1 Banach Fixed-Point Theorem

Definition (1): A linear space (vector space) V over $\mathbb{K}$ is a set V together with addition and scalar multiplication.

Definition (2): We say that $W \subseteq V$ is a (linear) subspace of V if W is closed under addition and scalar multiplication. That is for all $u, v \in W$ and $\alpha \in \mathbb{K}$,

1. $u + v \in W$
2. $\alpha u \in W$

Example (3): Let Ω be a nonempty set. The set $\mathbb{K}^\Omega$ if all $\mathbb{K}$-valued functions defined on Ω is a linear space over $\mathbb{K}$ if for each $f, g \in \mathbb{K}^\Omega$ and $\alpha \in \mathbb{K}$, we define

1. $f + g$ to be the function $x \mapsto f(x) + g(x)$,
2. αf to be the function $x \mapsto \alpha f(x)$.

- We use $\mathbb{K}^n$ to denote the space of n-dimensional vectors with components in $\mathbb{K}$.
- We use $\mathbb{K}^\mathbb{N}$ to denote the infinite sequences with elements in $\mathbb{K}$.

Example (4):

a. The collection $B(\Omega) \subseteq \mathbb{K}^\Omega$ of $\mathbb{K}$-valued functions that are defined and bounded on Ω are a subspace of $\mathbb{K}^\Omega$

b. Let $\Omega_0 \subseteq \Omega$ be given. The collection

$$\{f \in \mathbb{K}^\Omega : \ f(x) = 0 \text{ for each } x \in \Omega \backslash \Omega_0\} \tag{1}$$

is a subspace of $\mathbb{K}^\Omega$.

Definition (5): Let V and F be linear spaces over $\mathbb{K}$. We call $A : V \to F$ a linear operator or linear map is for each $u, v \in V$ and $\alpha, \beta \in \mathbb{K}$, we have

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v). \tag{2}$$

If $F = \mathbb{K}$, then we may call a linear map a linear form or a linear functional.

Definition (6): Let X be a nonempty set.

a. A collection $\mathcal{T}$ of subsets of X is called a topology on X if it possesses the following properties:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. If $U_1, ..., U_n \in \mathcal{T}$, then $\bigcap_{j=1}^{n} U_j \in \mathcal{T}$.
3. If $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

b. If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a topological space. The elements of $\mathcal{T}$ are called the open sets. If $\mathcal{T}$ is understood, we may refer to X as a topological space.

c. If $F^c \in \mathcal{T}$, then F is a closed set. Given $B \subseteq X$, we define the Closure of B by

$$\overline{B} := \bigcap_{\substack{F \text{ is closed} \\ B \subseteq F}} F. \tag{3}$$

Definition (7):

a. Let X be a topological space. Given $x \in X$, any open set containing x is called a (open) neighborhood of x.

b. A neighborhood base at $x \in X$ is a collection $\mathscr{B}_x$ of neighborhoods of x such that any neighborhood of x contains some member of $\mathscr{B}_x$.

Remark: A topology can be completely specified by providing a neighborhood base at each $x \in X$. Then a set $U \subseteq X$ is open if and only if for each $x \in U$ there is a $V \in \mathscr{B}_x$ such that $V \subseteq U$, i.e. U is open if and only if for each $x \in U$, there is a neighborhood of x contained in U.

Definition (9): If X and Y are topological spaces and $f : X \to Y$ satisfies $f^{-1}(U)$ is open in X whenever U is open in Y, then f is continuous. If f is one-to-one, continuous, and has a continuous inverse, we call f a homeomorphism, and X and Y are homeomorphic.

Example (10): In the usual topology on $\mathbb{R}$, a neighborhood base at $x \in \mathbb{R}$ can be taken to be the collection of all open intervals that contain x.

Definition (11): A topological vector space (TVS) is a linear space V with a topology $\mathcal{T}$ on V such that addition and scalar multiplication are continuous. That is if V is a TVS, then for each $v \in V$ and $\alpha \in \mathbb{K}$, $u \mapsto u + v$ and $u \mapsto \alpha u$ are continuous.

Definition (12): Let V be a linear space over $\mathbb{K}$. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ with the following properties:

1. (Positivity) $\|u\| \geq 0$ for all $u \in V$ and $\|u\| = 0$ if and only if $u = 0$.

2. (Subadditivity or Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

3. (Homogeneity) For all $\alpha \in \mathbb{K}$ and $u \in V$, $\|\alpha u\| = |\alpha| \, \|u\|$.

$\|u - v\|$ is referred to as the distance between u and v.

Definition (13): Let V be a linear space equipped with a norm $\|\cdot\|$. We call $(V, \|\cdot\|)$ a normed space. If the norm is understood we just refer to V as the normed space.

Definition (14): Let V be a normed space. Given $u_0 \in V$ and $\epsilon > 0$, the set $B_\epsilon(u_0) := \{u \in V : \|u - u_0\| < \epsilon\}$ is called an ϵ-neighborhood of u_0.

Definition (15): Let V be a normed space. For each $u_0 \in V$, set $\mathscr{B}_{u_0} := \{B_\epsilon(u_0)\}_{\epsilon > 0}$. The collection $\{\mathscr{B}_{u_0}\}_{u_0 \in V}$ provides a neighborhood base at each $u_0 \in V$ and the topology specified by this neighborhood bases is called the norm topology.

Proposition (16): A normed space with the norm topology is a TVS.

Definition (17): Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a linear space V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there is a constant $c > 0$ such that

$$c^{-1} \|u\|_1 \leq \|u\|_2 \leq c \|u\|_1$$

for all $u \in V$.

Definition (18): Let $\{u_j\}_{j=1}^\infty \subseteq V$, with V a normed space. We say that $\{u_j\}_{j=1}^\infty$ converges to $u \in V$ if

$$\lim_{j \to \infty} \|u_j - u\| = 0.$$

We may also write $\lim\limits_{j \to \infty} u_j = u$ and $u_j \to u$ as $j \to \infty$.

Proposition (19): Let V be a normed space. Let $\{u_j\}_{j=1}^\infty, \{v_j\}_{j=1}^\infty \subseteq V$ and $u, v \in V$ be given.

a. If $u_j \to u$ as $j \to \infty$, then u is unique.

b. If $u_j \to u$ as $j \to \infty$, then $\{u_j\}_{j=1}^\infty$ is bounded, i.e. there is a $r \geq 0$ such that $\|u_j\| \leq r$, for all $j \in \mathbb{N}$.

c. If $u_j \to u$ as $j \to \infty$, then $\|u_j\| \to \|u\|$ as $j \to \infty$.

d. If $u_j \to u$ and $v_j \to v$ as $j \to \infty$, then $u_j + v_j \to u + v$ as $j \to \infty$.

e. If $\{\alpha_j\}_{j=1}^\infty \subset \mathbb{K}$ converges to $\alpha \in \mathbb{K}$ and $u_j \to u$ as $j \to \infty$, then $\alpha_j u_j \to \alpha u$ as $j \to \infty$.

Definition (20): A sequence $\{u_j\}_{j=1}^\infty \subseteq V$, with V a normed space, is called a Cauchy sequence if for each $\epsilon > 0$ there is $j_0(\epsilon) \in \mathbb{N}$ such that

$$\|u_j - u)k\| < \epsilon \text{ for all } j, k \geq j_0.$$

Proposition (21): In a normed space, each convergent sequence is Cauchy.

Definition (22): A normed space is called complete if every Cauchy sequence is convergent. Such spaces are called Banach Spaces (B-spaces).

Example (23):

a. The space $V = \mathbb{K}$ with norm $\|u\| := |u|$ for each $u \in \mathbb{K}$ is a Banach Space.

b. The space $= \mathbb{K}^n$ with norm

$$\|u\|_p := \begin{cases} \left(\sum_{j=1}^n |u_j|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \max\limits_{1 \leq j \leq n} |u_j|, & p = \infty \end{cases}$$

is a Banach space. For each $p_1, p_2 \in [0, \infty]$ the norms $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$ are equivalent.

c. The space $= \mathbb{K}^{\mathbb{N}}$ with norm

$$\|u\|_p := \begin{cases} \left(\sum\limits_{j=1}^{\infty} |u_j|^p\right)^{1/p}, & 1 \leq p < \infty, \\ \sup\limits_{1 \leq j \leq n} |u_j|, & p = \infty \end{cases}$$

is a Banach space. For each $p_1, p_2 \in [0, \infty]$ the norms $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$ are **not** equivalent.

Example (24): Let Ω be a nonempty set. The linear space $B(\Omega)$ with the uniform (sup) norm

$$\|u\|_\infty = \sup_{x \in \Omega} |u(x)|, \text{ for all } u \in B(\Omega),$$

is a Banach space.

Example (25): Let Ω be a topological space. The subspace $C_{\text{bd}}(\Omega) \subseteq B(\Omega)$ of bounded continuous functions from Ω into $\mathbb{K}$ is a Banach space with the uniform norm.

Proposition (26): Let $E \subseteq V$ be given with V a normed space. TFAE:

i. E is closed in the norm topology of V.

ii. E contains all of its limit points.

Theorem (27): A subspace W of a Banach space V is complete if and only if it is closed.

Example (28): Our argument in example 25 shows that if Ω is a topological vector space, then the subspace $C_{\text{bd}}(\Omega)$ of $B(\Omega)$ is closed in $B(\Omega)$.

Definition (29): Let V be a normed space, and let $E \subseteq V$ be given. An operator $A : E \to E$ (not necessarily linear) is called a <u>contraction</u> on E, or <u>contractive</u> on E, if there is an $\alpha \in [0, 1)$ such that

$$\|A(u) - A(v)\| \leq \alpha \|u - v\|, \text{ for all } u, v \in E.$$

Theorem (30): (Banach Fixed-Point Theorem) Let V be a Banach space. Assume that

a. $E \subseteq V$ is a closed set in V

b. $A : E \to E$ is contractive.

Then we have

i. (Existence and Uniqueness) The operator equation

$$Au = u \tag{1}$$

has a unique solution for some $u \in E$.

ii. (Convergence) Given $u_0 \in E$, the sequence $\{u_j\}_{j=1}^{\infty} \subseteq E$ defined by

$$u_j = A(u_{j-1}) \text{ for } j \in \mathbb{N} \tag{2}$$

converges to the unique solution to (1).

iii. (Error Estimates) For each $j \in \mathbb{N}$, we have an a priori estimate,

$$\|u_j - u\| \leq \frac{\alpha^j}{1 - \alpha} \|u_1 - u_0\|,$$

and an a posteriori estimate

$$\|u_j - u\| \leq \frac{\alpha}{1 - \alpha} \|u_j - u_{j-1}\|,$$

where α is the constant from definition 29.

iv. (Convergence Rate) For each $j \in \mathbb{N}$, we have

$$\|u_j - u\| \leq \alpha \|u_j - u\|.$$

Theorem (31): (Solutions to Linear Equations) Let $b \in \mathbb{K}$ be given. Suppose that $C \in \mathbb{K}^{n\times n}$ is an $n \times n$ matrix satisfying

$$\sum_{\ell=1}^{n} |C_{j\ell}| < 1 \quad \text{for each } j = 1, ..., n,$$

then the equation

$$u = Cu + b \tag{3}$$

has exactly one solution, $u \in \mathbb{K}^n$. Moreoverm given any $u^{(0)} \in \mathbb{K}^n$, we may define $\{u^{(j)}\}_{j=1}^{\infty} \subset \mathbb{K}^n$ by $u^{(j)} = Cu^{(j-1)} + b$ for $j \in \mathbb{N}$, and the following hold

i. $\lim_{j\to\infty} \left\|u^{(j)} - u\right\|_\infty = 0$

ii. $\left\|u^{(j)} - u\right\|_\infty \leq \frac{\alpha^j}{1-\alpha} \left\|u^{(1)} - u^{(0)}\right\|_\infty$

iii. $\left\|u^{(j)} - u\right\|_\infty \leq \frac{\alpha}{1-\alpha} \left\|u^{(j)} - u^{(j-1)}\right\|_\infty$,

with $\alpha = \max\limits_{1\leq j\leq n} \sum\limits_{\ell=1}^{n} |C_{j\ell}|$.

Theorem (32): (Picard-Lindeloff Theorem) With $a, b > 0$ and $(x_0, u_0) \in \mathbb{R}^2$ given, set $R := \{(x, u) \in \mathbb{R}^2 : |x - x_0| < a, |u - u_0| < b\}$.

a. Suppose that $F : R \to \mathbb{R}$ is continuous and that $(x, u) \mapsto F(x, u)$ is also continuous on R.

b. Put

$$M := \max_{(x,u)\in R} |F(x, u)| \quad \text{and} \quad L := \max_{(x,u)\in R} \left|\frac{\partial}{\partial u} F(x, u)\right|.$$

Select $h > 0$ such that

$$h \leq a, hM \leq b, \text{ and } hL \leq 1.$$

Then the following hold:

i. There is a unique $u : [x_0 - h, x_0 + h] \to [u_0 - b, u_0 + b]$ that is continuously differentiable and is a solution to the IVP

$$\begin{cases} u'(x) = F(x, u(x)), \text{ for all } x \in [x_0 - h, x_0 + h], \\ u(x_0) = u_0. \end{cases} \tag{4}$$

ii. The solution u to (4) is also the unique solution to the integral equation

$$u(x) = u_0 + \int_{x_0}^{x} F(s, u(s))\, ds, \tag{5}$$

for all $x \in [x_0 - h, x_0 + h]$ satisfying $u(x) \in [u_0 - b, u_0 + b]$ for each $x \in [x_0 - h, x_0 + h]$.

iii. With $u_0(x) \equiv u_0$, define the sequence $\{u_j\}_{j=1}^{\infty}$ of functions on $[x_0 - h, x_0 + h]$ by

$$u_j(x) := u_0 + \int_{x_0}^{x} F(s, u(s))\, ds$$

for $x \in [x_0 - h, x_0 + h]$ and $j \in \mathbb{N}$. Then $\lim_{j\to\infty} \|u_j - u\|_\infty = 0$, where u is the solution identified in (i) and (ii).

iv. For each $j \in \mathbb{N}$, we have

$$\|u_j - u\|_\infty \leq \frac{\alpha^j}{1-\alpha} \|u_1 - u_0\|_\infty,$$

and

$$\|u_j - u\|_\infty \leq \frac{\alpha}{1-\alpha} \|u_j -_{j-1}\|_\infty,$$

with $\alpha := hL$.

Theorem (33): Let $[a, b] \subset \mathbb{R}$ be given. Assume the following:

a. The function $f \in C([a, b])$.

b. The function $F \in C_{\text{bd}}([a, b] \times [a, b] \times \mathbb{R})$ and $(x, y, u) \mapsto \frac{\partial}{\partial u} F(x, y, u)$ is continuous on $[a, b] \times [a, b] \times \mathbb{R}$. Put $L := \sup_{(x,y,u)\in[a,b]\times[a,b]\times\mathbb{R}} \left| \frac{\partial}{\partial u} F(x, y, u) \right|$.

c. The number $\lambda \in \mathbb{R}$ satisfies $(b - a)|\lambda|L < 1$.

Then the following hold:

i. There is a unique $u \in (C([a, b]), \|\cdot\|_\infty)$ that solves the integral equation

$$u(x) = f(x) + \lambda \int_a^b F(x, s, u(s))\, ds, \text{ for all } x \in [a, b]. \tag{7}$$

ii. With $u_0(x) \equiv 0$, define the sequence $\{u_j\}_{j=1}^{\infty} \subset C([a, b])$ by

$$u_j(x) = f(x) + \lambda \int_a^b F(x, s, u_{j-1}(s))\, ds, \text{ for all } x \in [a, b], j \in \mathbb{N}.$$

Then $\lim_{j\to\infty} \|u_j - u\| = 0$, where u is the solution identified in (i).

iii. For each $j \in \mathbb{N}$, we have

$$\|u_j - u\|_\infty \leq \frac{\alpha^j}{1-\alpha} \|u_1\|_\infty$$

and

$$\|u_j - u\|_\infty \leq \frac{\alpha}{1-\alpha} \|u_j - u_{j-1}\|_\infty ,$$

with $\alpha := (b-a)|\lambda|L$.

2 Brouwer Fixed-Point Theorem

Definition (35): Let V and W be normed spaces. With $E \subseteq V$, the operator $A : E \to W$ is

- sequentially continuous if for each $\{u_j\}_{j=1}^\infty \subseteq E \lim_{j\to\infty} u_j = u$ for some $u \in E$ implies $\lim_{j\to\infty} A(u_j) = A(u)$.
- continuous if for each $\epsilon > 0$ there exists $\delta(\epsilon, u) > 0$ such that $\|A(v) - A(u)\|_W < \epsilon$ whenever $\|v - u\|_V < \delta$.
- uniformly continuous if A is continuous and δ can be selected independent of u for each $\epsilon > 0$.
- Lipschitz continuous if there is a number $L \geq 0$ such that $\|A(u) - A(v)\|_W \leq L \|u - v\|_V$, for all $u, v \in E$.

Proposition (36): Let V and W be normed spaces and let $E \subseteq V$ be given. Given the operator $A : E \to W$, we have:

1. Lipschitz continuity implies uniform continuity
2. Uniform continuity implies continuity
3. sequential continuity and continuity are equivalent.

Proposition (37): Let V, W, and X be normed spaces. Suppose $E \subseteq V$ and that $A : E \to W$ and $B : A(E) \to X$ are both continuous, then $C : E \to X$ defined by $C := B \circ A$ is also continuous.

Remark (38): Analogues to proposition 37 hold if the operators A and B are both uniformly continuous or both Lipschitz continuous.

Definition (39): Let $E \subseteq V$ be given with V a normed space. The set E is

- relatively sequentially compact (relatively compact) if each sequence $\{u_j\}_{j=1}^\infty \subseteq E$ has a subsequence $\{u_{j_k}\}_{k=1}^\infty \subseteq \{u_j\}_{j=1}^\infty$ such that $\lim_{k\to\infty} u_{j_k} = u$ for some $u \in V$.

- sequentially compact (compact) if each sequence $\{u_j\}_{j=1}^{\infty} \subseteq E$ has a subsequence $\{u_{j_k}\}_{k=1}^{\infty} \subseteq \{u_j\}_{j=1}^{\infty}$ such that $\lim_{k\to\infty} u_{j_k} = u$ for some $u \in E$.
- bounded if there is an $r \geq 0$ such that $\|u\| \leq r$ for all $u \in E$.

Proposition (40): A set $E \subseteq V$, with V a normed space, is compact if and only if E is closed and relatively compact.

Proposition (41): Relatively compact sets are bounded in normed spaces.

Proposition (42): In finite dimensions:

- closed and bounded sets are compact.
- bounded sets are relatively compact.

Proposition (43): Let V and W be normed spaces and let $E \subseteq V$ be compact. If $A : E \to W$ is continuous, then A is uniformly continuous.

Theorem (44): (Brouwer Fixed-Point Theorem) Let $n \in \mathbb{N}$ be given. Set $B := \{x \in \mathbb{R}^n : \|x\| < 1\}$. If $f : \overline{B} \to \overline{B}$ is continuous, then there is an $x \in \overline{B}$ such that $f(x) = x$.

Example (45): In one dimension, a continuous function $f : [0,1] \to [0,1]$ has a fixed point. This can be proved using the Intermediate Value Theorem (IVT). Consider $x - f(x)$. The function $x - f(x)$ is still continuous on $[0,1]$, and $0 - f(0) \leq 0$ and $1 - f(1) \geq 0$. By the IVT, there exists $x \in [0,1]$ such that $f(x) = x$.

Theorem (46): (Stone-Weierstrass Theorem) Suppose that $f : E \to \mathbb{K}$ is a continuous function on $E \subset \mathbb{K}^n$, which is compact. Then for each $\epsilon > 0$, there is a polynomial $P : E \to \mathbb{K}$ such that $\|f - P\|_\infty < \epsilon$.

Lemma (47): (C^1 No Retraction Principle) There does not exist a continuously differentiable map $f : \overline{B} \to \partial B$ that satisfies $f(x) = x$ for all $x \in \partial B$. Here $B :=\}x \in \mathbb{R}^n : \|x\|_2 < 1\}$.

Theorem (48): (No-Retraction Theorem) There is no continuous map from $f : \overline{B} \to \partial B$ satisfying $f(x) = x$ for all $x \in \partial B$.

Theorem (49): Let V be a normed space and let $E \subseteq V$ be a set that is homeomorphic to

$$\overline{B} := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\},$$

for some $n \in \mathbb{N}$. If $A : E \to E$ is continuous, then there is a $u \in E$ such that $A(u) = u$.

Definition (50): A set $E \subseteq V$, with V a linear space, is called convex if for each $u, v \in E$, we find $\lambda u + (1-\lambda)v \in E$ for each $\lambda \in [0,1]$.

Definition (51): Let V be a linear space and $E \subseteq V$ be a convex set. We call $f : E \to \mathbb{R}$ convex if for each $u, v \in E$

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v),$$

for each $\lambda \in [0,1]$.

Definition (52): Let V be a linear space and let $E \subseteq V$ be given. Then

- $\text{span}\,(E) = \bigcap_{\substack{E \subseteq W \\ W \text{ is a subspace}}} W$
- $\text{co}\,(E) = \bigcap_{\substack{E \subseteq W \\ W \text{ is convex}}} W$

We call span (E) the span of E and co (E) the convex hull of E.

Proposition (53): Let $E \subseteq V$, with V a linear space, be given. Then

$$\text{co}\,(E) = \left\{ \sum_{j=1}^{k} \lambda_j u_j : \, k \in \mathbb{N}, \, \{u_j\}_{j=1}^{k} \subseteq E, \, \sum_{j=1}^{k} \lambda_j = 1 \right\}.$$

Definition (54): Let V and W be normed spaces. With $E \subseteq V$ given, let $A : E \to W$ be an operator. We say that A is compact if

(i) A is continuous.

(ii) For each bounded set $F \subseteq E$, the set $A(F)$ is a relatively compact set in W.

Proposition (55): (Schauder Approximation Theorem) Let V and W be Banach spaces, and let $E \subseteq V$ be a bounded set. Suppose that $A : E \to W$ is a compact operator. Then for each $j \in \mathbb{N}$ there is a continuous operator $A_j : E \to W$ satisfying

(i) $\sup_{u \in E} \|A(u) - A_j(u)\|_W \leq \dfrac{1}{j}$.

(ii) $\dim\,(\text{span}\,(A_j(E))) < \infty$.

(iii) $A_j(E) \subseteq \text{co}\,(A(E))$.

Theorem (56): (Schauder Fixed Point Theorem) Let V be a Banach space, and $E \subseteq V$ be a nonempty, closed, bounded, convex set. If $A : E \to E$ is compact, then A has a fixed point.

Theorem (57): (Arzela-Ascoli Theorem) Let $[a,b] \subset \mathbb{R}$ be given. Suppose that $E \subseteq C([a,b], \|\cdot\|_\infty)$ satisfies

(i) E is bounded, so there is an $r \geq 0$ such that $\|u\|_\infty \leq r$ for each $u \in E$.

(ii) E is equicontinuous; i.e. for each $\epsilon > 0$ there is a $\delta > 0$ such that for each $u \in E$

$$|x_1 - x_2| < \delta \quad \text{implies} \quad |u(x_1) - u(x_2)| < \epsilon.$$

Then E is relatively compact in $C([a,b], \|\cdot\|_\infty)$.

Lemma (58): With $[a,b] \subset \mathbb{R}$ and $r > 0$, set

$$Q := [a,b] \times [a,b] \times [-r,r].$$

Suppose that $F : Q \to \mathbb{R}$ is continuous. Set

$$E = \left\{u \in C([a,b], \|\cdot\|_\infty) : \|u\|_\infty \leq r\right\},$$

and define $A : E \to C([a,b], \|\cdot\|_\infty)$ by

$$(A(u))(x) := \int_a^b F(x,s,u(s))\,ds,$$

for each $x \in [a,b]$. Then A is compact.

Theorem (59): Let $[a,b] \subseteq \mathbb{R}$ and $r > 0$ be given. Set

$$Q := [a,b] \times [a,b] \times [-r,r].$$

Assume the following hold:

(a) The function $F : Q \to \mathbb{R}$ is continuous.

(b) With $L := \max_{(x,y,u)\in Q} |F(x,y,u)|$, suppose $\lambda \in \mathbb{R}$ satisfies $|\lambda|(b-a)L \leq r$.

Set

$$E = \left\{u \in C([a,b], \|\cdot\|_\infty) : \|u\|_\infty \leq r\right\}.$$

Then there is a solution $u \in E$ to the integral equation

$$u(x) = \lambda \int_a^b F(x,s,u(s))\,ds. \tag{9}$$

Theorem (60): (Peano's Theorem) Let $a, b > 0$ and $(x_0, u_0) \in \mathbb{R}^2$ be given. Set $R := \{(x,u) \in \mathbb{R}^2 : |x - x_0| \leq a, |u - u_0| \leq b\}$.

(a) Suppose $F : R \to \mathbb{R}$ is continuous.

(b) Put $L := \max_{(x,y)\in R} |F(x,u)|$ and select $h > 0$ such that

$$h \leq a \quad \text{and} \quad hL \leq b.$$

Then there is a solution $u \in C^1([x_0 - h, x_0 + h])$ to the initial value problem

$$\begin{cases} u'(x) = F(x, u(x)), & x \in [x_0 - h, x_0 + h] \\ u(x_0) = u_0. \end{cases} \tag{10}$$

Theorem (61): Suppose that V is a Banach space and that $A : V \to V$ is compact. Assume there is $r \geq 0$ such that for each $\beta \in [0, 1]$ the following holds: If $v \in V$ satisfies

$$v = \beta A(v), , \tag{12}$$

then $\|v\| \leq r$. Then there is $u \in V$ satisfying

$$u = A(u). \tag{13}$$

Remark (62): The theorem above does not require a solution to (12) to be known to exist. The theorem only requires a priori estimates for solution to (12) if such solutions exist.

3 Hahn-Banach Theorem

Definition (61):

(a) A set F is called <u>ordered</u> if there is a relation '$\leq$ ' satisfying the following:

  (i) $u \leq u$ for each $u \in F$.

  (ii) $u \leq v$ and $v \leq w$ implies $u \leq w$ for $u, v, w \in F$.

  (iii) $u \leq v$ and $v \leq u$ implies $u = v$ for $u, v \in F$.

(b) A <u>maximal element</u> of an ordered set F is an element $m \in F$ such that whenever $u \in F$ satisfies $m \leq u$, it must be that $m = u$.

(c) An ordered set F is called <u>totally ordered</u> if for each $u, v \in F$ we have either $u \leq v$ or $v \leq u$.

Lemma (62): (Zorn's Lemma) Let F be a nonempty ordered set with the property that each totally ordered subsset of $\mathcal{T}$ of F has a maximal element. Then F has a maximal element.
Theorem (63): (Hahn-Banach Theorem) Let V be a linear space over $\mathbb{R}$, and suppose that $p : V \to \mathbb{R}$ satisfies

(a) (Positive Homogeneity) $p(\alpha u) = \alpha p(u)$ for $u \in V$ and $\alpha > 0$.

(b) (Subadditivity) $p(u + v) \leq p(u) + p(v)$ for $u, v \in V$.

Then if $W \subseteq V$ is a subspace and $\ell : W \to \mathbb{R}$ is a linear functional such that $\ell(w) \leq p(w)$ for all $w \in W$, ℓ can be extended to a linear functional to all of V where $\ell(v) \leq p(v)$ for all $v \in V$.

Definition (64): Let V be a linear space and suppose that $K \subseteq V$ is a convex set. The gauge of K $p_K : V \to \mathbb{R}$ is given by

$$p_K(u) = \inf\left\{\alpha > 0 : \frac{u}{\alpha} \in K\right\}.$$

Proposition (65): If $K \subseteq V$ is a convex set of a linear space V such that $0 \in \text{int}\,(K)$, then p_K has positive homegeneity and is subadditive.

Definition (66): Let V be a linear space and suppose that $\ell : V \to \mathbb{R}$ is a linear functional. For each $\alpha \in \mathbb{R}$, the set $\{u \in V : \ell(u) = \alpha\}$ is called a hyperplane in V. The sets $\{u \in V : \ell(u) < \alpha\}$ and $\{u \in V : \ell(u) > \alpha\}$ are called (open) half spaces. The closed half spaces are $\{u \in V : \ell(u) \leq \alpha\}$ and $\{u \in V : \ell(u) \geq \alpha\}$.

Theorem (67): (Hyperplane Separation Theorem) Let K be a nonempty, open, convex set in a linear space V over $\mathbb{R}$. Suppose that $y \notin K$. Then there exists a hyperplane that separates y from K; i.e. there is a linear functional $\ell : V \to \mathbb{R}$ and a number $\alpha \in \mathbb{R}$ such that

$$u \in K \Rightarrow \ell(u) < \alpha \quad \text{and} \quad \ell(y) = \alpha.$$

Theorem (68): (Extended Separation Theorem) Suppose that $H, K \subseteq V$ are disjoint convex sets of the linear space V. Suppose that $\text{int}\,(H) \neq \emptyset$. Then there is a hyperplane separating H and K; i.e. there is a nontrivial linear functional $\ell : V \to \mathbb{R}$ and an $\alpha \in \mathbb{R}$ such that

$$\ell(u) \leq \alpha \leq \ell(v), \qquad \text{for all } u \in H, v \in K.$$

4 Dual Spaces

Definition (69): A linear operator $A : V \to W$ is bounded if there is an $L \geq 0$ such that

$$\|Au\|_W \leq L\,\|u\|_V\,, \qquad \text{for all } u \in V.$$

Proposition (70): If $A : V \to W$ is a linear operator, then the following are equivalent:

(i) A is continuous on V.

(ii) A is continuous at 0.

(iii) A is bounded.

Definition (71): We use $\mathscr{L}(V,W)$ to denote the space of bounded linear operators from V to W. The function $|||\cdot||| : \mathscr{L}(V,W) \to [0,\infty)$ is given by

$$|||A||| = \sup\{\|Au\|_W : \|u\|_V \leq 1\}$$

is called the <u>operator norm</u>.

Remark (72):

(·) $(\mathscr{L}(V,W), |||\cdot|||)$ is a bounded linear space.

(·) Given $A \in \mathscr{L}(V,W)$ we find

$$\begin{aligned} |||A||| &= \sup\left\{\frac{\|Au\|_W}{\|v\|_V} : u \neq 0\right\} \\ &= \inf\left\{L \in \mathbb{R} : \|Au\|_W \leq L\,\|u\|_V\,,\ \text{for all } u \in V\right\}. \end{aligned}$$

(·) If W is a Banach space, then so is $\mathscr{L}(V,W)$ even if V is incomplete.

Definition (73): The space $(\mathscr{L}(V,\mathbb{R}), |||\cdot|||)$ is called the (topological or continuous) <u>dual space</u>. We use V^* or $(V^*, \|\cdot\|_{V^*})$ to denote the dual of V.

Remark (74): Since $\mathbb{R}$ is a Banach space, so is V^* even if V is incomplete.

Theorem (75):

(i) If $W \subseteq V$ is a closed subspace and $u \in V\backslash W$, then there is an $f \in V^*$ such that $f(u) \neq 0$ and $f(w) = 0$ for all $w \in W$. Moreover f may be selected so that $\|f\|_{V^*} = 1$ and $f(u) = \inf_{w\in W} \|u - w\|_V$.

(ii) If $u \in V\backslash\{0\}$, then there is an $f \in V^*$ such that $\|f\|_{V^*} = 1$ and $f(u) = \|u\|_V$.

(iii) V^* can be used to separate points in V; i.e. if $u_1, u_2 \in V$ and $u_1 \neq u_2$ there is an $f \in V^*$ such that $f(u_1) \neq f(u_2)$.

(iv) For each $u \in V$, define $\hat{u} \in V^{**}$ by $\hat{u}(f) = f(u)$ for each $f \in V^*$. The map $u \mapsto \hat{u}$ is a linear isometry from V into V^{**}; i.e.

$$\|u\|_V = \|\hat{u}\|_{V^*} = \sup\{|\hat{u}(f)| : \|f\|_{V^*} \leq 1\} = \sup\{|f(u)| : \|f\|_{V^*} \leq 1\}.$$

Remark 76:

- V^{**} is a Banach space, even if V is not complete.

- Define $\hat{V} := \{\hat{u} : u \in V\}$. We can identify $\hat{V}$ with V itself, so the map $u \mapsto \hat{u}$ embeds V into V^{**}. By definition V, which is identified with $\hat{V}$, is a dense subset of $\overline{\hat{V}}$ (the closure of $\hat{V}$ in V^{**}). We call $\overline{\hat{V}}$ the completion of V. If V is complete, then $V \cong \hat{V} = \overline{\hat{V}}$. In general $\overline{\hat{V}} \subseteq V^{**}$.

- It is standard convention to just identify $\hat{u}$ with u itself and $\hat{V}$ with V itself. So we say $V \subseteq V^{**}$.

Definition (77): V is called reflexive if $V = V^{**}$.

5 Functions of Bounded Variation

Unless otherwise stated, the numbers $a.b \in \mathbb{R}$ satisfy $a < b$. Given a partition $P := \{a = x_0 < x_1 < ... < x_m = b\}$ of $[a, b]$, we define

$$\|P\| = \max_{j=1,\ldots,n} (x_j - x_{i-1}).$$

Definition (78): A function $g \in B([a, b])$ is said to be of bounded variation if

$$|g|_{\mathrm{BV}([a,b])} := \sup \left\{ \sum_{j=1}^{m} |g(x_j) - g(x_{j-1})| : m \in \mathbb{N} \text{ and } P \text{ is a partition of } [a, b] \right\} < \infty.$$

The number $|g|_{\mathrm{BV}}$ is called the total variation of g on $[a, b]$. Defining

$$\|\cdot\|_{\mathrm{BV}([a,b])} := \|\cdot\|_\infty + |\cdot|_{\mathrm{BV}([a,b])},$$

the space $(\mathrm{BV}([a, b]), \|\cdot\|_{\mathrm{BV}})$ is a normed space.

Jordan's Decomposition Theorem: A function $g \in B([a, b])$ is of bounded variation if and only if there are two non-decreasing functions $g_1, g_2 \in B([a, b])$ such that $g = g_1 - g_2$.

Definition (79): Given $f \in C([a, b])$ and $g \in \mathrm{BV}([a, b])$, we define the Stieltjes integral

$$\int_a^b f(x)\, dg(x) := \lim_{\substack{\|P\| \to 0 \\ P \text{ is a part.}}} \sum_{j=1}^{m} f(x_j)[g(x_j) - g(x_{j-1})].$$

Remark 80:

- The space $(\mathrm{BV}([a, b]), \|\cdot\|_{\mathrm{BV}})$ is a Banach space.

- If $f \in C([a, b])$ and $g \in \mathrm{BV}([a, b])$, then

$$\left| \int_a^b f(x)\, dg(x) \right| \leq \|f\|_\infty |g|_{\mathrm{BV}([a,b])} \leq \|f\|_\infty \|g\|_{\mathrm{BV}([a,b])}.$$

Proposition (81): We find $F \in (C([a,b]), \|\cdot\|_\infty)^*$ if and only if there is a $g \in \mathrm{BV}([a,b])$ such that

$$F(u) = \int_a^b u(x)\, dg(x), \text{ for all } u \in C([a,b]).$$

Moreover $\|F\|_{C([a,b])^*} = |g|_{\mathrm{BV}([a,b])}$.

Notation: We may use $\langle \cdot, \cdot \rangle_V$ to denote the duality pairing between V and V^*. If $f \in V^*$ and $u \in V$, then $\langle f, u \rangle_V = f(u)$.

Remark 82: Proposition 81 allows us to identify $C([a,b])$ with $\mathscr{M}([a,b])$ (radon measures on $[a,b]$. Given $g \in \mathrm{BV}([a,b])$ and $u \in C([a,b])$, we find $dg \in \mathscr{M}([a,b])$ and

$$\langle dg, u \rangle_{C([a,b])} = \int_a^b u(x)\, dg(x).$$

Notation: Fix $u_0 \in V$, with V a normed space over $\mathbb{R}$, and let L be a closed subspace of V. Let us call the problem of finding $u \in L$ such that

$$\|u - u_0\|_V = \inf_{v \in L} \|v - u_0\| = \alpha. \tag{17}$$

the <u>primal problem</u>. The <u>dual problem</u> requires finding $u^* \in L^*$ such that $\|u^*\|_{V^*} \leq 1$ and

$$\langle u^*, u_0 \rangle = \sup_{\substack{v^* \in L \\ \|v^*\|_{V^*} \leq 1}} \langle V^*, u_0) \rangle = \beta. \tag{18}$$

Here $L^\perp := \{v^* \in V^* : \langle v^*, v \rangle = 0 \text{ for all } v \in L\}$.

Theorem (83): Given $u_0 \in V$. The following hold:

(i) $\alpha = \inf_{v \in L} \|v - u_0\|_V = \sup_{\substack{v^* \in L^\perp \\ \|v^*\| \leq 1}} \langle v^*, u_0 \rangle = \beta.$

(ii) The dual problem in (18) has a solution u^*.

(iii) Given a solution u^* to the dual problem, an element $u \in L$ is a solution to the primal problem (17) if and only if

$$\langle u^*, u_0 - u \rangle_V = \|u - u_0\|_V.$$

Definition (84): Let Ω be a nonempty set. For each $x_0 \in \Omega$, define $\delta_{x_0} \in B(\Omega)^*$ by $\delta_{x_0} := u(x_0)$ for each $u \in B(\Omega)$. The functional δ_{x_0} is called the <u>dirac mass at x_0</u>. Clearly $\|\delta_{x_0}\|_{B(\Omega)^*} = 1$.

Lemma (85): Let $u^* \in C([a,b])^*$ satisfying $\|u^*\|_{C([a,b])^*} \neq 0$ be given. Suppose that

$$\langle u^*, u \rangle_{C([a,b])} = \|u^*\|_{C([a,b])^*} \|u\|_\infty$$

for some $u \in C([a,b])$ such that $x \mapsto |u(x)|$ attains its maximum at exactly one point. Then $u^* = \alpha \delta_{x_0}$ with $|\alpha| = \|u^*\|_{C([a,b])^*}$.

Proposition (90): Let V be a normed space and let $\{u_j\}_{j=1}^{\infty} \subseteq V$ be given.

(a) If $u_j \rightharpoonup u$ as $j \to \infty$, then u is uniquely determined.

(b) If $u_j \rightharpoonup u$ as $j \to \infty$, then $\{u_j\}_{j=1}^{\infty}$ is bounded in V.

(c) If $u_j \rightharpoonup u$ as $j \to \infty$, then for each subsequence $\{u_{j_k}\}_{k=1}^{\infty} \subseteq \{u_j\}_{j=1}^{\infty}$ we have $u_{j_k} \rightharpoonup u$.

(d) If $u_j \to u$, then $u_j \rightharpoonup u$.

Definition (91): Let V be a Banach space, we say that $E \subseteq V$ is weakly sequentially compact if each sequence $\{u_j\}_{j=1}^{\infty} \subseteq E$ has a subsequence $\{u_{j_k}\}_{k=1}^{\infty}$ such that $u_{j_k} \rightharpoonup u$ in V for some $u \in E$.

Theorem (92): Suppose that V is a reflexive Banach space. Then the closed unit ball in V is weakly sequentially compact.

Corollary (93): Suppose that V is a reflexive Banach space. If $\{u_j\}_{j=1}^{\infty} \subset V$ is a sequence in V that is bounded, then there is a subsequence that converges weakly, i.e. there exists $\{u_{j_k}\}_{k=1}^{\infty}$ and $u \in V$ such that $u_{j_k} \rightharpoonup u$ in V.

Definition (94): We say that a topological space X is separable if there is a countable subset $E \subseteq X$ such that $\overline{E} = X$; i.e. E is a countable dense subset.

Theorem (95): Let V be a normed space. If V^* is separable, then so is V.

Theorem (96): Let V be a reflexive Banach space. If $W \subseteq V$ is a closed subspace, then W is also reflexive.

Definition (97): Let V be a normed space. Let $F : E \to \mathbb{R}$ with $E \subseteq V$ be given. We say that $u_0 \in E$ is

1. A (global or absolute) minimum for F in E if $F(u_0) = \inf\limits_{u \in E} F(u)$
2. A local minimizer for F if there is an open neighborhood U of u_0 such that $U \subseteq E$ and $F(u_0) = \inf\limits_{u \in U} F(u)$, in which case $F(u_0)$ is called a local minima.

Definition (98): Let V be a normed space. We say that $E \subseteq V$ is sequentially weakly closed if whenever $\{u_j\}_{j=1}^{\infty} \subseteq E$ converges weakly to $u \in V$, we find $u \in E$. (Examples of sequentially weakly closed sets that are not weakly closed do exist.)

Definition (99): Let V be a normed space. With $E \subseteq V$ we say that $F : E \to \mathbb{R}$ is sequentially weakly lower semicontinuous (swlsc) if for each $u \in E$ and any weakly convergent sequence to u, $\{u_j\}_{j=1}^{\infty} \subseteq E$, we have $\liminf\limits_{j \to \infty} F(u_j) \geq F(u)$.

Theorem (100): Let V be a reflexive Banach space, and let $E \subseteq V$ be a sequentially weakly closed set. Suppose that $F : E \to \mathbb{R}$ satisfies

1. F is sequentially weakly lower semicontinuous on E.

2. If E is not bounded, then $F(u) \to \infty$ as $\|u\|_V \to \infty$.

Then there is a minimizer for F in E.

Theorem (101): (Mazur's Lemma) Let V be a Banach space. Suppose that $K \subseteq V$ is convex and closed in the norm topology. Then K is sequentially weakly closed.

Remark: Actually Mazur's Lemma states that convex sets that are strongly closed are also weakly closed, i.e. K is closed with respect to the weak topology.

Definition (102): Let U be an open neighborhood of $u_0 \in V$. Let $F : U \to \mathbb{R}$ be given. Given $h \in V$, define $\Psi : I \to \mathbb{R}$ by $\Psi(t) = f(u_0 + th)$, where $I \subseteq \mathbb{R}$ is an open interval containing 0. The n^{th} variation of F at u_0 in the direction of h is defined, provided it exists, by

$$\delta^n F(u_0; h) := \Psi^n(0).$$

In particular, the first variation is $\Psi'(0)$.

Definition (103):

1. The functional F in definition 102 has a Gâteaux derivative, $F'(u_0) \in V^*$ at u_0 if $\delta F(u_0; h)$ exists for each $h \in V$ and

$$\delta F(u_0; h) = \langle F'(u_0), h \rangle_V, \text{ for all } h \in V.$$

2. The Gâteaux derivative $F'(u_0)$ is called a Fréchet derivative if

$$\lim_{\|h\| \to 0} \frac{F(u_0 + h) - F(u_0) - \langle F'(u_0), h \rangle}{\|h\|} = 0.$$

Clearly if F has a Fréchet derivative at u_0, then F is continuous at u_0.

Definition (104): With $U \subseteq V$ an open set, we say that $F : U \to \mathbb{R}$ is a C^1-functional is the Fréchet derivative $F'(u) \in V^*$ exists for each $u \in U$ and the mapping $u \mapsto F'(u)$ is continuous in U.

Definition (105): With $U \subseteq V$ an open neighborhood of $u_0 \in V$ and $F : U \to \mathbb{R}$ given, we call u_0 a critical point of F if $\delta F(u_0; h)$ exists for each $h \in V$ and $\delta F(u_0; h)$ exists for each $h \in V$ and $\delta F(u_0; h) = 0$ for all $h \in V$.

Theorem (106): Let $U \subseteq V$ be a neighborhood of $u_0 \in V$ and let $F : U \to \mathbb{R}$ be given. Then the following hold:

1. (Necessary Condition) If F has a local minimum at u_0, then u_0 is a critical point for F. So,

$$\delta F(u_0; h) = 0 \text{ for all } h \in V. \tag{24}$$

If F is Gâteaux differentiable at u_0 then u_0 is a solution to the following operator equation

$$F'(u_0) = 0. \qquad \text{(Euler Equation)}$$

2. (Sufficient Condition) The functional F has a local minimum at u_0 if each of the following hold:

   (a) Condition (24) must be satisfied.

   (b) The second variation $\delta^2 F(u_0; h)$ exists for all u in an open neighborhood of u_0 and all $h \in V$ and there is $\gamma > 0$ such that

   $$\delta^2 F(u_0; h) \geq \gamma \|h\|_V^2 \text{ for all } h \in V.$$

   (c) For each $\epsilon > 0$ there is a $\zeta(\epsilon) > 0$ such that

   $$|\delta^2 F(u; h) - \delta^2 F(u_0; h)| \leq \epsilon \|h\|^2$$

   for all $h \in V$ and all $u \in U$ satisfying $\|u - u_0\|_V < \zeta(\epsilon)$.

Theorem (107): Let $U \subseteq V$ be an open neighborhood of $u_0 \in V$. Let W be a subspace of V. Set

$$\mathscr{A} = \{u \in V : [u - u_0] \in W\}.$$

So $\mathscr{A}$ is an affine space. Suppose $F : \mathscr{A} \to \mathbb{R}$ has a local minimum at $\mathscr{A}$ at u_0, i.e. there is an $\epsilon > 0$ such that $F(u) \geq F(u_0)$ for all $u \in \mathscr{A}$ such that $\|u - u_0\|_V < \epsilon$. Then provided it exists, we have $\delta F(u_0; h) = 0$ for all $h \in W$.
Example (108):

5.1 Game Theory

Definition (109): The ordered pair $(x_0, y_0) \in E \times F$ is called an optimal strategy pair if (x_0, y_0) is a saddle point for the gain function f with respect to $E \times F$; i.e.

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0) \quad \text{for all } (x, y) \in E \times F.$$

The value of $f(x_0, y_0)$ is called the value of the game.

Example (110):

Definition (111): Let $E \subseteq V$ be convex. We call $f : E \to \mathbb{R}$ strictly convex if for each $u, v \in E$ such that $u \neq v$, we have

$$f(\lambda u + (1-\lambda)v) < \lambda f(u) + (1-\lambda)f(v) \text{ for all } \lambda \in (0,1).$$

Theorem (112): (Von Neumann's Saddle Point Theorem) Let V be a strictly convex and reflexive Banach space and suppose that

1. $f : E \times F \to \mathbb{R}$ is given with E and F begin nonempty, closed, convex, and bounded sets in V.
2. For each $y \in F$, the functional $x \mapsto f(x, y)$ is convex and lower semicontinuous.
3. For each $x \in E$, the functional $y \mapsto -f(x, y)$ is convex and lower semicontinuous.

Then the functional f has a saddle point (x_0, y_0) with respect to $E \times F$ and

$$f(x_0, y_0) = \min_{x \in E} \max_{y \in F} f(x, y) = \max_{y \in F} \min_{x \in E} f(x, y). \tag{27}$$

Moreover (27) holds for all saddle points of f.

Remark (113): Hypothesis (i) in Theorem 112 my be replaced by

(ia) $f : E \times F \to \mathbb{R}$ is given with E and F nonempty, closed, and convex subsets of V.

(ib) If E is not bounded, there is a $v_0 \in F$ such that $f(x, v_0) \to \infty$ as $\|x\|_V \to \infty$, and if F is not bounded, there is a $u_0 \in E$ such that $f(u_0, y) \to -\infty$ as $\|y\|_V \to \infty$

Definition (114): A normed space V is strictly convex if the unit ball $\{u \in V : \|u\|_V \leq 1\}$ is strictly convex.

Definition (116): Let V be a Banach space and suppose $F : V \to \mathbb{R}$ has a Gâteaux dertivative $F'(u) \in V^*$ for each $u \in V$. We say that F satisfies the Palais-Smale condition (PS) if $\{u_j\}_{j=1}^{\infty} \subseteq V$ has a convergent subsequence whenever the following are true:

(i) $\{F(u_j)\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$.

(ii) $\lim_{j\to\infty} \|F'(u_j)\|_{V^*} = 0$.

Theorem (117): (Mountain Pass Theorem) Suppose that V is a Banach space and that $H : V \to \mathbb{R}$ is a C^1-functional that satisfies (PS). Assume that

(i) there is an $\alpha, R > 0$ such that $H(u) \geq \alpha$ for all $u \in V$ such that $\|u\|_V = R$.

(ii) $H(0) < \alpha$

(iii) There is a $u_0 \in V$ such that $\|u_0\|_V > R$ and $H(u_0) < \infty$.

Set $M := \{p \in C([0,1], V) : p(0) = 0 \text{ and } p(1) = u_0\}$. Then there is a $u_* \in V$ that is a critical point for H; i.e. $H'(u_0) = 0$. Moreover

$$H(u_*) = \inf_{p \in M} \sup_{t \in [0,1]} H(p(t)) =: c_* \quad \text{and} \quad c_* \geq \alpha.$$

Definition (118): Suppose that $E \subseteq V$ is a nonempty closed subset of a Banach space V. We will call $u \in E$ an (ϵ, λ)-quasiminimizer of $F : E \to \mathbb{R}$ if $F(u) < F(v) + \frac{\epsilon}{\lambda} \|u - v\|_V$ for all $v \in E$ such that $u \neq v$.

Theorem (119): (Ekeland's Variational Principle) Let V be a Banach space and suppose that $E \subseteq V$ is non-empty and closed. Assume that $F : E \to \mathbb{R}$ is lower semicontinuous on E. Let $\epsilon, \lambda > 0$ be given. Given $v \in E$ such that

$$F(v) \leq \inf_{u \in E} F(u) + \epsilon,$$

there is an (ϵ, λ)-quasiminimizer $u_* \in E$ of F such that $\|u - v\|_V \leq \lambda$ and $F(u_*) \leq F(v)$.

Corollary (120): Suppose that $F : V \to \mathbb{R}$ is lower semicontinuous and suppose that the Gâteaux derivative $F'(u) \in V^*$ exists for all $u \in V$. Then for each $\epsilon > 0$, there is a $u_\epsilon \in V$ such that

$$F(u_\epsilon) \leq \inf_{u \in V} F(u) + \epsilon \quad \text{and} \quad \|F'(u_\epsilon)\|_{V^*} \leq \epsilon.$$

Theorem (121): The BVP

$$\begin{cases} -\Delta u(x) = g(u(x)), & \text{for } x \in \Omega \\ u(x) = 0, & \text{for } x \in \partial\Omega \end{cases}$$

has at least one non-trivial weak solution.

6 Gelerkin's Method

Theorem (122): Suppose that V is a separable, reflexive Banach space. Assume that $A : V \to V^*$ satisfies the following:

(i) A is monotone, i.e. $\langle A(u) - A(v), u0v \rangle_V \geq 0$.

(ii) A is continuous on each finite dimensional subspace of V.

(iii) A is coercive, i.e.

$$\lim_{\|u\|_V \to \infty} \frac{\langle A(u) - A(v) \rangle}{\|u\|_V} = \infty.$$

Then the operator equation

$$A(u) = b \tag{4}$$

has a solution for each $b \in V^*$.

Lemma (123): Let $R > 0$ be given. Suppose that $\{g_j\}_{j=1}^n \subset C(\overline{B}_R)$ satisfies the following condition: For each $x = (x_1, ..., x_n) \in \mathbb{R}^n$ with $\|x\|_2 = R$ we find

$$\sum_{j=1}^{n} g_j(x) x_j \geq 0.$$

Then the system of equations

$$g_j(x) = 0, \quad \text{for all } j \in \{1, ..., n\}$$

has a solution in $\overline{B}_R$.

Lemma (124): (Browder and Minty's Monotonicity Trick) Suppose that V is a reflexive separable Banach space. Assume that $A : V \to V^*$ satisfies

(i) A is monotone

(ii) A is continuous on finite dimensional subspaces.

Let $b \in V^*$ be given. If $\{u_j\}_{j=1}^{\infty} \subset V$ such that $u_j \rightharpoonup u$ in V, $A(u_j) \rightharpoonup b$ in V^*, and $\langle A(u_j), u_j \rangle_V \to \langle b, u \rangle_V$. Then $A(u) = b$.

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