Joe's Math 825/826 Qualifying Exam Review Sheet

I note here that I did not produce any of the material in this review. Although I have modified the order in which some things appear, the layout of the material follows Davidson's and Donsig's Real Analysis with Real Applications, cited as [1] throughout. My goal in producing this quick reference was to review the material on the Math 825/826 Qualifying Exams at the University of Nebraska-Lincoln. I post it on my website as a reference for other students planning to take the Math 825/826 Qualifying Exam.

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1 Metric Spaces

1.1 Definitions and Examples

1. [1] **Definition:** A **metric space** is a set X with a function $d : X \times X \to [0, \infty)$,called a **metric**, which satisfies the following properties:

   (a) (positive definite) $d(x, y) = 0$ if and only if $x = y$,

   (b) (symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$,

   (c) (triangle inequality) $d(x, z) = d(x, y) + d(y, z)$ for all $x, y, z \in X$.

2. [1] **Discrete Metric:**

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}.$$

3. [1] **P-adic Metric on $\mathbb{Z}$:**

$$d(m,n) = \begin{cases} 0, & \text{if } m = n \\ 2^{-d}, & \text{if } m \neq n \end{cases},$$

   where d is the highest power of 2 that divides $m - n$.

4. [1] **Hausdorff Metric:** If x is a closed subset of $\mathbb{R}^n$, let $K(X)$ denote the collection of all nonempty compact subsets of X. Then, the Hausdorff metric is

$$d_H(A,B) = \max\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}.$$

5. [1] **Theorem:** Let $A_\epsilon = \{x \in \mathbb{R}^n : d(x, A) \leq \epsilon\}$. Then, $d_H(A, B) \leq \epsilon$ if and only if $A \subset B_\epsilon$ and $B \subset A_\epsilon$.

6. [1] **Definition:** The **ball** $B^d_r(x)$ of radius $r > 0$ about a point x is defined as $\{y \in X : d(x, y) < r\}$.

7. [1] **Definition:** A subset U is **open** if for every $x \in U$, there is an $r > 0$ so that $B_r(x) \subset U$.

8. [1] **Definition:** A sequence $(x_n)_{n=1}^\infty$ is said to converge to x if $\lim_{n\to\infty} d(x, x_n) = 0$.

9. [1] **Definition:** A set C is **closed** if it contains all limit points of sequences of points in C.

10. [1] **Definition:** A sequence $(x_n)_{n=1}^\infty$ in a metric space (X, d) is a **Cauchy sequence** if for every $\epsilon > 0$, there is an integer N so that $d(x_i, x_j) < \epsilon$ for all $i, j \geq N$.

11. [1] **Definition:** A metric space is **complete** if every Cauchy sequence converges in X.

12. [1] **Theorem:** A closed subset of a complete metric space is complete if and only if it is closed.

13. [1] When X is a closed subset of $\mathbb{R}^n$, the metric space $(K(X), d_H)$ is complete.

14. [1] **Definition:** Two metrics ρ and σ on a set X are **topologically equivalent**if for each $x \in X$ and $r > 0$, there is an $s = s(r, x) > 0$ so that $B^\rho_s(x) \subset B^\sigma_r(x)$ and $B^\sigma_s(x) \subset B^\rho_r(x)$.

15. [1] **Theorem:** Two topologically equivalent metrics have the same open and closed sets and the same convergents sequences.

16. [1] **Definition:** A function f from a metric space (X, ρ) into a metric space (Y, d) is **continuous** if for every $x_0 \in X$ and $\epsilon > 0$, there is a $\delta > 0$ so that $d(f(x), f(x_0)) < \epsilon$ whenever $\rho(x, x_0) < \delta$.

17. [1] **Theorem:** Let f map a metric space (X, ρ) into (Y, σ). The following are equivalent:

   (a) f is continuous on X,

(b) for every convergent sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = a$ in X, we have $\lim_{n\to\infty} f(x_n) = f(a)$,

(c) $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for every open set U in Y.

18. [1] The space $C_b(X)$ of all bounded continuous functions on a metric space X with the sup norm $\|f\| = \sup\{|f(x)| : x \in X\}$ is complete.

19. [1] **Theorem:** A set U is open in (X, d) if and only if $X \backslash U$ is closed.

1.2 Compactness and Metric Spaces

1. [1] **Definition:** A collection of open sets $\{U_\alpha : \alpha \in A\}$ in X is called an **open cover** of $Y \subset X$ if $Y \subset \cup_{\alpha\in A} U_\alpha$.

2. [1] **Definition:** A **subcover** of $\{U_\alpha : \alpha \in A\}$ is a subcollection $\{U_\alpha : \alpha \in B\}$ for some $B \subset A$ that is still a cover.

3. [1] A collection of closed sets $\{C_\alpha : \alpha \in A\}$ has the **finite intersection property** if every finite subcollection has nonempty intersection.

4. [1] **Definition:** A metric space X is **totally bounded** if for every $\epsilon > 0$, there are finitely many points $x_1, ...x_n \in X$ so that $\{B_\epsilon(x_i) : 1 \leq i \leq n\}$ is an open cover.

5. [1] **Theorem:** (**Borel-Lebesgue Theorem**) For a metric space X, the following are equivalent.

   (a) (Compact) Every open cover of X has a finite subcover.

   (b) Every collection of closed subsets of X with the finite intersection property has nonempty intersection.

   (c) (Sequentially Compact) Every sequence of points in X has a convergent subsequence.

   (d) X is complete and totally bounded.

6. [1] **Theorem:** Let C be a compact subset of a metric space (X, d). If f is a continuous function from (X, d) into (Y, σ), then the image set $f(C)$ is compact.

7. [1] **Definition:** A metric space is **seperable** if it has a countable dense subset.

8. [1] **Theorem:** Every compact metric space is seperable.

9. [1] **Theorem: Cantor's Intersection Theorem** A decreasing sequence of nonempty compact subsets $A_1 \supset A_2 \supset A_3 ...$ of a metric space (X, ρ) has nonempty intersection.

1.3 Connected Sets

1. [1] **Definition:** A subset A of a metric space X is **not connected** if there are disjoint open sets U and V such that $A \subset U \cup V$ and $A \cap U \neq \emptyset \neq A \cap V$. Otherwise, A is said to be **connected**.

2. [1] **Definition:** A set A is **totally disconnected** if for every pair of distinct points $x, y \in A$, there are two disjoint open sets U and V so that $x \in U$ and $y \in V$ and $A \subset U \cup V$.

3. [1] **Theorem:** The continuous images of a connected set is connected.

4. [1] **Definition:** A **path** in a metric space is a continuous image of a closed interval.

5. [1] **Definition:** A subset S of a metric space X is said to be **path connected** if for each pair of points x and y in S, there is a path in S connecting x and y, meaning that there is a continuous function f from $[0, 1]$ into S such that $f(0) = x$ and $f(y) = 1$.

6. [1] **Corollary:** Every path connected set is connected.

2 Normed Vector Spaces and Inner Product Spaces

2.1 Norm Definition and Topology

1. [1] **Definition:** A **normed vector space**, $(V, \|\cdot\|)$, is a vector space, V, and a function, $\|\cdot\|$ acting on V, where $\|\cdot\|$ takes on values in $[0, +\infty)$ and has the following properties:
   (a) (Positive Definite) $\|x\| = 0$ if and only if $x = 0$,
   (b) (Homogeneous) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$,
   (c) (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.
2. [1] **Important Norms:**
   (a) (Traffic Norm) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.
   (b) (Max, Infinity, or Sup Norm) $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$.
   (c) (Uniform Norm) $\|f\| = \sup_{x\in K} |f(x)|$.
   (d) (L^p Norm) $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}$, where $p \in [1, \infty)$.
3. [1] **Definition:** In a normed vector space $(V, \|\cdot\|)$, we say that a sequence $(v_n)_{n=1}^\infty$ **converges** to $v \in V$ if $\lim_{n\to\infty} \|v_n - v\| = 0$. Equivalently, for every $\epsilon > 0$, there is an integer $N > 0$ so that $\|v_n - v\| < \epsilon$ for all $n \geq N$. This is written $\lim_{n\to\infty} v_n = v$.
4. [1] For recursively defined sequences, find a fixed point and show the sequence converges to this point. (Typically involves induction.)
5. [1] **Definition:** A sequence $(v_n)_{n=1}^\infty$ is a **Cauchy sequence** provided for every $\epsilon > 0$, there is an integer $N > 0$ so that $\|v_n - v_m\| < \epsilon$ for all $n, m \geq N$.
6. [1] **Definition:** We say $(V, \|\cdot\|$ is **complete** if every Cauchy sequence in V converges to some vector $v \in V$.
7. [1] **Definition:** A complete normed vector space is called a **Banach Space**.
8. [1] **Definition:** A point $\mathbf{x}$ is a **limit point** of a subset C of V if there is a sequence $(\mathbf{x}_n)_{n=1}^\infty$ with $\mathbf{x}_n \in C$ such that $\mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n$.
9. [1] **Definition:** A subset C of V is **closed** if it contains all of its limit points.
10. [1] **Definition:** If $C$is a subset of V, the **closure** of C is the set $\overline{C}$ consisting of all limit points of C.
11. [1] **Definition:** The ball about $\mathbf{a}$ in V of radius r is the set $B_r(\mathbf{a}) = \{\mathbf{v} \in V : \|v - a\| < r\}$.
12. [1] **Definition:** A subset C of V is **open** if for every $\mathbf{a} \in U$, there is some $r = r(\mathbf{a}) > 0$ so that the ball $B_r(\mathbf{a}) \subset U$.
13. [1] **Theorem:** A set $C \subset V$ is open if and only if the complement of C, C^c is closed.
14. [1] **Proposition:** A sequence x_n in a normed vector space V converges to a vector x if and only if for each open set U containing x, there is an integer N so that $x_n \in U$ for all $n \geq N$.
15. [1] **Definition:** A subset K os a normed vector space V is **compact** if every sequence $(x_n)_{n=1}^\infty$ of points in K has a subsequence $(x_{n_i})_{i=1}^\infty$ which converges to a point in K.
16. [1] **Theorem:** A compact subset of a normed vector space is closed and bounded. (Note: Heine-Borel only holds in $\mathbb{R}^n$.)

17. [1] **Theorem:** The following are equivalent for a normed vector space $(V, \|\cdot\|)$:
    (a) $(V, \|\cdot\|)$ is complete.
    (b) Every decreasing sequence of closed balls has a nonempty intersection.
    (c) Every decreasing sequence of closed balls with radii going to zero has nonempty intersection.
18. [1] **Definition:** A normed vector space V is **strictly convex** if $\|u\| = \|v\| = \|(u+v)/2\| = 1$ for vectors $u, v \in V$ implies that $u = v$.

2.2 Inner Product Spaces

1. [1] **Definition:** An **inner product space** is a vector space V with a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying the following properties:
    (a) (Positive Definite) $\langle x, x \rangle \geq 0$ for all $x \in V$ and $\langle x, x \rangle = 0$ only if $x = 0$.
    (b) (Symmetry) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$
    (c) (Bilinearity) For all $x, y, z \in V$ and scalars $\alpha, \beta \in \mathbb{R}$, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.
2. [1] Every inner product forms a norm by
$$\|x\| = \langle x, x \rangle^{1/2}.$$
3. [1] **Theorem:** (**Cauchy-Schwarz Inequality**) For all vectors x, y in an inner product space V,
$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$
Equality holds if and only if x and y are colinear. Proof follows from considering $\langle x - ty, x - ty \rangle$, where $t = \pm \|x\| / \|y\|$.
4. [1] **Corollary:** An inner product space V satisfies the triangle inequality
$$\|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in V.$$
Equality implies x and y are colinear. (Proof follows from considering $\|x + y\|^2$ and using Cauchy-Schwarz inequality.)
5. [1] Norms induced by inner product spaces are strictly convex.

2.3 Orthonormal Sets

1. [1] **Definition:** Two vectors x and y are **orthogonal** if $\langle x, y \rangle = 0$.
2. [1] **Definition:** A collection of vectors $\{e_n : n \in S\}$ in V is called **orthonormal** if $\|e_n\| = 1$ for all $n \in S$ and $\langle e_n, e_m \rangle = 0$ for $n \neq m \in S$.
3. [1] **Definition:** An orthonormal set $\{e_n : n \in S\}$ in V is an **orthonormal basis** if it spans V.
4. [1] **Proposition:** An orthonormal set is linearly independent. So, an othonormal basis for a finite-dimensional inner product space is a basis.
5. [1] (**Gram-Shmidt Process**) Provides an orthonormal set with the same span as some original set of vectors $\{x_1, x_2, ..., x_n\}$. Set $y_1 = x_1$. Then, set
$$y_{k+1} = x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, y_i \rangle \frac{y_i}{\|y_i\|^2} \quad \text{for } 1 \leq k < n.$$

6. [1] **Lemma:** Let $\{e_1, ...e_n\}$ be an orthonormal set in an innder product space V. If M is the subspace spanned by $\{e_1, ...e_n\}$, then every vector $x \in M$ can be written uniquely as $\sum_{i=1}^{n} a_i e_i$, where $a_i = \langle x, e_i \rangle$. In other words, the set $\{e_1, ...e_n\}$ is linearly independent.
   Moreover, for each y in V with $\langle y, e_i \rangle = \beta_i$ and each $x = \sum_{j=1}^{n} \alpha_j e_j$ in M,
$$\langle x, y \rangle = \sum_{i=1}^{n} \alpha_i \beta_i.$$
   In particular, $\|x\|^2 = \sum_{j=1}^{n} \alpha_j^2$. (Tells us every finite dimensional vector space behaves like $\mathbb{R}^n$.
7. [1] **Corollary:** If V is an inner product space of finite dimension n, then it has an orthonormal basis $\{e_i : 1 \leq i \leq n\}$ and the inner product is given by
$$\left\langle \sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j \right\rangle = \sum_{i=1}^{n} \alpha_i \beta_i$$
   and the norm is
$$\left\| \sum_{i=1}^{n} \alpha_i e_i \right\| = \left(\sum_{i=1}^{n} \alpha_i^2 \right)^{1/2}.$$

2.4 Orthogonal Expansions

1. [1] **Definition:** A **projection** is a linear map P such that $P^2 = P$.
2. [1] **Definition:** An **orthogonal projection** is a projection where $\ker P = \{v \in V : Pv = 0\}$ is orthogonal to $\text{Ran}P = PV$.
3. [1] **Pythagorean Identity:** (Only if P is an orthogonal projection)
$$\|x\|^2 = \|Px\|^2 + \|(I - P)x\|^2.$$
4. [1] **Theorem:** (**Projection Theorem**) Let $\{e_1, ..., e_n\}$ be an orthonormal set in an innder product space V and let M be the subspace spanned by $\{e_1, ..., e_n\}$. Define $P : V \to M$ by $Py = \sum_{j=1}^{n} \langle y, e_j \rangle e_j$, for each $y \in V$. Then, P is the orthogonal projection onto M and
$$\|y\|^2 = \sum_{j=1}^{n} \langle y, e_j \rangle^2.$$
   Moreover, for all $v \in M$,
$$\|y - v\|^2 = \|y - Py\|^2 + \|Py - v\|^2.$$
   In particular, Py is the closest vector in M to y.
5. [1] **Theorem:** (**Bessel's Inequality**) Let $S \subseteq N$ and let $\{e_n : n \in S\}$ be an orthonormal set in an inner product space V. For $x \in V$,
$$\sum_{n \in S} |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

6. [1] **Definition:** A complete inner product space is called a **Hilbert Space**.

7. [1] **Definition:** In a Hilbert space, the **closed span** of a set of vectors S, denoted $\overline{\text{span } S}$, is the closure of the linear subspace spanned by S.

8. [1] **Definition:** A Hilbert space is **seperable** if every orthonormal set is countable.

9. [1] **Theorem:** (**Parseval's Theorem**) Let $S \subset N$ and $E = \{e_n : n \in S\}$ be an orthonormal set in a Hilbert space H. Then, the subspace $M = \overline{\text{span } E}$ consists of all vectors $x = \sum_{n \in S} \alpha_n e_n$, where the coefficient sequence $(a_n)_{n=1}^{\infty}$ belongs to l^2. Further, if x is a vector in H, then x belongs to M if and only if

$$\sum_{n \in S} |\langle x, e_n \rangle|^2 = \|x\|^2 .$$

10. [1] **Corollary:** Let $E = \{e_n : n \in S\}$ be an orthonormal set in a Hilbert space H. Then there is a continuous linear orthogonal projection P_E of H onto $M = \overline{\text{span } E}$ given by $P_E x = \sum_{n \in S} \langle x, e_n \rangle e_n$.

11. [1] **Corollary:** If $E = \{e_i : i \geq 1\}$ is an orthonormal basis for a Hilbert space H, every vector $x \in H$ may be uniquely expressed as $x \sum_{i=1}^{\infty} \alpha_i e_i$, where $\alpha_i = \langle x, e_i \rangle$.

2.5 Finite-Dimensional Normed Spaces

1. [1] **Lemma:** If $\{v_1, ..., v_n\}$ is a linearly independent set in a normed vector space $(V, \|\cdot\|)$, then there exists positive constants $0 < c < C$ so that for all $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{R}^n$, we have

$$c \|\mathbf{a}\|_2 \leq \left\| \sum_{i=1}^{n} a_i v_i \right\| \leq C \|\mathbf{a}\|_2 .$$

2. [1] **Corollary:** Suppose that V is an n-dimensional normed space with basis $\{v_1, ..., v_n\}$. Then, the maps T, where $T\mathbf{a} = \sum_{i=1}^{n} a_i v_i$, and T^{-1}, where $T^{-1} \sum_{i=1}^{n} a_i v_i = \mathbf{a}$, are both Lipschitz, and therefore continuous. Thus, a set A in $\mathbb{R}^n$ is closed, bounded, open, or compact if and only if $T(A)$ is closed, bounded.open, or compact respecitively in V.

3. [1] **Corollary:** A subset of a finite-dimensional normed vector space is compact if and only if it is closed and bounded.

4. [1] **Corollary:** A finite-dimensional subspace of a normed vector space is complete, and in particular it is closed.

5. [1] Let $(V, \|\cdot\|)$ be a normed vector space, and let W be a finite dimensional subspace of V. Then, for any $v \in V$ there is at least one closest point $w^* \in W$ so that $\|v - w*\| = \inf\{\|v - w\| : w \in W\}$.

3 Topology of $\mathbb{R}^n$

3.1 Definitions and Basic Results

1. [1] **Inner Product:** (Linear)

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

2. [1] **Norm:**

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

3. [1] **Distance:**

$$\|x - y\| = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$$

4. [1] **Triangle Inequality:**

$$\|x + y\| \leq \|x\| + \|y\|$$

Equality holds if and only iff either $x = 0$ or $y = cx$ with $c \geq 0$. Proof follows from considering $\|x + y\|^2$ and using Cauchy-Schwarz Inequality.

5. [1] **Lemma:** Let $\{v_1, ..v_m\}$ be an orthonormal set in $\mathbb{R}^n$. Then,

$$\left\|\sum_{i=1}^{n} a_i v_i\right\| = \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2}.$$

An othonormal set in $\mathbb{R}^n$ is linearly independent. So, an orthonormal basis for $\mathbb{R}^n$ is a basis and has exactly n elements.

6. [1] **Pythagorean Formula:**

$$\|x + y\| = \left(\|x\|^2 + \|y\|^2\right)^{1/2}$$

7. [1] **Parallelogram Law:** (All norms formed from inner products must satisfy this.)

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

3.2 Convergence and Completeness in $\mathbb{R}^n$

1. [1] **Lemma:** Let $(\mathbf{x}_k)$ be a sequence in $\mathbb{R}^n$. Then, $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$ if and only if $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{a}\| = 0$.

2. [1] **Lemma:** A sequence $\mathbf{x}_k = (x_{k,1}, ..., x_{k,2})$ in $\mathbb{R}^n$ converges to a point $\mathbf{a} = (a_1, .., a_n)$ if and only if

$$\lim_{k\to\infty} x_{k,i} = a_i \text{for} 1 \leq i \leq n.$$

3. [1] $\mathbb{R}^n$ is a Banach Space!

3.3 Closed, Open, and Heine-Borel

1. [1] **Proposition:**

   (a) If $A, B \subset \mathbb{R}^n$ are closed, then $A \cup B$ is closed.

   (b) If $\{A_i : i \in I\}$ is a family of closed subsets of $\mathbb{R}^n$, then $\cap_{i\in I} A_i$ is closed.

   (c) If U, V are open subsets of $\mathbb{R}^n$, then $U \cap V$ is an open subset of $\mathbb{R}^n$.

   (d) If $\{U_i : i \in I\}$ is a family of open subsets of $\mathbb{R}^n$, then $\cup_{i\in I} U_i$ is open.

2. [1] **Definition:** The **interior** of X, or int$X$ is the largest open subset of X. If the interior of a set X is empty, then we say X has **empty interior**.

3. [1] **Definition:** A set S is **dense** in a set T if $T \subset \overline{S}$. Another way to think of this is to show every element of T is a limit point of a sequence in S.

4. [1] **Definition:** A point $\mathbf{x}$ is a **cluster point** of a subset A of $\mathbb{R}^n$ if there is a sequence $(\mathbf{a}_n)_{n=1}^{\infty}$ with $\mathbf{a}_n \in A \backslash \{x\}$ such that $\mathbf{x} = \lim_{n\to\infty} \mathbf{a}_n$. In other words, $\mathbf{x}$ is a limit point of a sequence in A that doesn't contain $\mathbf{x}$.

5. [1] **Definition:** A subset S of $\mathbb{R}^n$ is called **bounded** provided there is a real number M such that $S \subset B_M(0)$, or equivalently $\sup_{x\in S} \|x\| < \infty$.

6. [1] **Lemma:** If C is a closed subset of a compact subset of $\mathbb{R}^n$, then C is compact.

7. [1] **Lemma:** The cube $[a, b]$ is a compact subset of $\mathbb{R}^n$.

8. [1] **Theorem:** (**Heine-Borel**) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

9. [1] **Theorem:** (**Cantor's Intersection Theorem** If $A_1 \supset A_2 \supset A_3 ...$ is a decreasing sequence of nonempty compact subsets of $\mathbb{R}^n$, then $\cap_{k\geq 1} A_k$ is not empty.

10. [1] **Definition:** A set whose closure has no interior is **nowhere dense**.

11. [1] **Definition:** A point x of a set A is **isolated** if there is an $\epsilon > 0$ so that $B_\epsilon(x) \cap A = \{x\}$.

12. [1] C is a perfect set if every point in C is the limit of a sequence of other points in C. (Every point is a cluster point.)

13. [1] **Properties of the Cantor Set:**
    (a) It is compact.
    (b) It has empty interior, or is nowhere dense.
    (c) It has not isolated points.
    (d) Furthermore, It is a perfect set.
    (e) The decimal expansion is given by
    $$x = \sum_{k\geq 0} 3^{-k} x_k.$$
    (f) It has measure zero.

4 Sequences in $\mathbb{R}$

1. [1] **Theorem:** (**The Squeeze Theorem**) Suppose that three sequences $(a_n), (b_n)$, and (c_n) satisfy
$$a_n \leq b_n \leq c_n \text{ for all } n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \infty.$$
Then, $\lim_{n\to\infty} b_n = L$.

2. [1] **Definition:** If S is a nonempty subset of $\mathbb{R}$ that is bounded above, the **supremum**, or **least upper bound**, is the number L such that
$$s \leq L \text{ for all } s \in S$$
and whenever L' is another upper bound for S, then $L' \geq L$.

3. [1] **Definition:** If S is a nonempty subset of $\mathbb{R}$ that is bounded below, the **infimum**, or **greatest lower bound** is the number L such that
$$s \geq L \text{ for all } s \in S$$
and whenever L' is another lower bound for S, then $L' \leq L$.

4. [1] **Theorem:** (**Least Upper Bound Principle**) Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum. Similarly, every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.

5. [1] **Theorem:** (**Monotone Convergence Theorem**) A monotone increasing sequence that is bounded above converges. A monotone decreasing sequence that is bounded below converges.

6. [1] **Definition:** The **limit superior** of a bounded sequence $(a_n)_{n=1}^{\infty}$ is given by
$$\lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

7. [1] **Definition:** The **limit inferior** of a bounded sequence $(a_n)_{n=1}^{\infty}$ is given by
$$\lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

8. [1] **Lemma:** (**Nested Interval's Lemma**) Suppose that
$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$$
are nonempty closed intervals such that I_{n+1} is contained in I_n for each $n \geq 1$. Then, the intersection $\cap_{n\geq 1} I_n$ is nonempty.

9. [1] **Theorem:** (**Bolzano-Weiestrass Theorem**) Every bounded sequence of real numbers has a convergent subsequence.

10. [1] **Theorem:** (**Completeness Theorem**) A sequence of real numbers converges if and only if it is Cauchy.

5 Series

5.1 Definitions and Basic Results

1. [1] **Definition:** If $(a_n)_{n=1}^{\infty}$ is a sequence of numbers, then the **infinite series** with terms a_n is the formal expression $\sum_{n=1}^{\infty} a_n$..

2. [1] **Definition:** We say a series **converges**, or is **summable**, if the sequence of partial sums $(s_n)_{n=1}^{\infty}$, where $s_n = \sum_{k=1}^{n} a_k$, converges. That is, $\lim_{n\to\infty} s_n = L < \infty$.

3. [1] **Definition:** A series that does not converge is said to **diverge**.

4. [1] (**Harmonic Series**)
$$\sum_{k=1}^{\infty} \frac{1}{k}$$

5. Don't forget to check for a series that can be written as a telescoping series in which terms will cancel.

6. [1] **Theorem:** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

7. [1] **Theorem:** (**Cauchy Criterion for Series**) The following are equivalent for a series $\sum_{n=1}^{\infty} a_n$.

   (a) The series converges.

   (b) For every $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $\left|\sum_{k=n+1}^{\infty} a_k\right| < \epsilon$ for all $n \geq N$.

   (c) For every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that $\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon$ if $m, n \geq N$.

5.2 Convergence Tests for Series

1. [1] **Proposition:** If $a_k \geq 0$ for $k \geq 1$ and $s_n = \sum_{k=1}^{n} a_k$, then either

   (a) $(s_n)_{n=1}^{\infty}$ is bounded above, in which case $\sum_{n=1}^{\infty} a_n$ converges,

or

   (b) $(s_n)_{n=1}^{\infty}$ is unbounded, in which case $\sum_{n=1}^{\infty} a_n$ diverges.

2. [1] **Theorem:** (**Geometric Series**) A geometric series converges if $|r| < 1$. Moreover, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

3. [1] **Theorem:** (**Comparison Test**) Consider two sequences of real numbers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ with $|a_n| \leq b_n$ for all $n \geq 1$. If $(b_n)_{n=1}^{\infty}$ is summable, then $(a_n)_{n=1}^{\infty}$ is summable and

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} b_n.$$

   If $(a_n)_{n=1}^{\infty}$ is not summable, then $(b_n)_{n=1}^{\infty}$ is not summable.

4. [1] **Theorem:** (**Root Test**) Suppose that $a_n \geq 0$ for all $n \geq 1$ and let $l = \limsup \sqrt[n]{a_n}$. If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges, and if $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

5. [1] **Theorem:** (**Ratio Test**) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of positive terms. If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Conversely, if $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

6. [1] **Theorem:** (**Leibniz Alternating Series Test**) Suppose that $(a_n)_{n=1}^{\infty}$ is a monotone decreasing sequence $a_1 \geq a_2 \geq a_3 \geq ... \geq 0$ and that $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

7. [1] **Corollary:** Suppose that $(a_n)_{n=1}^{\infty}$ is a monotone decreasing sequence $a_1 \geq a_2 \geq a_3 \geq ... \geq 0$ and that $\lim_{n\to\infty} a_n = 0$. Then the difference between the sum of the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and the N^{th} partial sum is at most $|a_N|$.

8. [1] **Theorem:** (**Limit Comparison Test**) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $b_n \geq 0$ such that $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

9. [1] **Theorem:** (**Integral Test**) Let $f(x)$ be a positive monotone decreasing function on $[1,\infty)$. Then, $(f(n))_{n=1}^{\infty}$ is summable if and only if $\int_1^{\infty} f(x)dx < \infty$.

5.3 Absolute and Conditional Convergence

1. [1] **Definition:** A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

2. [1] **Definition:** A series that converges but does not converge absolutely is **conditionally convergent**.

3. [1] **Proposition:** An absolutely convergent series converges.

4. [1] **Definition:** A **rearrangement** of a series $\sum_{n=1}^{\infty} a_n$ is the series $\sum_{n=1}^{\infty} a_{\pi(n)}$, where $\pi(n)$ is a permutation of the natural numbers $\mathbb{N}$.

5. [1] **Proposition:** For an absolutely convergent series, every rearrangement converges to the same limit.

6. [1] **Lemma:** Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Denote the positive terms as $b_1, b_2, b_3, ...$ and the other terms as $c_1, c_2 c_3, ...$

   (a) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then so are both $\sum_{n=1}^{\infty}$ and $\sum_{n=1}^{\infty} |c_n|$, and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |c_n|$.

   (b) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |c_n|$ both diverge.

7. [1] **Theorem:** (**Rearrangement Theorem**) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent series, then for every real number L, there is a rearrangement that converges to L.

8. [1] **Lemma:** (**Summation by Parts**) Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences of real numbers and define $X_n = \sum_{k=1}^{n} x_k$ and $Y_n = \sum_{k=1}^{n} y_k$. Then

$$\sum_{n=1}^{m} x_n Y_n + \sum_{n=1}^{m} X_n y_{n+1} = X_m Y_{m+1}.$$

9. [1] **Theorem:** (**Dirichlet's Test**) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers with bounded partial sums:

$$\left|\sum_{k=1}^{n} a_k\right| \leq M < \infty \text{ for all } n \geq 1.$$

   If $(b_n)_{n=1}^{\infty}$ is a sequence of positive numbers decreasing monotonically to 0, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

10. [1] **Theorem:** (**Abel's Test**) If $\sum_{n=1}^{\infty} a_n$ converges and $(b_n)_{n=1}^{\infty}$ is a monotonic convergent sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

6 Functions

6.1 Limits and Continuity

1. [1] **Definition:** Let $S \subset \mathbb{R}^n$ and let f be a continuous function from S into $\mathbb{R}^m$. If $\mathbf{a}$ is a limit point of $S \backslash \{a\}$, then the point $\mathbf{v} \in \mathbb{R}^m$ is the **limit** of f at $\mathbf{a}$ if for every $\epsilon > 0$, there is a $\delta > 0$ so that
$$\|f(x) - v\| < \epsilon \text{ whenever } 0 < \|x - a\| < \delta \text{ and } x \in S.$$
2. [1] **Definition:** Let $S \subset \mathbb{R}^n$ and let f be a function from S into $\mathbb{R}^m$. We say that f is **continuous at a** $\in S$ if for every $\epsilon > 0$, there is $\delta > 0$ such that, for all $\mathbf{x} \in S$ with $\|x - a\| < \delta$, we have $\|f(x) - f(a)\| < \epsilon$.
3. [1] **Definition:** f is **continuous on** S if it is continuous at each point $\mathbf{a} \in S$.
4. [1] **Definition:** If f is not continuous, we say it is **discontinous**.
5. [1] **Definition:** A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is called a **Lipschitz function** if there is a constant C such that
$$\|f(x) - f(y)\| \leq C \|x - y\| \text{ for all } x, y \in S.$$
The **Lipschitz constant** of f is the smallest choice of C for which the previous condition holds.
6. [1] **Proposition:** Every Lipschitz function is continuous.
7. [1] **Corollary:** Every linear transformation A from $\mathbb{R}^n$ to $\mathbb{R}^m$ is Lipschitz, and therefore is continuous.

6.2 Discontinuous Functions

1. [1] **Heaviside Function**
$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$
2. [1] **Definition:** We say that L is the **right-hand limit** at a if for every $\epsilon > 0$, there is a $\delta > 0$ so that
$$|f(x) - L| < \epsilon \text{ for all } a < x < a + \delta.$$
Similarly, we have define a **left-hand limit**.
3. [1] **Definition:** A function f has a jump discontinuity at a if the right-hand limit at a differs from the left-hand limit at a.
4. [1] **Definition:** Say that the limit of a function $f(x)$ as x approaches a is $+\infty$ if for every integer N, there is a $\delta > 0$ so that
$$f(x) > N \text{ for all } 0 < |x - a| < \delta.$$
Similarly we can define a limit of $-\infty$.
5. [1] **Characteristic Function**
$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

6.3 Properties of Continuous Functions, Compactness, and Extreme Values

1. [1] **Definition:** A subset $V \subset S \subset \mathbb{R}^n$ is **open in** S or relatively open (with respect to S) if there is an open set U in $\mathbb{R}^n$ such that $U \cap S = V$. Another way to say this is, V is open in S if for every $\mathbf{v} \in V$, there is an $\epsilon > 0$ so that $B_\epsilon(\mathbf{v}) \subset V$.
2. [1] **Reference** See theorem in 1.1.17 on this sheet for a rebriefing of continuity, sequential continuity, and topological continuity.
3. [1] **Theorem:** Suppose that f maps a domain $S \subset \mathbb{R}^n$ into $T \subset \mathbb{R}^m$, and g maps T into $\mathbb{R}^l$. If f is continuous at $\mathbf{a} \in S$ and g is continuous at $f(\mathbf{a}) \in T$, then the function $g \circ f$ is continuous at $\mathbf{a}$. Thus, if f and g are continuous, then so is $g \circ f$.
4. [1] **Theorem:** Let C be a compact subset of $\mathbb{R}^n$, and let f be a continuous function from C into $\mathbb{R}^m$. Then, the image set $f(C)$ is compact.
5. [1] **Theorem:** (**Extreme Value Theorem**) Let C be a compact subset of $\mathbb{R}^n$, and let f be a continuous function from C into $\mathbb{R}$. Then there are points $\mathbf{a}$ and $\mathbf{b}$ in C attaining the minimum and maximum values of f on C. That is,

$$f(\mathbf{a}) \le f(\mathbf{x}) \le f(\mathbf{b}) \text{ for all } \mathbf{x} \in C.$$

6.4 Uniform Continuity

1. [1] **Definition:** A function f from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is **uniformly continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ so that

$$\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon \text{ whenever } \|\mathbf{x} - \mathbf{a}\| < \delta, \mathbf{x}, \mathbf{a} \in S.$$

2. [1] **Proposition:** Every Lipschitz functions is uniformly continuous.
3. [1] **Corollary:** Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous.
4. [1] **Corollary:** Let f be a differentiable real-valued function on $[a, b]$ with a bounded derivative ; that is, there is $M > 0$ so that $|f'(x)| \le M$ for all $a \le x \le b$. Then, f is uniformly continuous on $[a, b]$.
5. [1] **Theorem:** Suppose that $C \subset \mathbb{R}^n$ is compact and $f : C \to \mathbb{R}^n$ is continuous. Then f is uniformly continuous on C.

6.5 Intermediate Value Theorem and Monotone Functions

1. [1] **Theorem:** If f is a continuous real-valued function on $[a, b]$ with $f(a) < l < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = l$.
2. [1] **Corollary:** If f is a continuous real-valued function on $[a, b]$, then $f([a, b])$ is a closed interval.
3. [1] **Proposition:** If f is an increasing function on the interval (a, b), then the one-sided limits of f exist at each point $c \in (a, b)$, and

$$\lim_{x \to c^-} f(x) = L \le f(c) \le \lim_{x \to c+} f(x) = M.$$

   For decreasing functions the inequalities are simply reversed.
4. [1] **Corollary:** The only type of discontinuity a monotone function on an interval can have is a jump discontinuity.
5. [1] **Corollary:** If f is a monotone function on $[a, b]$ and the range of f intersects every nonempty open interval in $[f(a), f(b)]$, then f is continuous.
6. [1] **Theorem:** A monotone function on $[a, b]$ has at most countably many discontinuities.

7. [1] **Theorem:** Let f be a continuous strictly increasing function on $[a, b]$. Then, f maps $[a, b]$ one-to-one and onto $[f(a), f(b)]$. Moreover, the inverse function f^{-1} is also continuous and strictly increasing.

8. If you have time take a look at the Cantor Function in the text p.137.

7 Calculus of Functions of One Variable

7.1 Differentiation

1. [1] **Definition:** A real-valued function $f : (a, b) \to \mathbb{R}$ is **differentiable at a point** $x_0 \in (a, b)$ if

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x_0 \to x} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. We call this limit $f'(x_0)$.

2. [1] **Definition:** A function f is **differentiable on an interval** $[a, b]$ if it is differentiable at each point x_0 in the interval.

3. [1] **Definition:** The **tangent line** to f at x_0 is the linear function $T(x) = f(x) + f'(x_0)(x - x_0)$.

4. [1] **Proposition:** If f is differentiable at x_0, then it is continuous at x_0. So every differentiable function is continuous.

5. [1] **Lemma:** Let f be a function on $[a, b]$ that is differentiable at x_0. Let $T(x)$ be the tangent line to f at x_0. Then T is the unique linear function with the property that

$$\lim_{x\to x_0} \frac{f(x) - T(x)}{x - x_0} = 0.$$

6. [1] **Corollary:** If $f(x)$ is a function on (a, b) and $x_0 \in (a, b)$, then the following are equivalent:
(a) f is differentiable at x_0.
(b) There is a linear function $T(x)$ and a function $\epsilon(x)$ on (a, b) such that $\lim_{x\to x_0} \epsilon(x) = 0$ and $f(x) = T(x) + \epsilon(x)(x - x_0)$.
(c) There is a function $\psi(x)$ on (a, b) such that $f(x) = f(x_0) + \psi(x)(x - x_0)$ and $\lim_{x\to x_0} \psi(x)$ exists.
If these hold, then the linear function $T(x)$ in (2) is the tangent line, and the limit $\lim_{x\to x_0} \psi(x)$ in (3) equals $f'(x_0)$.

7. [1] **Theorem:** (**Chain Rule**) Suppose that f is defined on $[a, b]$ and has range contained in $[c, d]$. Let g be defined on $[c, d]$. Suppose that f is differentiable at $x_0 \in (a, b)$ and g is differentiable at $f(x_0)$. Then the composition $g(f(x))$ is defined and $(g(f(x_0)))' = g'(f(x_0))f'(x_0)$.

8. [1] **Definition:** We say f is **left-differentiable** at x_0 if $\lim_{h\to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}$ exists. Similarly, we can define **right-differentiable**.

9. [1] **Theorem:** (**Fermat's Theorem**) Let f be a continuous function on an interval $[a, b]$ that takes its maximum or minimum value at a point x_0. Then,
(a) x_0 is an endpoint, or
(b) f is not differentiable at x_0, or
(c) f is differentiable at x_0 and $f'(x_0) = 0$.

10. [1] **Theorem:** (**Rolle's Theorem**) Suppose that f is a function that is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b) = 0$. Then, there is a point $c \in (a, b)$ such that $f'(c) = 0$.

11. [1] **Theorem:** (**Mean Value Theorem**) Suppose that f is a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

12. [1] **Corollary:** Let f be a differentiable function on $[a, b]$.
(a) If $f'(x)$ is (strictly) positive, then f is (strictly) increasing.
(b) If $f'(x)$ is (strictly) negative, then f is (strictly) decreasing.
(c) If $f'(x) = 0$ at every $x \in (a, b)$, then f is constant.

13. [1] **Theorem:** (**Intermediate Value Theorem for Derivatives**) Let f be differentiable on $[a, b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then, there exists a point $c \in (a, b)$ such that $f'(c) = k$.

14. [1] **Theorem:** (**Darboux's Theorem**) If f is differentiable on $[a, b]$ and $f'(a) < L < f'(b)$, then there is a point x_0 in (a, b) at which $f'(x_0) = L$.

7.2 Riemann-Stieltjes Integration

1. [1] **Definition:** Let f be a bounded function defined on an interval $[a, b]$. A **partition** of $[a, b]$ is a finite set $P = \{a = x_0 < x_1 < ... < x_n = b\}$.

2. [1] **Definition:** Consider bounded functions $f, g : [a, b] \to \mathbb{R}$. Given a partition of $[a, b]$, $P = \{a = x_0 < x_1 < ... < x_n = b\}$, and an evaluation sequence, $X = (x'_1, x'_2, ..., x'_n)$, for P, the Riemann-Stieltjes sum for f with respect to g using P and X is
$$I_g(f, P, X) = \sum_{j=1}^{n} f(x'_j)[g(x_j) - g(x_{j-1})].$$

3. [1] **Definition:** We say f is **Riemann-Stieltjes Integrable** with respect to g, $f \in R(g)$, if there is a number L so that for all $\epsilon > 0$, there is a partition P_ϵ so that, for all partitions P with $P \supset P_\epsilon$ and all evaluations sequences X for P, we have
$$|I_g(f, P, X) - L| = \left| \sum_{j=1}^{n} f(x'_j)[g(x_j) - g(x_{j-1})] - L \right| < \epsilon.$$

4. [1] **Theorem:** If $f_1 \in R(g)$ and $f_2 \in R(g)$ on $[a, b]$ an $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in R(g)$ on $[a, b]$ and
$$\int_a^b (c_1 f_1 + c_2 f_2) dg = c_1 \int_a^b f_1 dg + c_2 \int_a^b f_2 dg.$$
If $f \in R(g_1)$ and $f \in R(g_2)$ on $[a, b]$, and $c_1, c_2 \in \mathbb{R}$, then $f \in R(c_1 g_1 + c_2 g_2)$ on $[a, b]$ and
$$\int_a^b f d(c_1 g_1 + c_2 g_2) = c_1 \int_a^b f dg_1 + c_2 \int_a^b f dg_2.$$
Finally, if $a < b < c$ and f is R-S integrable with respect to g on both $[a, b]$ and $[b, c]$, then it is R-S integrable with respect to g on $[a, c]$ and
$$\int_a^c f dg = \int_a^b f dg + \int_b^c f dg.$$

5. [1] **Theorem:** Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded and $\psi : [c, d] \to [a, b]$ is a strictly increasing, continuous function onto $[a, b]$. Let $F = f \circ \psi$ and $G = g \circ \psi$. If $f \in R(g)$ on $[a, b]$, then $F \in R(G)$ on $[c, d]$ and
$$\int_c^d F dG = \int_a^b f dg.$$

6. [1] **Theorem:** Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. If $f \in R(g)$ on $[a, b]$, then $g \in R(f)$ on $[a, b]$ and

$$\int_a^b f dg + \int_a^b g df = g(b)f(b) - g(a)f(a).$$

7. [1] **Theorem:** If $f : [a, b] \to \mathbb{R}$ has $f \in R(g)$, where $g : [a, b] \to \mathbb{R}$ is C^1, then fg' is Riemann Integrable on $[a, b]$ and

$$\int_a^b f dg = \int_a^b f(x)g'(x)dx.$$

8. [1] **Definition:** Let f be a bounded function on $[a, b]$ and $P = \{a = x_0 < x_1 < ... < x_n = b\}$ be a partition of $[a, b]$. Then we defined the **maximum** and **minimum of** f on each interval $[x_{j-1}, x_j]$ of P by

$$M_j(f, P) = \sup_{x_{j-1} \leq x \leq x_j} f(x) \quad \text{and} \quad m_j(f, P) = \inf_{x_{j-1} \leq x \leq x_j} f(x).$$

9. [1] **Definition:** Now, we define the **upper sum with respect to** g and the **lower sum with respect to** g to be

$$U_g(f, P) = \sum_{i=1}^n M_i(f, P)[g(x_j) - g(x_{j-1})],$$
$$L_g(f, P) = \sum_{i=1}^n m_i(f, P)[g(x_j) - g(x_{j-1})].$$

10. [1] **Definition:** A partition R is a **refinement** of P provided that $P \subset R$.

11. [1] If P and Q are two partitions, then R is a **common refinement** of P and Q provided that $P \cup Q \subset R$.

12. [1] **Lemma:** (**Refinement Lemma**) Let f, g be bounded functions on $[a, b]$ and P, R partitions of $[a, b]$. Assume that g is increasing. If R is a refinement of P, then

$$L_g(f, P) \leq L_g(f, R) \leq U_g(f, R) \leq U_g(f, P).$$

13. [1] **Corollary:** Let f, g be bounded functions on $[a, b]$ and assume that g is increasing. If P and Q are any two partitions of $[a, b]$,

$$L_g(f, P) \leq U_g(f, Q).$$

14. [1] **Definition:** Define

$$L_g(f) = \sup_P L_g(f, P) \quad \text{and} \quad U_g = \inf_P U_g(f, P).$$

15. [1] **Theorem:** (**Riemann-Stieltjes Condition**) Let f, g be bounded functions on $[a, b]$ and assume that g is increasing on $[a, b]$. The following are equivalent:
(1) f is Riemann-Stieltjes intergrable with respect to g.
(2) $U_g(f) = L_g(f)$
(3) for each $\epsilon > 0$, there is a partition P so that

$$U_g(f, P) - L_g(f, P) < \epsilon.$$

Moreover, if (2) holds, the common value equals $\int f dg$.

7.3 Variation of Functions

1. [1] **Definition:** Given a function $f : [a, b] \to \mathbb{R}$ and a partition of $[a, b]$, say $P = \{a = x_0 < x_1 < ... < x_n = b\}$, then the **variation** of f over P is

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

2. [1] **Definition:** The (**total**) **variation** of f on $[a, b]$ is

$$V_a^b f = \sup\{V(f, P) : P \text{ a partition of}[a, b]\}.$$

3. [1] **Definition:** We say f is of bounded variation on $[a, b]$ if $V_a^b f$ is finite.
4. [1] **Lemma:** If $a < b < c$ and $f : [a, c] \to \mathbb{R}$ is given, then

$$V_a^c f = V_a^b f + V_b^c f.$$

5. [1] **Theorem:** If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.
6. [1] **Theorem:** If $f : [a, b] \to \mathbb{R}$ is bounded by M, $g : [a, b] \to \mathbb{R}$ is of bounded variation and $f \in R(g)$ on $[a, b]$, then

$$\left| \int_a^b f dg \right| \leq M \cdot V_a^b g.$$

7. [1] **Theorem:** If $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$ is increasing, then $f \in R(g)$ on $[a, b]$.
8. [1] **Corollary:** If $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$ is of bounded variation, then $f \in R(g)$ on $[a, b]$.

7.4 Riemann Integration

1. [1] **Definition:** Set $\Delta_j = x_j - x_{j-1}$. Then, the **mesh** of a partition P is $\text{mesh}(P) = \max_{1 \leq j \leq n} \Delta_j$.
2. [1] **Definition:** The **upper and lower sums** of f with respect to the partition P by

$$U(f, P) = \sum_{j=1}^{n} M_j(f, P)\Delta_j \quad \text{and} \quad L(f, P) = \sum_{j=1}^{n} m_j(f, P)\Delta_j.$$

3. [1] **Definition:** We define the **Riemann Sum** by

$$I(f, P, X) = \sum_{j=1}^{n} f(x_j')\Delta_j.$$

4. [1] **Definition:** A bounded function f on a finite interval $[a, b]$ is called **Riemann integrable** if $L(f) = U(f)$. In this case, we write $\displaystyle\int_a^b f(x)dx$ for the common value.
5. [1] **Theorem:** (**Riemann's Condition**) Let $f(x)$ be a bounded function on $[a, b]$. The following are equivalent:
(1) f is Riemann integrable.
(2) For each $\epsilon > 0$, there is a partition P so that $U(f, P) - L(f, P) < \epsilon$.

6. [1] **Corollary:** Let f be a bounded real-valued function on $[a, b]$. If there is a sequence of partitions of $[a, b]$, P_n, so that

$$\lim_{n\to\infty} U(f, P_N) - L(f, P_n) = 0,$$

then f is Riemann integrable. Moreover, if X_n is any choice of points $x'_{n,j}$ selected from each interval of P_n, then

$$\lim_{n\to\infty} I(f, P, X) = \lim_{n\to\infty} \sum_{j=1}^{n} f(x'_j)\Delta_j = \int_a^b f(x)dx.$$

7. [1] **Theorem:** Let f be a bounded function on $[a, b]$. The following are equivalent:
(1) f is Riemann integrable.
(2) For each $\epsilon > 0$, there is a partition P so that $U(f, P) - L(f, P) < \epsilon$.
(3) For every $\epsilon > 0$, there is a $\delta > 0$ so that every partition Q such that $\text{mesh}(Q) < \delta$ satisfies

$$U(f, Q) - L(f, Q) < \epsilon.$$

(4) For every $\epsilon > 0$, there is a $\delta > 0$ so that every partition Q such that $\text{mesh}(Q) < \delta$ and every choice of a set $X = (x'_1, x'_2, ..., x'_n)$, where $x'_j \in [x_{j-1}, x_j]$ satisfies $\left|I(f, Q, X) - \int_a^b f(x)dx\right| < \epsilon$.

8. [1] **Theorem:** Every monotone function on $[a, b]$ is Riemann integrable.

9. [1] **Theorem:** Every continuous function on $[a, b]$ is Riemann integrable.

7.5 Fundamental Theorem of Calculus

1. [2] p. 286 **Theorem:** (**Fundamental Theorem of Calculus**) Let f be Riemann integrable on $[a, b]$, and let

$$F(x) = \int_a^x f(t)\,dt \quad \text{for } a \le x \le b.$$

Then F is uniformly continuous function on $[a, b]$. Furthermore, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

2. [1] **Definition:** A function f on $[a, b]$ has an **antiderivative** if there is a continuous function $F(x)$ on $[a, b]$ such that $F'(x) = f(x)$ for every point $x \in (a, b)$.

3. [2] p. 288 **Corollary:** (**Fund. Thm. Part II**) If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

4. [1] **Lemma:** Suppose that f is an integrable function on $[a, b]$ bounded by M. Then

$$\left|\int_a^b f(t)dt\right| \le M(b - a).$$

5. [3] p. 134 **Theorem:** (**Integration by Parts**) Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in R$, and $G' = g \in R$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

6. [1] **Theorem:** (**Mean Value Theorem**) If f is a continuous function on $[a, b]$, then there is a point $c \in (a, b)$ such that $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = f(c)$.

7.6 Improper Integrals and Cauchy Principle Values

1. [2] p. 289 **Definition:** Let f be defined on $(a, b]$ and integrable on $[c, b]$ for every $c \in (a, b]$. If $\lim_{c \to a^+} \int_c^b f$ exists, then the **improper integral** of f on $(a, b]$, denoted by $\int_a^b f$, is given by
$$\int_a^b f = \lim_{c \to a^+} \int_c^b f.$$

2. [2] p. 290 **Definition:** Let f be defined on $[a, \infty)$ and integrable on $[a, c]$ for every $c > a$. If $\lim_{c \to \infty} \int_a^c f$ exists, then the **improper integral** of f on $[a, \infty)$, denoted by $\int_a^\infty$, is given by
$$\int_a^\infty f = \lim_{c \to \infty} \int_a^c f.$$

8 Sequences and Series of Functions

8.1 Sequences of Functions

1. [1] **Definition:** Let $(f_n)_{n=1}^\infty$ be a sequence of functions from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$. This sequence **converges pointwise** to a function f if
$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in S.$$

2. [1] **Definition:** Let $(f_n)_{n=1}^\infty$ be a sequence of functions from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$. This sequence **converges uniformly** to a function f if for every $\epsilon > 0$, there is an integer N so that
$$\|f_n(x) - f(x)\| < \epsilon \text{ for all } x \in S \text{ and } n \geq N$$

3. [1] **Theorem:** For a sequence of functions (f_n) in $C_b(S, \mathbb{R}^m)$, (f_n) converges uniformly to f if and only if
$$\lim_{n \to \infty} \|f_n - f\|_\infty = 0.$$

4. [1] **Theorem:** (**Dini's Theorem**) Suppose that f and f_n are continuous functions on $[a, b]$ such that $f_n \leq f_{n+1}$ for all $n \geq 1$ and (f_n) converges to f pointwise. Then (f_n) converges to f uniformly.

5. [1] **Theorem:** Let (f_n) be a sequence of continuous functions mapping a subset S of $\mathbb{R}^n$ into $\mathbb{R}^m$ that converges uniformly to a function f. Then f is continuous.

6. [1] **Theorem:** (**Completeness Theorem for** $C(K)$) If K is a compact set, the space $C(K)$ of all continuous functions on K with the sup norm is complete.

8.2 Uniform Convergence and Integration

1. [1] **Theorem:** (**Integral Convergence Theorem**) Let (f_n) be a sequence of continuous functions on the closed interval $[a, b]$ converging uniformly to $f(x)$ and fix $c \in [a, b]$. Then the functions
$$F_n(x) = \int_c^x f_n(t)dt \text{ for } n \geq 1$$
converges uniformly on $[a, b]$ to the function $F(x) = \int_c^x f(t)dt$.

2. [1] **Corollary:** Suppose that (f_n) is a sequence of continuously differentiable functions on $[a,b]$ such that (f_n') converges uniformly to a function g and there is a point $c \in [a,b]$ so that $\lim_{n\to\infty} f_n(c) = \gamma$ exists. Then (f_n) converges uniformly to a differentiable function f with $f(c) = \gamma$ and $f' = g$.

3. [1] **Proposition:** Let $f(x,t)$ be a continuous function on $[a,b]\times[c,d]$. Define $F(x) = \int_c^d f(x,t)dt$. Then F is continuous on $[a,b]$.

4. [1] **Theorem:** (**Leibniz's Rule**) Suppose that $f(x,t)$ and $\frac{\partial}{\partial x} f(x,t)$ are continuous functions on $[a,b]\times[c,d]$. Then $F(x) = \int_c^d f(x,t)dt$ is differentiable and

$$F'(x) = \int_c^d \frac{\partial}{\partial x} f(x,t)dt.$$

8.3 Series of Functions

1. [1] **Theorem:** Let (f_n) be a sequence of continuous functions from a subset S of $\mathbb{R}^n$ into $\mathbb{R}^m$. If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly, then it is continuous.

2. [1] **Theorem:** A series of functions converges uniformly if and only if it is uniformly Cauchy.

3. [1] **Theorem:** (**Weierstrass M-Test**) Suppose that $f_n(x)$ is a sequence of functions on $S \subset \mathbb{R}^k$ into $\mathbb{R}^m$ and (M_n) is a sequence of real numbers so that

$$\|f_n\|_\infty = \sup_{x\in S} \|f_n(x)\| \le M_n \text{ for all } x \in S.$$

If $\sum_{n=1}^{\infty} M_n$ converges, then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on S.

4. Sometimes it is helpful to take the derivative of a series of functions to find the maxes (min) of each function in the sequence and then show the series of maxes (mins) converges.

8.4 Power Series

1. [1] **Theorem:** (**Hadamard's Theorem**) Given a power series $\sum_{n=1}^{\infty} a_n x^n$, there is R so in $[0,+\infty)$ so that the series converges for all x with $|x| < R$ and diverges for all x with $|x| > R$. Moreover, the series converges uniformly on each closed interval $[a,b]$ contained in $(-R,R)$.
   finally, if $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$, then

$$R = \begin{cases} +\infty & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha = +\infty, \\ \frac{1}{\alpha} & \text{if } \alpha \in (0,+\infty). \end{cases}$$

2. [1] **Theorem:** (**Term-by-Term Operations on Series**) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R, f is differentiable on $(-R,R)$, and for $x \in (-R,R)$,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Further, $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ has radius of convergence R, and for $x \in (-R, R)$,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

8.5 Taylor Series and Weierstrass Approximation Theorem

1. [1] **Definition:** If f has n derivatives at a point $a \in [A, B]$, the **Taylor polynomial** of order n for f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

2. [1] **Theorem:** (**Taylor's Theorem**) Let $f(x)$ belong to $C^n[A, B]$, and furthermore assume that $f^{(n+1)}$ is defined and $|f^{(n+1)}| \leq M$ for $x \in [A, B]$. Let $a \in [A, B]$, and let $P_n(x)$ be the Taylor polynomial of order n for f at a. Then for each $x \in [A, B]$, the error of approximation $R_n(x) = f(x) - P_n(x)$ satisfies

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}$$

3. [1] **Definition:** When f is C^∞, the **Taylor Series** of f about a is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

4. [1] **Definition:** Given a function f, a **uniform approximation** is a polynomial p such that

$$\|f - p\|_\infty = \max_{x \in [a,b]} |f(x) - p(x)| < \epsilon.$$

5. [1] **Theorem:** (**Weierstrass Approximation Theorem**) Let f be any continuous real-valued function on $[a, b]$. Then there is a sequence of polynomials p_n that converges uniformly to f on $[a, b]$.

8.6 Arzela-Ascoli and Banach Contraction Principle

1. [1] **Definition:** A family of functions F mapping a set $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is **equicontinuous at a point** $a \in S$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|f(x) - f(a)\| < \epsilon \text{ whenever } \|x - a\| < \delta \text{ and } f \in F.$$

2. [1] **Definition:** The family F is **equicontinuous on a set** S if it is equicontinuous at every point in S.

3. [1] **Definition:** The family F is **uniformly equicontinuous** on S if for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|f(x) - f(y)\| < \epsilon \text{ whenever } \|x - y\| < \delta, x, y \in S \text{ and } f \in F.$$

4. [1] **Proposition:** If F is an equicontinuous family of functions on a compact set, then it is uniformly equicontinuous.

5. [1] **Corollary:** Let K be a compact subset of $\mathbb{R}^m$. Then K contains a sequence $\{x_i : i \geq 1\}$ that is dense in K. Moreover, for any $\epsilon > 0$, there is an integer N so that $\{x_i : 1 \leq i \leq N\}$ forms an ϵ-net for K.

6. [1] **Theorem:** (**Arzela-Ascoli Theorem**) Let K be a compact subset of $\mathbb{R}^n$. A subset F of $C(K, \mathbb{R}^m)$ is compact if and only if it is closed, bounded, and equicontinuous.

7. [1] **Definition:** Suppose that X is a subset of a normed linear space, and T is a continuous map from X into iiself. The pair (X, T) is called a **discrete dynamical system**.

8. [1] **Definition:** For each point $x \in X$, the **forward orbit** of x is the sequence $O(x) := \{T^n x : n \geq 0\}$.

9. [1] **Definition:** A fixed point x^* is called an **attractive fixed point** or a **sink** if there is an open neighborhood $U = (a, b)$ containing X^* so that for every point x in (a, b), the orbit $O(x)$ converges to x^*.

10. [1] **Definition:** A fixed point x^* is called a **repelling fixed point** or a **source** if there is a neighborhood $U = (a, b)$ containing x^* so that for every point x in (a, b) except for x^* itself, the orbit $O(x)$ always leaves the interval U.

11. [1] **Lemma:** Suppose that T is a C^1 dynamical system with a fixed point x^*. If $|T'(x^*)| < c < 1$, then x^* is an attractive fixed point. Moreover, there is an interval $U = (x - \delta, x + \delta)$ about x^* so that for every $x_0 \in U$, the sequence $x_n = T^n x_0$ satisfies

$$|x_n - x^*| \leq c^n |x_0 - x^*| \leq \frac{c^n}{1-c} |x_1 - x_0|.$$

If $|T'(x^*)| > 1$, then x^* is a repelling fixed point.

12. [1] **Definition:** Let X be a subset of a normed vector space $(V, \|\cdot\|)$. A map $T : X \to X$ is called a **contraction** on X if there is a positive constant $c < 1$ so that

$$\|Tx - Ty\| \leq c \|x - y\| \text{ for all } x, y \in X.$$

That is, T is Lipschitz with constant $c < 1$.

13. [1] **Theorem:** (**Banach Contraction Principle**) Let X be a closed subset of a complete normed vector space $(V, \|\cdot\|)$. If T is a contraction map of X into X, then T has a unique fixed point x^*. Furthermore, if x is any vector in X, then $x^* = \lim_{n\to\infty} T^n x$ and

$$\|T^n x - x^*\| \leq c^n \|x - x^*\| \leq \frac{c^n}{1-c} \|x - Tx\|,$$

where c is the Lipschitz constant for T.

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