\{(x, Tx) \in X \times Y : x \in X\}$. Put the product topology on $X \times Y$. To say $\mathcal{G}(T)$ is closed means that if $x_n \to x$ and $Tx_n \to y$, then $(x, y) \in \mathcal{G}(T)$, i.e. $Tx = y$.

Theorem (118): (**Closed Graph**) If X and Y are Banach spaces, $T : X \to Y$ is linear, and $\mathcal{G}(T)$ is closed in $X \times Y$. Then T is bounded.

Theorem (119): Let X, Y, and Z be Banach spaces. Suppose $\mathscr{F} \subseteq \mathcal{B}(X, Y)$ is total, i.e. if $Fx = 0$ for all $F \in \mathscr{F}$, then $x = 0$. Suppose $T : Z \to X$ is such that $F \circ T \in \mathcal{B}(Z, Y)$, for all $F \in \mathscr{F}$. Then $T \in \mathcal{B}(Z, X)$.

Fact: Let X and Y be Banach spaces. Then if this says that if $h : X \to Y$ satisfies $\varphi \circ h \in X^*$, for all $\varphi \in Y^*$, then $h \in \mathcal{B}(X, Y)$.

Definition (120): Let X be a Banach soace and $M \subseteq X$ a closed subspace. Call M complemented in X if there exists a closed subspace $N \subseteq X$ such that $M + N = X$ and $M \cap N = (0)$. When this occurs, we call N a complement of M.

Definition (121): Suppose M and N are complementary closed subspaces. Define $E_M : X \to X$ by $E_M(x) = m$, where $x = m+n$, $m \in M$, $n \in N$. Notice that $E_M \circ E_M = E_M$ and $I_X - E_M = E_N$.

Theorem (122): $E_M \in \mathcal{B}(X)$.

Theorem (123): If $E \in \mathcal{B}(X)$, $E^2 = E$, then $\ker(E)$ and $\text{Range}(E)$ are closed subspaces of X which are complementary and $E = E_{\text{Range}(E)}$.

Theorem (124): (Uniform Boundedness Principle) Suppose $\mathscr{F} \subseteq \mathcal{B}(X, Y)$, where X is a Banach spaces and Y normed space. If for every $u \in X$, $\sup_{T \in \mathscr{F}} \|Tu\| < \infty$, then $\sup_{T \in \mathscr{F}} \|T\| < \infty$.

Theorem (125): Suppose $\mathscr{F} \subseteq \mathcal{B}(X, Y)$ where X and Y are normed spaces. $\mathscr{F}$ is equicontinuous if and only if $\sup_{T \in \mathscr{F}} \|T\| < \infty$, i.e. uniformly bounded for $\mathscr{F}$ if and only if $\mathscr{F}$ is eqiucontinuous.

Corollary (126): Suppose X is a normed space and $A \subseteq X$. Then A is a bounded set if and only if for all $f \in X^*$, $\sup\{|f(a)| : a \in A\} < \infty$.

Corollary (127): Suppose X is a Banach space. Then $A \subseteq X^*$ is bounded if and only if for all $x \in X$, $\sup\{|f(x)| : f \in A\} < \infty$.

Theorem (128): Suppose X is a Banach space and Y is a normed space. Assume $A \subseteq \mathcal{B}(X, Y)$ is such that for all $x \in X$, $g \in Y^*$, $\sup\{|g(Tx)| : T \in A\} < \infty$. Then $\sup_{T \in A} \|T\| < \infty$.

Theorem (129): (Banach-Steinhaus) Suppose X and Y are Banach spaces and $\{A_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{B}(X, Y)$ such that $\lim_{n\to\infty} A_n x$ exists for all $x \in X$. Define $A : X \to Y$ by $Ax = \lim_{n\to\infty} A_n x$. Then A is a linear map, A is bounded, and $\sup \|A_n\| < \infty$.

Theorem (130): Let $\mathcal{H}$ be a Hilbert space and ξ be an orthonormal basis. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$. Then $\langle x_n, y_n \rangle \to 0$ for all $y \in \mathcal{H}$ if and only if $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$ and $\langle x_n, e \rangle \to 0$ for all $e \in \xi$.

Theorem (131): Let $\mathcal{H}$ be a Hilbert space, $(x_n) \subseteq \mathcal{H}$, and ξ be an orthonormal basis. Then (x_n) converges weakly in $\mathcal{H}$ to 0 if and only if $(x_n)_{n=1}^{\infty}$ is bounded and for all $e \in \xi$, $\langle x_n, e \rangle \to 0$.

Theorem (132): Let X be normed and consider the weak topology on X.

1. $+ : X \times X \to X$ is continuous.
2. $\cdot : \mathbb{F} \times X \to X$ is continuous.
3. If $\mathscr{F}$ is a family of semi-norms on X which is separating, then the topology from $\mathscr{F}$ is Hausdorff.

Definition (133): A nonempty set Λ is called a directed set if there is a relation $\leq$ on Λ such that

(a) $x \leq y \leq z$ implies $x \leq z$

(b) If $x_1, x_2 \in \Lambda$, there exists $x_3 \in \Lambda$ such that $x_1 \leq x_3$ and $x_2 \leq x_3$

(c) $x \leq x$ for all $x \in \Lambda$

Definition (134): Given a set X, a net in X is a function $(x_\lambda)_{\lambda \in \Lambda}$, i.e. a function $x : \Lambda \to X$ write x_λ instead of $x(\lambda)$.

Definition (135): Suppose (X, τ) is a topological space. We say that a net (x_λ) converges to $x \in X$ if whenever G is open with $x \in G$, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in G$ for all $\lambda \geq \lambda_0$. ((x_λ) is eventually in G.)

Fact: Let (X, τ) and (Y, σ) be topological spaces and $f : X \to Y$ be a function. Then f is continuous if and only if whenever (x_λ) is a convergent net in X, say $x_\lambda \to x$, we have $f(x_\lambda) \to f(x)$.

Definition (136): A topological vector space is a vector space over $\mathbb{F}$ together with a Hausdorff topology τ such that

(a) $+ : V \times V \to V$ is continuous

(b) $\cdot : \mathbb{F} \times V \to V$ is continuous

Fact: Let V be a topological vector space. Suppose $\rho : V \to \mathbb{R}$ is a semi-norm. Then ρ is a continuous on V if and only if ρ is continuous at 0.

Theorem (137): Suppose V is a vector space and $\mathscr{F}$ is a separating family of seminorms on V. Let τ be the smallest topology on V which makes each of the seminorms in $\mathscr{F}$ continuous. Then (V, τ) is a topological vector space and the sets $G_{\rho,\epsilon}(y)$ where

$$G_{\rho,\epsilon}(y) = \{x \in V : \rho(x - y) < \epsilon\}$$

is a subbase for τ.

Example (138): For X a Banach space, let $\mathscr{F} = \{|f(\cdot)| : f \in X^*\}$. This gives the $\sigma(X, X^*)$-topology.

Example (139): Suppose X is a Hausdorff space. Let V be a vector space of functions on X. For each subset $K \subseteq X$, with K compact, let $\rho_K(f) = \sup\limits_{x \in K} |f(x)|$. Then $\{\rho_K : K \subseteq X \text{ is compact}\}$. is a separating family. The topology, τ, arising from this family is called the topology of uniform convergence on compact set.

Example (140): Let $0 < p < 1$ and (X, Ω, μ) be a σ-finite measure space. Let

$$V = L^p(X\mu) = \{f : X \to \mathbb{F} : \int_\Omega |f|^p \, d\mu < \infty\}.$$

Define $\Delta(f) = \int_X |f|^p \, d\mu$. Under the metric $d(f, g) = \Delta(f - g)$, V is a topological vector space.

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3.1 Locally Convex Topological Vector Spaces

Definition (141): Let V be a topological vector space. A set $B \subseteq V$ such that $B \neq \emptyset$ is <u>balanced</u> if whenever $\alpha \in \mathbb{F}$, with $|\alpha| \leq 1$, $\alpha B \subseteq B$.

Proposition (142): Let V be a topological vector space.

(a) Every neighborhood of 0 contains a balanced neighborhood.

(b) Every convex neighborhood of 0 contains a convex, balanced neighborhood of 0.

Remark: The proof shows:

(a) If C is convex, so is C°, the interior of C.

(b) If C is balanced, then C° is balanced.

Definition (143): A topological vector space is locally convex if every neighborhood of 0 contains a convex neighborhood of 0.

Corollary (144):

(a) Every topological vector space has a balanced local base at 0.

(b) Every locally convex topological vector space has a balanced, convex local base at 0.

Definition (145): Let X be a vector space. A set $E \subseteq X$ is <u>absorbing</u> if for all $x \in X$, there is an $\epsilon > 0$ such that $tx \in E$ for all $0 \leq t < \epsilon$.

Proposition (145): If V is a neighborhood of 0 in a topological vector space and $0 < r_1 < r_2 < \ldots.$ such that $r_j \to \infty$, then $X = \bigcup\limits_{n=1}^{\infty} r_n V$.

Corollary (146): Any open neighborhood of 0 is absorbing.

Theorem (147): Let (V, τ) be a topological vector space. Then V is locally convex if and only if there exists a separating family $\mathcal{F}$ of semi-norms such that τ is the smallest topology on V making all semi-norms in $\mathcal{F}$ continuous.

Definition (148): Let X be a vector space and $E \subseteq X$ be convex and absorbing. Define the Minkowski gauge functional by

$$g_E(x) = \inf\{\alpha \in \mathbb{R} : \alpha > 0, \frac{1}{\alpha}x \in E\},$$

which exists as an honest to goodness number.

Facts:

(a) Observe $g_E(x) \geq 1$ if $x \notin E$ and $g_E(x) \leq 1$ if $x \in E$.

(b) If E is an open, convex, and absorbing set, then

$$E = \{x \in X : g_E(x) < 1\}$$

(c) g_E satisfies:

  (i) $g_E(x+y) \leq g_E(x) + g_E(y)$
  (ii) For $\alpha \geq 0$, $g_E(\alpha x) = \alpha g_E(x)$
  (iii) Moreove, if E is balanced, then g_E is a semi-norm on X.

Remark: If X is a locally convex topological vector space (LCTVS) and if E is a convex, balanced, open neighborhood of 0, then g_E is a continuous semi-norm. (Use this to prove Theorem 147.)

Theorem (149): Let X, Y be TVS's. Suppose $T : X \to Y$ is linear. If T is continuous at 0, then T is continuous. Moreover, if $W \subseteq Y$ is a neighborhood of 0, there exists a neighborhood $V \subseteq X$ of 0 such that $y - x \in V$ implies $Tx - Ty \in W$. (Statement about uniform continuity.)

Theorem (150): Suppose X is a TVS, $\varphi : X \to \mathbb{F}$, and $\varphi \neq 0$ is linear. TFAE:

1. φ is continuous at 0.
2. φ is continuous.
3. $\ker\varphi$ is closed.
4. $\ker\varphi$ is not dense in X.

5. There exists an open neighborhood V of 0 such that $\sup_{v\in V}|\varphi(v)| < \infty$ (i.e. φ is bounded on V).

Definition (151): X^* is the set of all continuous linear functionals on X.

Example (152):

1. Let $1 \le p < \infty$. Then, $(L^p)^* = L^q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.
2. Let $0 < p < 1$. Then $L^p([0,1])^* = \{0\}$.
3. Let $0 < p < 1$. Then $(\ell^p)^* = \ell^\infty$.

Theorem (153): Let X be a LCTVS. Then X^* separates points.

Definition (154): A metric d on a vector space is <u>invariant</u> if $d(x+z, y+z) = d(x,y)$, for all $x, y, z \in X$. Also, a TVS is <u>metrizable</u> if there exists a metric d such that $\tau = \tau_d$, where τ is the original topology.

Theorem (154): A LCTVS is metrizable if and only if there exists a countable family $\mathscr{F}$ of seminorms which give the topology. Moreover, when this occurs, the metric can be taken to be invariant.

Fact: A TVS is metrizable if and only if it has a countable local base (at 0).

Definition (155): Let X be a TVS. A set $E \subseteq X$ is <u>bounded</u> if for every open set $G \subseteq X$, there exists $s > 0$ such that $E \subseteq tG$ for all $t > s$.

Proposition (156): Let X be a LCTVS. Then X is normable if and only if X contains a bounded, non-empty, open set.

3.2 Test Functions and Distributions

Definition (157):

1. Let D^α have the usual meaning for a partial differential operator.
2. Define $C^\infty(\Omega) := \{f : \Omega \to \mathbb{F} : D^\alpha f \in C_{\mathbb{F}}(\Omega) \text{ for all } \alpha\}$.
3. Recall $\text{supp}(f) = \overline{f^{-1}(\mathbb{F}\backslash\{0\})}$.
4. For $K \subseteq \Omega$ with K compact, define

$$\mathscr{D}_K := \{f \in C^\infty(\Omega) : \text{supp } f \subseteq K\}.$$

5. The space of test functions, $\mathscr{D}(\Omega)$, is

$$\mathscr{D}(\Omega) = \bigcup_{\substack{K \text{ compact} \\ K \subseteq \Omega}} \mathscr{D}_K.$$

Fact: For $\varphi \in \mathscr{D}(\Omega)$ and $N \in \mathbb{N}$,

$$\|\varphi\|_N = \max\{|(D^\alpha \varphi)(x)| : x \in \Omega, |\alpha| \leq N\}$$

is a norm on $\mathscr{D}(\Omega)$.

Definition (158): For $K \subseteq \Omega$ with K compact, define τ_K to be the topology on $\mathscr{D}_K$ with local base $\{V_N\}_{N\in\mathbb{N}}$ where $V_N = \{\varphi \in \mathscr{D}_K : \|\varphi\|_N < \frac{1}{N}\}$.

Remark: $\varphi_n \in \mathscr{D}_K$ converges to $\varphi \in \mathscr{D}_K$ if and only if $\|\varphi_n - \varphi\|_N \to 0$ for all N.

Definition (159): Let τ be the topology on $\mathscr{D}(\Omega)$ determined by the local base

$$\beta = \{W \subseteq \mathscr{D}(\Omega) : W \text{ is convex, balanced, and for all compact } K \subseteq \Omega, W \cap \mathscr{D}_K \in \tau_K\}.$$

Theorem (160): (See Rudin Functional Analysis for proofs.)

1. $(\mathscr{D}(\Omega), \tau)$ is a LCTVS.
2. The relative topology on $\mathscr{D}_K$ (from $\mathscr{D}(\Omega)$) is τ_K for all $K \subset \Omega$ with K compact.
3. If $\varphi_i \in \mathscr{D}(\Omega)$ is a sequence, then $\varphi_i \overset{\to}{\tau} 0$ if and only if there exists K compact such that $\{\varphi_i\} \subseteq \mathscr{D}_K$ and $D^\alpha \varphi_i \to 0$ uniformly as $i \to \infty$ for all α.
4. $D^\alpha : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is continuous for all α.
5. Every Cauchy sequence in $\mathscr{D}(\Omega)$ converges.

Definition (161): A distribution is a continuous linear functional on $\mathscr{D}(\Omega)$. The space of all distributions is denoted $\mathscr{D}'(\Omega)$.

Theorem (162): $\Lambda \in \mathscr{D}'(\Omega)$ if and only if for all compact sets $K \subseteq \Omega$, there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|\Lambda\varphi| \leq c\,\|\varphi\|_N\,, \text{ for all } \varphi \in \mathscr{D}_K.$$

Definition (163): Let α be a multi-index and $\Lambda \in \mathscr{D}'(\Omega)$ a distribution. Define $D^\alpha(\Lambda)$ by

$$(D^\alpha(\Lambda))(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi), \quad \varphi \in \mathscr{D}(\Omega).$$

3.3 Application of Hahn-Banach Theorem

Theorem (164): (Separating Hyperplane Theorem) Let X be a TVS and $A, B \subseteq X$ both be convex, non-empty, and $A \cap B = \emptyset$. The following hold:

1. If A is open, then there exists $\varphi \in X^*$ and $t_0 \in \mathbb{R}$ such that $\text{Re}\varphi(x) < t_0 \leq \text{Re}\varphi(y)$ for all $x \in A$ and $y \in B$.
2. If A is compact, B is closed, and X is a LCTVS, then there exists $\varphi \in X^*$ and $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and $\text{Re } \varphi(x) < t_1 < t_2 < \text{Re } \varphi(y)$ for all $x \in A$ and $y \in B$.

Lemma (165): Suppose X is a TVS and A, B are disjoint subsets with A compact and B closed. Then there exists a neighborhood V of 0 such that $(A + V) \cap (B + V) = \emptyset$. Moreover, when X is locally convex, V can be taken convex.

3.4 Consequences of the Separating Hyperplane Theorem

Theorem (166): If X is a LCTVS, then X^* separates points. (Repeated from before, but the proof is very simple here.)

Theorem (167): Let X be a LCTVS, $M \subseteq X$ a subspace, and $x_0 \in X \backslash \overline{M}$. Then there exists $\varphi \in X^*$ such that $\varphi(x_0) = 1$ and $\varphi|_M = 0$.

Remark: Let X be a LCTVS. Note that if M is a subspace of X, then $x_0 \in \overline{M}$ if and only if $\varphi(x_0) = 0$ whenever $\varphi \in X^*$ and $\varphi|_M = 0$.

Theorem (168): For $n \in \mathbb{Z}$ and $f \in E = \{f \in C(\mathbb{R}) : f(x) = f(x + 2\pi), \text{ for all } x \in \mathbb{R}\}$, $e_n \in T_f$ if and only if $\hat{f}(n) \neq 0$. Moreover, if $\hat{f}(n) \neq 0$ for all $n \in \mathbb{Z}$, then $T_f = E$.

Lemma (169): For $\varphi \in E^*$ and $f \in E$, define $g_{\varphi,f}(x) = \varphi(f_x)$. Then $g_{\varphi,f} \in E$ and $\hat{g}_{\varphi,f}(n) = \hat{f}(n)\varphi(e_n)$.

Theorem (170): Let X be a LCTVS and $M \subseteq X$ be any subspace. Suppose $f : M \to \mathbb{F}$ is a continuous linear functional. Then there exists $\varphi \in X^*$ such that $\varphi|_M = f$.

Theorem (171): Let X be a LCTVS and $B \subseteq X$ be a convex, balanced, and closed set. Suppose $x_0 \in X$ but $x_0 \notin B$. Then there exists $\varphi \in X^*$ such that $|\varphi(x)| \leq 1$ for all $x \in B$ and $\varphi(x_0) > 1$.

Example (172): Let X be a Banach Space. If dim $(X) = \infty$, then dim $(X) \geq c$ where $c =$ card $(\mathbb{R})$.

Example (173): Let X be a normed space. Then there is a set A and an isometric mapping of X into the continuous bounded functions on A, $C_b(A)$.

Lemma (174): Let V be a vector space. Suppose $f_1, ..., f_n$ are linear functionals on V. Put $N = \cap_{j=1}^n \text{ker } f_j$. Let φ be a linear functional on V. TFAE:

1. $\varphi \in \text{span}\{f_1, ..., f_n\}$
2. There exists $r \in [0, \infty)$ such that $|\varphi(x)| \leq r \max_{1 \leq j \leq n} |f_j(x)|$, for all $x \in V$.
3. $N \subseteq \ker \varphi$.

Theorem (175): Let X be a vector space over $\mathbb{F}$ and suppose X' is a subspace of linear functionals on X such that X' separates points of X. Let τ' be the locally convex topology on X arising from X'. Then $(X, \tau')^* = X'$.

Theorem (176): Suppose X is a LCTVS and $E \subseteq X$ is convex. Then, $\overline{E}^{\text{original}} = \overline{E}^{\sigma(X,X^*)}$.

Corollary (177): If X is a LCTVS,

1. A subspace is weakly closed if and only if it is originally closed.
2. A convex subset is originally dense if and only if it is weakly dense in X.

Proposition (178): Let X be a LCTVS which is metrizable. If $\{x_n\}$ is a sequence in X such that $x_n \to x$ weakly, then there exists a sequence $\{y_n\} \subseteq X$ such that

1. Each $y_k \in \text{co}\,(\{x_n : n \geq 1\})$.
2. $y_n \to x$ in the original topology.

Theorem (179): $J(X)$ is $\sigma(X^{**}, X^*)$-dense in X^{**}.

Definition (179): Let X be a LCTVS and $A \subseteq X$. The polar of A, denoted A° is

$$A^\circ = \{\varphi \in X^* : |\varphi(a)| \leq 1, \text{ for all } a \in A\}.$$

Likewise if $B \subseteq X^*$, the prepolar of B is

$$B_\circ = \{x \in X : |\varphi(x)| \leq 1, \text{ for all } \varphi \in B\}.$$

Sometimes people write $B_\circ$ as ${}^\circ B$.

Facts:

1. A° and $B_\circ$ are convex, balanced sets.
2. A° is weak*-closed.
3. $B_\circ$ is closed (original topology).

Theorem (180): (Bipolar Theorem) Let X be a LCTVS and suppose $A \subseteq X$. Then

$$(A^\circ)_\circ = \overline{\left\{\sum_{j=1}^{n} \alpha_j v_j : n \in \mathbb{N},\, v_1, ... v_n \in A,\, \alpha_j \in \mathbb{F} \text{ with } \sum_{j=1}^{n} |\alpha_j| = 1\right\}},$$

that is the balanced, closed, convex hull of A.

Theorem (181): (Bipolar Theorem′) Let X be a TVS. If $B \subseteq X^*$. Then, $(B^\circ)_\circ$ = weak*-closed, balanced, convex hull of B.

Theorem (182): Let X be a LCTVS.

1. If $M \subseteq X$ is a subspace, then $(M^\perp)_\perp = \overline{M}$.
2. If $W \subseteq X^*$ is a subspace, then $(W_\perp)^\perp = \overline{W}^{w^*}$.

Lemma (183): If p is a seminorm on a vector space X and $M \subseteq X$ is a subspace, define $\overline{p} : X/M \to [0, \infty)$ by

$$\overline{p}(x + M) = \inf\{p(x + m) :\, m \in M\}.$$

Then $\overline{p}$ is a seminorm on X/M. If X is a LCTVS, M is a closed subspace, and $\mathscr{P}$ is a family of seminorms on X which gives the topology, then $\overline{\mathscr{P}} = \{\overline{p} : p \in \mathscr{P}\}$ is a family of seminorms which gives the quotient topology on X/M.

Theorem (184): Let X be a LCTVS and $M \subseteq X$ be a closed subspace. Also let $Q : X \to X/M$ be the quotient map.

1. The map $f \in (X/M)^* \to f \circ Q$ is a linear bijection between $(X/M)^*$ and $M^\perp$ and is a linear homeomorphism from $\left((X/M)^*, \sigma\left((X/M)^*, X/M\right)\right)$ onto $\big(M^\perp,$ relative $\sigma(X^*, X)$ − topology$\big)$.
2. Let $\rho : X^* \to M^*$ be the restriction map. Then $\widetilde{\rho} : X^*/M^\perp \to M^*$ is a bijection and is a homeomorphism from $\big(X^*/M^\perp, \sigma\left(X^*, X - \text{quotient map}\right)\big)$ onto $(M^*, \sigma(M^*, M))$, where the quotient topology is given by the previous lemma.

3.5 Banach-Alaglou and Applications

Theorem (185): (Banach-Alaglou) Let X be a TVS and let G be a neighborhood of 0. Let $K = G^\circ = \{\varphi \in X^* :\, |\varphi(x)| \le 1,$ for all $x \in G\}$. Then K is weak*-compact.

Theorem (186): (Tychonoff Theorem) Let $A \neq \emptyset$. For each $\alpha \in A$, let (X_α, τ_α) be a compact topological space. Then $X = \prod_{\alpha \in A} X_\alpha$ is compact, where X is equipped with the product topology.

Fact: Suppose X is a separable TVS and suppose $K \subseteq X^*$ is w*-compact. Then K is metrizable (in w*-topology).

Corollary (187): If X is a separable Banach space, and K is the closed unit ball of X^*, then the w*-topology on K is metrizable.

Note: The w*-topology on X^* need not be metrizable.

Theorem (188): Ket X be a Banach space and B be the closed unit ball of X^*. Then we know B is compact in the $\sigma(X^*, X)$-topology. Recall the map $\Phi : X \to C(B)$ given by $\Phi(x)(b) = b(x)$. We saw this was an isometry. Now $\Phi(x) : B \to \mathbb{C}$ is continuous. When X is separable, we can get a linear isometry of X into $C[0,1]$.

Fact: If A is a compact metric space, there exists a continuous surjection $h :$ Cantor Set $\to A$.

Theorem (189): (Characterization of Reflexivity) Let X be a Banach space. Then X is reflexive if and only if the closed unit ball, $B = \{x \in X : \|x\| \leq 1\}$, is compact in the weak topology on X.

Corollary (190): Let X be a Banach space and $M \subseteq X$ a closed subspace.

a) If X is reflexive, so is X/M.

b) If X is reflexive, so is M.

Corollary (191): Suppose X is a reflexive Banach space, $M \subseteq X$ is a norm closed subspace, and $N \subseteq X^*$ is a norm closed subspace. Then $M^{\perp}$ is reflexive, and $N_{\perp}$ is reflexive.

Definition (192): A Banach space X is uniformly convex if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that whenever $x, y \in X$ with $\|x\|, = \|y\| = 1$ and $\left\|\frac{x+y}{2}\right\| > 1 - \delta(\epsilon)$, then $\|x - y\| < \epsilon$.

Theorem (193): A uniformly convex Banach space is reflexive.

Definition (194): A Banach space X is smooth if the unit sphere has a unique tangent plane at each point.

Theorem (195): X is uniformly convex if and only if X^* is uniformly smooth.

Corollary (196): Any uniformly smooth Banach space is reflexive.

Definition (197): Let V be a vector space over $\mathbb{F}$ and $\emptyset \neq K \subseteq V$. A subset $\emptyset \neq S \subseteq K$ is called an extreme set of K if whenever an element $x \in S$ is written $x = ty + (1-t)z$, $t \in [0,1]$, and $y, z \in K$, we have $y, z \in S$. A point $x \in K$ is called an extreme point of K if $\{x\}$ is an extreme set of K.

Remark: If K is convex, an extreme set $S \subseteq K$ is called a face.

Theorem (198): (Minkowski 1900) If $K \subseteq \mathbb{R}^n$ is compact and convex. Then $K = \text{co}$ (extreme points).

Lemma (199): Let (X, τ) be a TVS such that X^* separates points. Suppose $A, B \subseteq X$ are disjoint, nonempty, compact, convex subsets of X. Then there exists $\varphi \in X^*$ such that

$$\sup_{a \in A} \text{Re}\varphi(a) < \inf_{b \in B} \text{Re}\varphi(b).$$

3.6 Consequences of Krein-Millman

Theorem (200): (Krein-Millman) Suppose X is a TVS such that X^* separates points. If $K \subseteq X$ with K compact and convex, then

$$K = \text{co}\,(\text{extreme points of } K).$$

Theorem (201): If X is a TVS and V is a neighborhood of 0. Then

$$V^\circ = \overline{\text{co}}^{w^*}\{\text{extreme points of } V^\circ\}.$$

Corollary (202): If X is a Banach space, then the closed unit ball of X^* is the w^*-closed convex hull of its extreme points.

Theorem (203): If S is compact and Hausdorff, identify $C(S)^*$ with $\mathcal{M}(S)$ (via $\mu \in \mathcal{M}(S)$ by $\varphi_\mu(f) = \int_S f\,d\mu$), then the extreme points of the closed unit ball at $C(S)^*$ are the measures of the form $\lambda\delta_s$, where $\lambda \in \mathbb{F}, |\lambda| = 1$, and δ_s is the points mass concentrated at $s \in S$.

Theorem (204): (Stone-Weierstrass) Let S be a compact, Hausdorff space. Suppose $\mathcal{A} \subseteq C(S)$ is an algebra such that

a) $1 \in \mathcal{A}$

b) $\mathcal{A}$ separates points of S

c) If $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$.

d) $\mathcal{A}$ is norm closed.

Then, $\mathcal{A} = C(S)$.

4 Adjoints Again:

Properties of Adjoints:

a) $\|T\| = \|T^*\|$

b) $(X + T)^* = S^* + T^*$

c) $(\lambda T)^* = \lambda T^*$

d) $\mathcal{N} = \mathcal{R}(T^*)_\perp$

e) $(ST)^* = T^* S^*$

f) If T^{-1} exists and $T^{-1} \in \mathcal{B}(Y, X)$, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

g) For X, Y that are Banach spaces, if T^* is invertible, so is T.

Definition (205): Let X, Y be Banach spaces. $T : X \to Y$ is bounded below if there exists $c > 0$ such that for all $x \in X$ $\|Tx\| \geq c\,\|x\|$.

Theorem (206): If T is bounded below, Range T is closed. If T is bounded below, T is one-to-one.

Theorem (207): Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$, then

a) If T is bounded below, then T^* is onto.

b) If T is onto, then T^* is bounded below.

c) If T^* is onto, then T is bounded below.

d) If T^* is bounded below, then T is onto.

Theorem (208): Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$.

a) T is bounded below if and only if T^* is onto.

b) T is onto if and only if T^* is bounded below.

c) T is one-to-one if and only if $R(T^*)$ is weak*-dense in X^*.

d) $R(T)$ is dense if and only if T^* is one-to-one.

Theorem (209): (Closed Range Theorem) Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Then TFAE:

a) $R(T)$ is (norm) closed in Y.

b) $R(T^*)$ is weak*-closed in X^*.

c) $R(T^*)$ is norm closed in X^*.

Theorem (210): (Banach-Stone Theorem) Let Q and S be compact Hausdorff spaces. Suppose $T : C(Q) \to C(S)$ is an isometric linear surjection. Then there exists a homeomorphism $\tau : S \to Q$ and a continuous function $\alpha : S \to T$ such that for all $s \in S$ and for all $f \in C(Q)$, $(Tf)(s) = \alpha(s)f(\tau(s))$.

Theorem (211): (Brouwer Fixed Point Theorem) If $B \subseteq \mathbb{R}^n$ is the closed unit ball and $f : B \to B$ is continuous, then f has a fixed point.

Theorem (212): (Schauder Fixed Point Theorem) Let X be a LCTVS and $K \subseteq X$ be compact, convex, and non-empty. If $f : K \to K$ is continuous, then f has a fixed point.
Corollary (213): Suppose X is a Banach space and K is convex, closed, and non-empty with $K \subseteq X$. Suppose $C \subseteq K$ is compact. If $f : K \to C$ is continuous, then f has a fixed point.
Lemma (214): If X is a LCTVS and $E \subseteq X$ us a totally bounded set, then both $\text{co}(E)$ and $\overline{\text{co}}(E)$ are totally bounded.

5 Classification of Spectra

Let X be a Banach space and $T \in \mathcal{B}(X)$.
Definition (215):

a) The <u>spectrum</u> of T is $\sigma(T) = \{\lambda \in \mathbb{F} : \lambda I - T \text{ is not invertible}\}$.

b) The <u>point spectrum</u> is $\sigma_{\text{pt}}(T) = \{\lambda \in \mathbb{F} : \lambda I - T \text{ is not one-to-one}\}$.

c) The <u>continuous spectrum</u> is $\sigma_{\text{cont}}(T)\{\lambda \in \mathbb{F} : \lambda I - T \text{ is one-to-one and } R(\lambda I - T) \text{ is proper and dense}\}$.

d) The <u>residual spectrum</u> is $\sigma_{\text{res}}(T) = \{\lambda \in \mathbb{F} : \lambda I - T \text{ is one-to-one and } \overline{R(\lambda I - T)} \neq X\}$

e) The <u>approximate point spectrum</u> is $\pi(T) = \{\lambda \in \mathbb{F} : \lambda I - T \text{ is not bounded below}\}$.

f) The <u>compression spectrum</u> is $\Gamma(T) = \{\lambda \in \mathbb{F} : R(\lambda I - T) \text{ is not dense in } X\}$.

Remark:

a) $\sigma_{\text{pt}}(T), \sigma_{\text{cont}}(T)$, and $\sigma_{\text{res}}(T)$ are all pairwise disjoint subsets of the spectrum, and

$$\sigma(T) = \sigma_{\text{pt}}(T) \cup \sigma_{\text{cont}}(T) \cup \sigma_{\text{res}}(T).$$

b) $\sigma(T) = \pi(T) \cup \Gamma(T)$, but $\pi(T)$ and $\Gamma(T)$ are not disjoint necessarily.

Example (216): Let $X = \ell^p$ with $1 \leq p \leq \infty$. Let $T : X \to X$ be given by

$$T(x_1, x_2, ...) = (x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, ...).$$

Now $R(T)$ is not dense and T is not bounded below. So $0 \in \Gamma(T) \cap \pi(T)$.
Fact: Observe that the following hold:

(a) $\sigma_{\text{res}}(T) \subseteq \Gamma(T)$

(b) $\sigma_{\text{cont}}(T) \cap \Gamma(T) = \emptyset$. Hence, $\sigma_{\text{cont}}(T) = \pi(T)$.

(c) $\sigma_{\text{pt}}(T) \cap \Gamma(T)$ can be non-empty.

(d)

T		T^*
$\gamma(T)$	$\Leftrightarrow$	$\sigma_{\text{pt}}(T^*)$
σ_{pt}	$\Rightarrow$	$\Gamma(T^*)$
$\sigma_{\text{pt}}(T)$	$\Rightarrow$	$\sigma_{\text{pt}}(T^*) \cup \sigma_{\text{res}}(T^*)$
$\sigma_{\text{cont}}(T)$	$\Rightarrow$	$\sigma_{\text{cont}}(T^*) \cup \sigma_{\text{res}}(T^*)$
$\sigma_{\text{res}}(T)$	$\Rightarrow$	$\sigma_{\text{pt}}(T^*)$

If X is reflexive, $\sigma_{\text{cont}}(T) \cup \sigma_{\text{cont}}(T^*)$.

Definition (217): Let A be a Banach space. Suppose there exists a function $\varphi : A \times A \to A$ such that $(A, +, \cdot)$ is an algebra, (a vector space with a ring structure), such that for all $x, y \in A$ $\|xy\| \leq \|x\| \, \|y\|$. In addition, if A has a multiplicative identity e, we say that A is a unital Banach algebra. Note then $\|e\| \geq 1$. We make the assumption that $\|e\| = 1$. This is a Banach algebra.

Definition (218): The spectrum of $a \in A$ is

$$\sigma(a) = \{\lambda \in \mathbb{F} : \lambda e - a \text{ is not invertible.}\}.$$

The resolvent of a is $\mathbb{F} \backslash \sigma(a) = \rho(a)$.

Facts:

(a) If $x \in A$ and $\|x\| < 1$, then $e - x$ is invertible and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$

(b) For $x \in A$, $\rho(x)$ is open.

(c) Also define $f : \rho(x) \to A$ by $f(\lambda) = (\lambda e - x)^{-1}$. Then f is continuous.

(d) If $|\lambda| > \|x\|$, then $\lambda \in \rho(x)$.

(e) $\sigma(x)$ is closed and bounded, and hence compact. Also $\sigma(x) = \{z \in \mathbb{F} : |z| \leq \|x\|\}$.

(f) Let $\mathbb{F} = \mathbb{C}$. For every $\varphi \in A^*$, the map, $_\varphi : \rho(x) \to \mathbb{C}$ given by $h_\varphi(\lambda) = \varphi\left((\lambda e - x)^{-1}\right)$ is analytic.

Theorem (219): Suppose A is a unital complex Banach algebra. If $x \in A$, then $\sigma(x) \neq \emptyset$.
Example (220): Let (M, Ω, μ) be a finite measure space and $F : M \to \mathbb{C}$ a fixed L^∞ function. Let $X = L^p(M, \mu)$, $1 \leq p \leq \infty$. For $f \in L^p$, define $(TF)(x) = F(x)f(x)$. Then

(a) $\sigma(T) = \text{ ess. range } := \{\lambda \in \mathbb{C} : \text{ for all } \epsilon > 0\ \mu(\{m \in M : |F(m) - \lambda| < \epsilon\}) > 0\} = \pi(T)$.

(b) $\sigma_{\text{pt}} = \{\lambda \in \mathbb{F} : \mu(\{x \in M : F(x) = \lambda\}) > 0\}$

(c) If $1 \leq p \leq \infty$, then $\sigma_{\text{res}}(T) = \emptyset$.

(d) For $p = \infty$, $\sigma(T) = \pi(T) = \Gamma(T)$.

5.1 Compact Operators

Definition (221): Let X, Y be Banach spaces and U_x = the open unit ball of X. Then $T \in \mathcal{B}(X, Y)$ is compact if $\overline{T(U)}$ is compact. The old literature calls this completely continuous.
Fact: T is compact if and only if for every bounded subset $B \subseteq X$, $T(B)$ is totally bounded.
Theorem (222): Let $T, P \in \mathcal{B}(X, Y)$, $S \in \mathcal{B}(Y, Z)$, and $\mathcal{B}(W, X)$.

(a) If $R(T)$ is finite dimensional, then T is compact.

(b) If T is compact and $R(T)$ is closed, then dim $(R(T)) < \infty$.

(c) If P, T are compact operators, then $P + T$ is compact.

(d) If T is compact, then ST and TR are compact.

(e) If $\{T_n\} \in \mathcal{B}(X, Y)$ with T_n compact for all $n \in \mathbb{N}$ and $T_n \to T$ in norm, then T is compact.

(f) T is compact if and only if T^* is compact.

(g) If dim $X = \infty$ and T is compact, then T is not invertible.

(h) Let $T \in \mathcal{B}(X)$ and T be compact. If $\lambda \neq 0$, then ker $(\lambda I - T)$ is finite dimensional.

Main Properties of Spectral Theory: Let X be a Banach space and $T \in \mathcal{B}(X)$ be compact.

(a) $\sigma(T)$ is at most countable with 0 as its only possible limit point.

(b) If $\lambda \neq 0$ and $\lambda \in \sigma(T)$, then $\lambda \in \sigma_{\text{pt}}(T)$.

(c) If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then $R(\lambda I - T)$ is closed.

Recall: If X is a Banach space and $M \subseteq X$ such that M is a closed subspace. Then another closed subspace $N \subseteq X$ is a complement to M if

(a) $X = M + N$

(b) $M \cap N = (0)$

If M is complemented with complement N, $P_M(x) = P_{M,N}(x) = m$, where $x = m + n$.

Fact: Suppose X is a TVS and $M \subseteq X$ is a closed subspace.

(a) If X is a LCTVS and M is finite dimensional, then M has a complement.

(b) If dim $X/M < \infty$, then M is complemented.

Definition (223): Note dim X/M is called the codimension of M.

Proposition (224): Assume X is a Banach space and $T \in \mathcal{B}(X)$ is compact. If $\lambda \neq 0$, $R(\lambda I - T)$ is closed.

Theorem (225): Let X be a Banach space and $T \in \mathcal{B}(X)$ a compact operator. Then 0 is the only possible limit point of $\sigma_{\text{pt}}(T)$.

Corollary (226): If $S > 0$, $\sigma_{\text{pt}}(T) \cap \{\lambda : |\lambda| > S\}$ is finite.

Proposition (227): For $T \in \mathcal{B}(X)$ compact with X a Banach space, $\sigma(T)$ is at most countable with $\lambda = 0$ as the only possible accumulation point.

Proposition (228): Let X be a Banach space and $T \in \mathcal{B}(X)$ be a compact operator. If $\lambda \in \mathbb{F}\backslash\{0\}$ and $\lambda \in \sigma_{\text{pt}}(T)$, then $\mathcal{R}(\lambda I - T) \neq X$.

Proposition (229): Let X be a Banach space and $T \in \mathcal{B}(X)$ be compact. If $\lambda \in \sigma(T)$ and $\lambda \neq 0$, then $\lambda \in \sigma_{\text{pt}}(T)$.

Theorem (230): Let X be a Banach space and $T \in \mathcal{B}(X)$ be compact. Let $\lambda \neq 0, \alpha = \dim\,(\ker\,(\lambda I - T)), \beta = \dim\,(X/\mathcal{R}(\lambda I - T)), \alpha^* = \dim\,(\ker\,(\lambda I - T^*))$, and $\beta^* = \dim\,(X^*/\mathcal{R}(\lambda I - T^*))$. Then $\alpha = \beta = \alpha^* = \beta^* < \infty$.

Definition (231): Let X be a Banach space. Then $T \in \mathcal{B}(X)$ is called a Fredholm Operator if $\mathcal{R}(T)$ is closed and dim (ker T) and dim $(X/\mathcal{R}(T))$ are both finite.

Theorem (232): (Atkinson's Theorem) Let X be a Banach space and $T \in \mathcal{B}(X)$. Let $Q : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the set of all compact operators on X, be the quotient map. Then T is Fredholm, if and only if $Q(T)$ is invertible in $\mathcal{B}(X)/\mathcal{K}(X)$.

Definition (233): For $T \in \mathcal{B}(X)$ that are Fredholm, define the Fredholm index by

$$\begin{aligned}\text{index }(T) &= \dim\,(\ker\,(T)) - \dim\,(X/\mathcal{R}(T)) \\ &= \dim\,(\ker\,(T)) - \dim\,(\ker\,(T^*))\end{aligned}$$

Theorem (234): If $T \in \mathcal{B}(X)$ is Fredholm, then for all $K \in \mathcal{K}(X)$, $T+K$ is Fredholm (Atkinson), and index $(T) =$ index $(T+K)$.

Corollary (235): If S, T are Fredholm, then

$$\text{index } (ST) = \text{index } (S) + \text{index } (T).$$

6 Vector Valued Functions

Definition (236): Let X be a Banach space and $f : [a,b] \to X$ be a step function. Define

$$\int_a^b f(t)\,dt = \sum_{j=1}^{n} f\left(\frac{t_j + t_{j-1}}{2}\right)(t_j - t_{j-1}),$$

where $\{t_j\}_{j=0}^n$ is the obvious partition of the step function.

Facts: Let X be a Banach space and $f : [a,b] \to X$ be a step function.

1. $\displaystyle\int_a^b f(t)\,dt$ does not depend on the partition.
2. If g is also a step function and $\alpha \in \mathbb{F}$, then
$$\int_a^b f + \alpha g\,dt = \int_a^b f\,dt + \alpha \int_a^b g\,dt.$$
3. If Y is a Banach space and $T \in \mathcal{B}(X,Y)$, then
$$T\left(\int_a^b f(t)\,dt\right) = \int_a^b T(f(t))\,dt.$$
4. $\left\|\int_a^b f(t)\,dt\right\| \le \int_a^b \|f(t)\|\,dt.$

Lemma (237): Let X be a Banach space. Suppose $f : [a,b] \to X$ is piecewise continuous. Then given $\epsilon > 0$ there exists a step function $g : [a,b] \to X$ such that $\|g(t) - f(t)\| < \epsilon$, for all $t \in [a,b]$.

Corollary (238): Let X be a Banach space and $f : [a,b] \to X$ be piecewise continuous. Then there exists step functions $f_n : [a,b] \to X$ s.t. $f_n \to f$ uniformly on $[a,b]$.

Theorem (239): If $\{f_n\}$ is a sequence of step functions, $f_n : [a,b] \to X$ with $f_n \to f$ uniformly on $[a,b]$, then $I_n = \displaystyle\int_a^b f_n(t)\,dt$ is a norm convergent sequence.

Definition (240): Given $f : [a,b] \to X$ is piecewise continuous, define

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \int_a^b f_n(t)\,dt,$$

where $f_n : [a,b] \to X$ are step functions such that $f_n \to f$ uniformly on $[a,b]$.

Remark:

1. The above definition doesn't depend on the choice of approximating sequence.
2. All the properties listed above for integrals of step functions are valid for integrals of piecewise continuous functions.
3. Suppose $f : [a, b] \to X$ is continuous. Define

   $$F(x) = \int_a^x f(t)\,dt.$$

   Then,

   (a) F is continuous.

   (b) F is differentiable and $F'(x) = f(x)$.
4. (Dominated Convergence Theorem) Let f_n, f be piecewise continuous functions from $[a, b]$ into X such that $f_n(t) \to f(t)$ a.e. in $[a, b]$ and there exists $g : [a, b] \to \mathbb{R}$ such that $g \in L^1$ and $\|f_n(t)\| \leq g(t)$, then

   $$\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt.$$

7 Contour Integrals

Let $\mathbb{F} = \mathbb{C}$ and $\emptyset \neq \Omega \subseteq \mathbb{C}$ be open. Let X be a Banach space and $f : \Omega \to X$ be continuous. Let Γ be a path in Ω , i.e. $\gamma : [a, b] \to \Omega$ with $x = \alpha(t), y = \beta(t)$, and $\gamma(t) = \alpha(t) + i\beta(t)$. Assume α, β are continuous and piecewise $\mathcal{C}^1$.

Definition (241): Define

$$\int_\Gamma f(z)\,dz := \lim_{\text{mesh } P\to 0} \sum_{j=1}^n f(z_j)\Delta z_j,$$

where P is a partition $a = t_0 < ... < t_n = b$, $\Delta z_j = \gamma(t_j) - \gamma(t_{j-1})$, and $z_j = \gamma(t_j)$. The usual arguments show

$$\int_\Gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Theorem (242): Let all of the assumptions at the beginning of the section hold. TFAE:

1. $f'(\lambda)$ exists for all $\lambda \in \Omega$, where $f'(\lambda) = \lim\limits_{\substack{h\to 0\\ h\in\mathbb{C}}} \dfrac{f(x+h) - f(x)}{h}$ (norm limit).

2. For all $\varphi \in X^*$, $\varphi(f)$ is a scalar-valued analytic functions in Ω.

3. $\int_\Gamma f(\lambda)\,d\lambda = 0$ for all simple closed curves $\Gamma \subseteq \Omega$ with interior $\Gamma \subseteq \Omega$.

4. The Cauchy Integral Formula holds, i.e. for all $\lambda \in \Omega$,

$$f(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - \lambda}\,d\xi,$$

where Γ is any simple closed curve in Ω with interior $\Gamma \subseteq \Omega$ and $\lambda \in \operatorname{int}\Gamma$, oriented in the counterclockwise direction.

5. For all $\lambda_0 \in \Omega$, there exists $\delta > 0$ such that if $|\lambda - \lambda_0| < \delta$, $f(\lambda) = \displaystyle\sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n$, where $a_n \in X$ and the series converges uniformly and absolutely when $|\lambda - \lambda_0| \leq \rho < \delta$.

Theorem (243): (Power Series for Banach Spaces) Let X be a Banach space and $\{a_n\}_{n=0}^{\infty}$ a sequence in X. Consider the Power Series

$$\sum_{n=0}^{\infty} a_n\lambda_n, \quad \lambda \in \mathbb{C}.$$

Put $R = \left(\overline{\lim}\,\|a_n\|^{1/n}\right)^{-1}$. Then the series converges if $|\lambda| < R$ and diverges if $|\lambda| > R$. Moreover the series converges uniformly on $\{z \in \mathbb{C} : |z| \leq \rho < R\}$.

8 Banach Algebras

Example (244):

1. Let X be a Banach space. Then $\mathcal{B}(X)$ is a Banach algebra.

2. Let X be a Banach space. $\mathcal{K}(X)$, the compact operators, is a Banach algebra without a unit when $\dim(X) = \infty$. $\mathcal{K}(X)$ is a subalgebra of $\mathcal{B}(X)$ and in fact is an ideal.

3. Let X be a compact Hausdorff space. Then $C(X)$ is a Banach algebra and is commutative.

4. Let X be a locally compact Hausdorff space. Then $C_0(X)$ is a commutative Banach algebra with no unit when X is not compact.

5. Let (X, μ) be a measure space. Then $L^\infty(X, \mu)$ is a commutative Banach algebra with a unit.

6. $(L^1(\mathbb{R}), *)$ is a Banach algebra, where

$$(f * g)(x) = \int_{\mathbb{R}} f(x - t)g(t)\,dt.$$

7. $(\mathcal{M}(\mathbb{R}), *)$ is a Banach space, where $\mathcal{M}(\mathbb{R})$ is the complex Baire measures on $\mathbb{R}$ and
$$(\mu * \nu)(f) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s+t)\, d\mu(s)\, d\nu(t), \quad f \in C_0(X).$$
8. $\mathbb{A}(\mathbb{D})$, the set of all continuous functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which extend to a continuous function on $\overline{\mathbb{D}}$, is a Banach algebra.
9. $H^\infty(\mathbb{D})$, the set of all bounded analytic functions on $\mathbb{D}$, is a Banach algebra.

Definition (245): Let A be a Banach algebra.

1. If A is unital, an element $x \in A$ is regular (invertible) if there exists $y \in A$ such that $xy = yx = e$.
2. If x is not regular, x is called singular (noninvertible).
3. $\sigma(x) =$ spectrum.
4. The spectral radius is $\max\{|z| : z \in \sigma(x)\}$.

Proposition (246): Let G be the multiplicative group of invertible (regular) elements of the unital Banach algebra A. Then G is open and then map $x \mapsto x^{-1}$ is a homomorphism.

Proposition (247): If A is a unital Banach algebra and $x \in A$, then $\sigma(x)$ is a nonempty compact subset of $\mathbb{C}$ and the resolvent function
$$f(\lambda) = (\lambda e - x)^{-1} \quad (\lambda \in \rho(x)),$$
is an analytic function vanishing at ∞ and satisfies $f(\lambda) - f(\mu) = (\mu - \lambda) f(\lambda) f(\mu)$.

Definition (248): For a unital Banach algebra, A, an element $x \in A$ is called aright topological divisor of 0 if there exists $x_n \in A$, with $\|x_n\| = 1$, and $x_n x \to 0$. Likewise we can define a left topological divisor of 0.

Lemma (249): Let A be a unital Banach algebra. Every boundary point of the group of regular elements of A is a two-sided topological divisor of zero.

Theorem (250): Suppose A is a complex unital Banach algebra. If A has no nontrivial topological divisors of zero (either left or right), then A is isometrically isomorphic to $\mathbb{C}$.

Corollary (251): (Gelfand-Mazur Theorem) If A is a unital Banach algebra (complex) and A is a division ring, then A is isometrically isomorphic to $\mathbb{C}$.

Theorem (252): Let A be a unital complex Banach algebra and $x \in A$. Then $\lim\limits_{n\to\infty} \|x^n\|^{1/n}$ exists and equals the spectral radius.

Proposition (253): Let A be a unital Banach algebra and $x, y \in A$. Then $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.

8.1 Relative Spectrum

Setup: Let A, B be unital Banach algebras with $A \subseteq B$ and where A, B have the same unit and norm.

Proposition (254): Let $e \in A \subseteq B$ be Banach algebras. Let $x \in A$. Then

1. $\sigma_B(x) \subseteq \sigma_A(x)$
2. $\partial\sigma_A(x) \subseteq \partial\sigma_B(x)$.

Corollary (255): If $\sigma_A(x)$ has empty interior or if $\rho_B(x)$ is connected, then $\sigma_A(x) = \sigma_B(x)$.

8.2 Ideals in Banach Spaces

Definition (256): Let A be a Banach space. J is a maximal (left,right, or two-sided) ideal if whenever J_1 is a (left, right, two-sided) ideal such that $J \subseteq J_1$, then either $J = J_1$ or $J_1 = A$.

Proposition (257): Let J be an ideal in the unital Banach algebra A.

1. If $J \neq A$, then J contains no invertible element.
2. $\overline{J}$ is an ideal and $\overline{J} \neq A$ if $J \neq A$.
3. If J is a maximal ideal, then $\overline{J} = J$.
4. Every ideal is contained in a maximal ideal . When it is proper, it is contained in a proper maximal ideal.
5. If J is a proper left ideal, no element of J has a left inverse.

Theorem (258): Let J be a closed two-sided proper ideal in the Banach algebra A. Then A/J is a Banach algebra under the quotient norm and is unital if A is unital.

8.3 Adjoining a Unit to a Banach Algebra

Formula: Let A be a Banach algebra. Define $A^+ = \mathbb{C} \times A$. Think of (λ, a) as $\lambda e + a$. Define

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab).$$

Then A^+ becomes a unital algebra with unit $(1, 0)$.

Remark: There are many ways to define a norm in general. You need to be careful in defining the norm in order to make sure that you stay within the same class of Banach algebras. One way to define this is $\|(\lambda, A)\| = |\lambda| + \|a\|$.

Definition (259): Given $x \in A$, define $\sigma(x) := \sigma_{A^+}(x)$.
Remark: If A had a unit and $x \in A$, then $\sigma_A(x) \cup \{0\} = \sigma_{A^+}(x)$.

9 (Riesz) Functional Calculus

Definition (260): Let A be a unital Banach algebra over $\mathbb{C}$. Let $x \in A$. A piecewise smooth neighborhood (PSN) of $\sigma(x)$ is a closed set $\sigma(x) \subseteq N \subseteq \mathbb{C}$ such that

1. $\sigma(x) \subseteq \operatorname{int} N = N^\circ$
2. ∂N is a finite union of disjoint piecewise smooth closed Jordan curves.

Definition (261): Given a closed set $N \subseteq \mathbb{C}$, a function f is analytic in a neighborhood of N if there exists an open set G such that $N \subseteq G \subseteq \operatorname{dom}(f)$ and $f|_G$ is analytic.
Definition (262): Let $\mathcal{H}(\sigma(x))$ be the set of all pairs (f, N_f) where N_f is a PSN of $\sigma(x)$ and f is analytic in a neighborhood of N_f.
Definition (263): Given $(f, N_f), (g, N_g) \in \mathcal{H}(\sigma(x))$, say that $(f, N_f) \sim (g, N_g) \Leftrightarrow$ there exists a PSN of $\sigma(x)$ with $N \subseteq N_f \cap N_g$ and $f|_N = g|_N$.
Definition (264): Note $\mathcal{H}(\sigma(x))/\sim$ is an algebra. For $(f, N_f) \in \mathcal{H}(\sigma(x))$ define

$$\tilde{f}(x) = \frac{1}{2\pi i} \int_{\partial N_f} f(\lambda)(\lambda e - x)^{-1}\, d\lambda.$$

Theorem (265): Let A be a unital Banach algebra and $x \in A$. The map $(f, N_f) \in \mathcal{H}(\sigma(x)) \mapsto \tilde{f}(x)$ has the following properties:

1. If $(f, N_f) \sim (g, N_g)$, then $\tilde{f}(x) = \tilde{g}(x)$.
2. $f \mapsto \tilde{f}(x)$ induces an algebra homomorphism from $\mathcal{H}(\sigma(x))/\sim$ into A.
3. Suppose $\{f_n\}$ and f are all defined in a PSN, N, of $\sigma(x)$ and are analytic in a neighborhood of N. Suppose $f_n(\lambda) \to f(\lambda)$ uniformly on N. Then $\left\|\tilde{f}_n(x) - \tilde{f}(x)\right\| \to 0$.
4. If $f(\lambda) = a_0 + a_1\lambda + ... + a_n\lambda^n$, then $\tilde{f}(x) = a_0 e + a_1 x + ... + a_n x^n$.
5. If $y \in A$ and $xy = yx$, then for every $(f, N_f) \in \mathcal{H}(\sigma(x))$ and $(g, N_g) \in \mathcal{H}(\sigma(y))$, we have

$$\tilde{f}(x)\tilde{g}(y) = \tilde{g}(y)\tilde{f}(x).$$

Corollary (266): If $f(\lambda)$ is analytic in a disk with center c and radius δ and $\sigma(x) \subseteq \{z : |z-c| < \delta\}$, and $f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - c)^n$, then

$$\tilde{f}(x) = \sum_{n=0}^{\infty} a_n(x - ce)^n.$$

Corollary (267): Suppose f is a rational function with poles off of $\sigma(x)$. Say

$$f(\lambda) = c\prod_{j=1}^{n}(\lambda - \lambda_j)\left(\prod_{k=1}^{m}(\lambda - \mu_k\right)^{-1},$$

then

$$\tilde{f}(x) = c\prod_{j=1}^{n}(x - \lambda_j e)\left(\prod_{k=1}^{m}(x - \mu_k e)\right)^{-1}.$$

Theorem (268): If $K \subseteq \mathbb{C}$ is compact and $\{\alpha_j\}$ is a set containing one point in each component of $S^2\backslash K$ (Riemann Sphere$\backslash$ compact set). Suppose $f : (\Omega)^{\text{open}} \to \mathbb{C}$ is analytic (and $K \subseteq \Omega$) and $\epsilon > 0$. Then there exists a rational function $R(x)$ such that (poles of R) $\subseteq \{\alpha_j\}$ and $|f(\lambda) - R(\lambda)| < \epsilon$, for all $\lambda \in K$.

Theorem (269): (Uniqueness of Functional Calculus) Let $x \in A$ and suppose $\theta : \mathcal{H}(\sigma(x)) \to A$ is a function satisfying properties (a)-(d) of the previous theorem. Then for all $f \in \mathcal{H}(\sigma(x))$, $\sigma(f) = \tilde{f}(x)$.

Theorem (270): (Spectral Mapping Theorem) Let $x \in A$ and $f \in \mathcal{H}(\sigma(x))$. Then $\sigma(\tilde{f}(x)) = f(\sigma(x))$.

Fact: Suppose $f \in \mathcal{H}(\sigma(x))$, where A is a unital Banach algebra and $x \in A$. Also suppose $g \in \mathcal{H}(\sigma(x))$. Then $g \circ f \in \mathcal{H}(\sigma(x))$ and $\widetilde{(g \circ f)}(x) = \tilde{g}(\tilde{f}(x))$.

Theorem (271): Let A be a unital Banach algebra. Suppose $x \in A$ and $\sigma(x) = C_1 \cup C_2$, where C_1, C_2 are disjoint nonempty closed sets. Then let $f \in \mathcal{H}(\sigma(x))$ satisfy $f^2 = f$, $f \equiv 1$ on C_1, and $f \equiv 0$ on C_2. Put $e_1 = \tilde{f}(x)$ and $e_2 = e - e_1 = \widetilde{(1-f)}(x)$. Then e_1 and e_2 are idempotents and $e_1 + e_2 = e$. If $A_1 = e_1Ae_1$, $A_2 = e_2Ae_2$, then A_1, A_2 are subalgebras of A and the identity of A_i is e_i (Caution: $\|e_1\|$ may not be 1). We have $xe_i = e_ix \in A_i$ and $\sigma_{A_i}(xe_1) = C_1$ and $\sigma_{A_2}(xe_2) = C_2$.

Remark: If C is a Banach algebra generated by $\{e, x\}$, then $\sigma_e(x) = \sigma_A(x)$ and $c = e_1C \oplus e_2C$, $x = e_1x + e_2x$.

Theorem (272): A unital Banach algebra over $\mathbb{C}$ and $x \in A$. Suppose $p \neq 0$ is a polynomial such that $\tilde{p}(x) = 0$. Then

1. $\sigma(x)$ is a finite set, namely $\sigma(x) = \{\lambda_1, ..., \lambda_n\}$.

2. Let e_i be the corresponding spectral idempotent to λ_i,
$$e_i = \frac{1}{2\pi i}\int_{\Gamma_i}(\lambda e - x)^{-1}\, d\lambda.$$
Then $\sigma_{e_iAe_i}(xe_i) = \{\lambda_i\}$ and $e_ie_j = e_je_i = 0$ if $i \neq j$, we have $\displaystyle\sum_{j=1} ne_j = e$.

3. $x = \displaystyle\sum_{j=1}^{n}(\lambda_j e_j + y_j)$ where $y_j \in e_jAe_j, y_j = xe_j - \lambda_j e_j$, and y_j is nilpotent.

Theorem (272): Let X be a complex Banach space and $T \in \mathcal{B}(X)$ be a compact operator. Let $\lambda \in \sigma(T)$, with $\lambda \neq 0$. Let Γ be a small circle centered at λ with int $(\Gamma) \cap \sigma(T) = \{\lambda\}$. Let
$$f = \frac{1}{2\pi i}\int_{\Gamma}(\mu I - T)^{-1}\, d\mu.$$
Then $f^2 = f$ and $\sigma_{f\mathcal{B}(X)f}(fT) = \{\lambda\}$. Let $f^\perp = I - f$. Also, $\sigma_{f^\perp\mathcal{B}(X)f^\perp}(Tf^\perp) = \sigma(T)\backslash\{\lambda\}$.
Facts:

1. ker $(\lambda I - T) \subseteq R(f)$ is finite dimensional.

2. f is a compact operator.

3. $(\lambda I - T)f$ is nilpotent. $R(f)$ is finite dimensional.

Fact: Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Assume T is normal. If $r(T)$ is the spectral radius, then $r(T) = \|T\|$.

10 Gelfand Theory

Notation: Throughout A is a unital, commutative, complex Banach algebra.
Definition (273): A multiplicative linear functional on A is a linear map $\varphi : A \to \mathbb{C}$ such that for all $x, y \in A$,
$$\varphi(xy) = \varphi(x)\varphi(y),$$
where $\varphi \not\equiv 0$.
Proposition (274): Suppose φ is a multiplicative linear functional on A. Then φ is bounded and $\varphi(e) = 1$.
Definition (275): Let Δ be the set of all nonzero multiplicative linear functionals on A. So, $\Delta \subseteq A^*$.

Proposition (276): Δ is closed in A^*.

Definition (277): For $x \in A$, define $\hat{x} : \Delta \to \mathbb{C}$ by $\hat{x}(\varphi) = \varphi(x)$. $\hat{x}$ is called the Gelfand transform of x.

Note: The Gelfand transform is a contractive algebra homomorphism.

Theorem (278): Suppose J is a proper ideal in A, then J is maximal if and only if there exists $\varphi \in \Delta$ such that $J = \ker \varphi$.

Corollary (279): $x \in A$ is invertible if and only if $\hat{x}$ never vanishes.

Fact: A unital commutative Banach algebra and $x \in A$. Then $\hat{x}(\Delta) = \sigma(x)$.

Definition (280): If A is a commutative unital Banach algebra, the radical of A is

$$\begin{aligned} \text{rad } (A) &= \{x \in A : \, r(x) = 0\} \\ &= \cap(\text{max ideals of A}) \\ &= \{x \in A : \, e - \lambda x \text{ is invertible for all } \lambda \in \mathbb{C}\}. \end{aligned}$$

Definition (281): A Banach algebra is semi-simple if Rad $(A) = (0)$.

Proposition (282): Let A be a commutative unital Banach algebra. For $x \in A$ and f is holomorphic ($f \in \mathcal{H}(\sigma(x))$). Then $\left(\hat{\tilde{f}}(x)\right) = f \circ \hat{c}$.

Remark: If A is singly generated, i.e. $A = \overline{\text{alg}}(e, x) = \overline{\text{poly}}(x)$. Then $\hat{x} : \Delta \to \sigma(x)$ is a homeomorphism.

Theorem (283): (Wiener) Suppose $f \in C(\mathbb{T})$ be such that for all $s \in \mathbb{T}$,

$$f(s) = \sum_{-\infty}^{\infty} \hat{f}(n) e^{ins},$$

where the sum converges absolutely for all s. If $f(s) \neq 0$ for all $s \in \mathbb{T}$, then $\dfrac{1}{f}$ has an absolutely convergent Fourier series, i.e. for all $s \in \mathbb{T}$,

$$\frac{1}{f} = \sum_{-\infty}^{\infty} a_n e^{ins}$$

converges absolutely.

Definition (284): Let A be a Banach algebra. An involution $*$ on A is a map $* : A \to A$ such that

1. $(x^*)^* = x$,
2. $(xy)^* = y^* X^*$,
3. $(\lambda x)^* = \overline{\lambda} x^*$,

4. $(x+y)* = x^* + y^*$.

Definition (285): A Banach algebra with involution is a pair $(A, *)$, where $*$ is an involution on A. Also called a Banach $*$-algebra.

Definition (286): Suppose A is a Banach algebra with involution, $x \in A$ is

1. self-adjoint or Hermitian if $x^* = x$
2. normal if $x^*x = xx^*$
3. unitary if $x^*x = xx^* = e$.

Proposition (287): In a Banach algebra, A, with involution, for any $x \in A$,

1. $x = \text{Re }(x) + i\text{Im }(x)$, where $\text{Re }(x) = \frac{x+x^*}{2}$ and $\text{Im }(x) = \frac{x-x^*}{2}$
2. $e = e^*$ when A has a unit.
3. x is invertible if and only if x^* is invertible.
4. $\lambda \in \sigma(x)$ if and only if $\overline{\lambda} \in \sigma(x^*)$.

Proposition (288): Suppose A is a commutative semi-simple Banach algebra. Then every involution on A is continuous.

Definition (289): A C^*-algebra is a Banach algebra with involution such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. This is called the C^* identity.

Theorem (290): (Gelfand Naimark) Let A be a unital commutative C^*-algebra with maximal ideal space Δ. Then the Gelfand transform from A into $C(\Delta)$ is an isometric isomorphism of A onto $C(\Delta)$, which satisfies $\widehat{(x^*)} = \hat{x}^*$.

11 Spectral Theory

References

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