1 Abstract Measure Theory

Let X be a nonempty set.

Definition (10):

1. A collection $\mathcal{T}$ of subsets of X is called a **topology** on X if it has the following properties:
   (a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
   (b) If $\{U_j\}_{j=1}^n \subseteq \mathcal{T}$, then $\bigcap_{j=1}^n U_j \in \mathcal{T}$.
   (c) If $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.
2. If $\mathcal{T}$ is a topology on X, then $(X, \mathcal{T})$ is called a **topological space**. The members of $\mathcal{T}$ are called **open sets** in X. The complements of members of $\mathcal{T}$ are called **closed sets**.
3. If X and Y are topological spaces and $f : X \to Y$ satisfies $f^{-1}(V)$ is open in X for all open sets $V \subseteq Y$, then f is **continuous**.

Example (11):

1. Given X; $\mathcal{T} = \{\emptyset, X\}$ is called the **trivial topology**.
2. Given X; $\mathcal{T} = \mathcal{P}(X)$, where $\mathcal{P}$ is the power set of X is called the **discrete topology**.

Definition (12): A collection $\mathcal{M}$ of subsets of X is a σ**-algebra** if it has the following properties:

1. $\emptyset \in \mathcal{M}$ and $X \in \mathcal{M}$
2. If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$.
3. If $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$, then $\bigcup_{j=1}^\infty E_j \in \mathcal{M}$.

Also, we have the following:

1. If $\mathcal{M}$ is a σ-algebra on X, then the pair $(X, \mathcal{M})$ is called a **measurable space**. If $\mathcal{M}$ is understood, then X itself will be called a measurable space. The members of $\mathcal{M}$ are called **measurable sets**.
2. If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces and $f : X \to Y$ satisfies $f^{-1}(E)$ is a measurable set in X for all $E \in \mathcal{N}$, then f is called $(\mathcal{M}, \mathcal{N})$**-measurable**. If $\mathcal{M}$ and $\mathcal{N}$ are understood, we just say f is measurable.

Example (13):

1. $X \neq \emptyset$, $\mathcal{M} = \{\emptyset, X\}$ (trivial measurable space)
2. $X \neq \emptyset$, $\mathcal{M} = \mathcal{P}$ (discrete measurable space)
3. $X = \mathbb{R}$, $\mathcal{M} = \{\emptyset, \{0\}, \mathbb{R}\backslash\{0\}, \mathbb{R}\}$
4. $X = \mathbb{R}$, $\mathcal{M} = \{\emptyset, \{0\}, \{1\}, \{0,1\}, \mathbb{R}\backslash\{0\}, \mathbb{R}\backslash\{1\}, \mathbb{R}\backslash\{0,1\}, \mathbb{R}\}$
5. $X \neq \emptyset$, $\mathcal{M} = \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$

Lemma (14): (Discjoint Dissection Lemma) If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{P}(X)$, then $\{F_k\}_{k=1}^{\infty} \subset \mathcal{P}(X)$ defined by

$$F_1 := E_1 \quad \text{and} \quad F_k := E_k \setminus \bigcup_{j=1}^{k-1} E_j$$

is a sequence of mutually disjoint sets such that

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k.$$

Theorem (15): If E is a collection of subsets of X, then there is a unique smallest σ-algebra $\mathcal{M}(E)$ containing E.
Definition (16): The unique smallest σ-algebra containing $E \subseteq \mathcal{P}(X)$ is called the **σ-algebra generated by** E and we denote it by $\mathcal{M}(E)$.
Definition (17): Let $(X, \mathcal{T})$ be a topological space. The σ-algebra $\mathcal{M}(\mathcal{T})$ is called the **Borel σ-algebra** on X and is denoted by $\mathcal{B}_{(X,\mathcal{T})}$ or $\mathcal{B}_{(X)}$. The elements of $\mathcal{B}_{(X)}$ are called **Borel sets**.
Proposition (18): Let $(X, \mathcal{T})$ be a topological space. Set

$$\begin{aligned} F_\sigma &:= \{\text{countable unions of closed sets in } X\} \\ G_\delta &:= \{\text{countable intersections of open sets in } X\} \\ F_{\sigma\delta} &:= \{\text{countable intersections } F_\sigma \text{ sets}\} \\ G_{\delta\sigma} &:= \{\text{countable unions of } G_\delta \text{ sets}\} \end{aligned}$$

Proposition (19): $\mathcal{B}_{(\mathbb{R})}$ is generated by any of the following:

1. $\xi_1 = \{(a, b) : a < b\}$,
2. $\xi_2 = \{[a, b] : a < b\}$,
3. $\xi_3 = \{(a, b] : a < b\}$, or $\xi_4 = \{[a, b) : a < b\}$,
4. $\xi_5 = \{(a, \infty) : a \in \mathbb{R}\}$, or $\xi_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,
5. $\xi_7 = \{[a, \infty) : a \in \mathbb{R}\}$, or $\xi_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Proposition (20): Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. If $\mathcal{N} = \mathcal{M}(\xi)$ for some $\xi \subseteq \mathcal{P}(X)$, then $f : X \to Y$is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \xi$.
Corollary (21): If X and Y are topological spaces and $f \in C(X, Y)$, then f is $(\mathcal{B}_{(X)}, \mathcal{B}_{(Y)})$-measurable.
Proposition (22): If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$. Then the following are equivalent:

1. f is measurable,
2. $f^{-1}((a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,
3. $f^{-1}([a, \infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,
4. $f^{-1}((-\infty, a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$,
5. $f^{-1}((-\infty, a]) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

An obvious analog holds for $f : X \to \overline{\mathbb{R}}$.

Definition (23): Let $\{(Y_\alpha, \eta_\alpha)\}_{\alpha \in A}$ be a family of measurable spaces. If $f_\alpha : X \to Y_\alpha$ is a map for each $\alpha \in A$. Then the σ-algebra on X generated by $\{f_\alpha\}_{\alpha \in A}$ is the σ-algebra generated by $\{f_\alpha^{-1}(E) : \alpha \in A \text{ and } E \in \eta_\alpha\}$. It is the smallest σ-algebra on X so that each f_α is measurable.

Proposition (24): Let $(X, \mathcal{M})$ be a measurable space and let $f, g : X \to \mathbb{R}$ be measurable. Then the following hold:

1. $|f|$ and f^2 are both measurable.
2. αf and $\alpha + f$ are measurable for all $\alpha \in \mathbb{R}$.
3. If f is never 0, then $\dfrac{1}{f}$ is measurable.
4. The set $\{x \in X : f(x) > g(x)\}$ is measurable.
5. $f + g$ and $f - g$ are measurable.
6. $f \cdot g$ is measurable.
7. If g is never zero, then $\dfrac{f}{g}$ is measurable.

Corollary (25): If $f : X \to \mathbb{C}$ is measurable, then so is $\text{sgn}(f)$, where

$$\text{sgn}(z) = \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0 \end{cases}.$$

Proposition (26): If $\{f_j\}_{j=1}^\infty$ is a sequence of $\overline{\mathbb{R}}$-valued functions on $(X, \mathcal{M})$, then

$$g_1(x) = \sup_{j \in \mathbb{N}} f_j(x) \qquad g_3(x) = \limsup_{j \to \infty} f_j(x)$$
$$g_2(x) = \inf_{j \in \mathbb{N}} f_j(x) \qquad g_4(x) = \liminf_{j \to \infty} f_j(x)$$

are measurable. If $\lim_{j\to\infty} f_j(x) = f(x)$ at each $x \in X$, then f is measurable.

Corollary (27): If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then so are $x \mapsto \max\{f(x), g(x)\}$ and $x \mapsto \min\{f(x), g(x)\}$.

Corollary (28): If $f : X \to \overline{\mathbb{R}}$ is measurable, then so are f^+, f^-, and $|f|$.

Definition (29): A **simple function** on X is a measurable function with a range consisting of a finite number of points in $\mathbb{R}$ (or $\mathbb{C}$). If $\varphi : X \to \mathbb{R}$ is a simple function with $\text{range}(\varphi) = \{a_1, a_2, ..., a_n\}$, then for each $j \in \mathbb{N}$ such that $1 \leq j \leq n$, the set $E_j = \varphi^{-1}(a_j)$ is measurable and $\bigcup_{j=1}^n E_j = X$. The **standard representation** for φ is

$$\varphi = \sum_{j=1}^n a_j \chi_{E_j}.$$

Theorem (30): Let $(X, \mathcal{M})$ be a measurable space.

1. If $f : X \to [0, \infty]$ is measurable, then there is a sequence $\{\varphi_j\}_{j=1}^\infty$ of simple functions suc that
   (a) $0 \leq \varphi_1 \leq \varphi_2 \leq ... \leq \varphi_j \leq ... \leq f$,
   (b) $\lim_{j\to\infty} \varphi_j(x) = f(x)$ for all $x \in X$,
   (c) $\varphi_j \to f$ uniformly on any set where f is uniformly bounded.

2. If $f : X \to \mathbb{C}$ is measurable, then there is a sequence $\{\varphi_j\}_{j=1}^{\infty}$ of simple functions suc that
   (a) $0 \leq |\varphi_1| \leq |\varphi_2| \leq ... \leq |\varphi_j| \leq ... \leq |f|$,
   (b) $\lim_{j\to\infty} \varphi_j(x) = f(x)$ for all $x \in X$,
   (c) $\varphi_j \to f$ uniformly on any set where f is uniformly bounded.

Definition (31): Let $(X, \mathcal{M})$ be a measurable space.

1. A **positive measure** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ with the following properties:
   (a) $\mu(\emptyset) = 0$,
   (b) (Countable Additivity) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ are mutually disjoint, then
$$\mu\left(\bigcup_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} \mu(E_j).$$
2. A **measure space** is a triple $(X, \mathcal{M}, \mu)$ with μ a measure on $\mathcal{M}$.

Definition (32): Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. μ is called **semifinite** if for each $E \in \mathcal{M}$ with $\mu(E) > 0$, there is an $F \in \mathcal{M}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$.
2. μ is called σ-finite if $X = \bigcup_{j=1}^{\infty} E_j$ for some $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ with $\mu(E_j) < \infty$ for each $j \in \mathbb{N}$.
3. A set $E \in \mathcal{M}$ is called σ-finite if $E = \bigcup_{j=1}^{\infty} E_j$ for some $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ with $\mu(E_j) < \infty$ for $j \in \mathbb{N}$.
4. μ is called **finite** if $\mu(X) < \infty$.
5. μ is called a **probability measure** if $\mu(X) = 1$.

Example (33):

1. (Counting Measure) Let X be nonempty. Define $\mu : \mathcal{P}(X) \to [0, \infty]$ by
$$\mu(E) := \begin{cases} \#\text{ of elements in } E, & \text{if } E \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$$
   Note: μ is semifinite. μ is σ-finite if and only if X is countable. $E \subseteq X$ is σ-finite if E is countable.
2. Let X be nonempty. Define
$$\mathcal{M} := \{E \subseteq X : E \text{ or } E^c \text{ is countable}\}$$
   and $\mu : \mathcal{M} \to [0, \infty]$ by
$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is countable} \\ 1, & \text{if } E \text{ is uncountable} \end{cases}$$
   Note μ is a finite measure and a probability measure if X is uncountable.

3. (Dirac Measre at x_0) Let $(X, \mathcal{M})$ be a measurable space. Let $\{x_0\} \in \mathcal{M}$. Define $\mu : \mathcal{M} \to [0, \infty]$ by

$$\mu(E) := \begin{cases} 1, & \text{if } \{x_0\} \subset E \\ 0, & \text{otherwise} \end{cases}$$

μ is a probability measure.

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Definition (34): If $(X, \mathcal{M}, \mu)$ is a measurable space, then a μ-null set, or null set, in X is a set such that its measure is 0, i.e. $E \subseteq X$ is a null set if and only of $E \in \mathcal{M}$ and $\mu(E) = 0$.

Definition (35): If a statement P is true for all $x \in E$ with $X \backslash E$ a null set, then we say that P holds **almost everywhere in X (a.e. in X), or for almost every $x \in X$.**

Definition (36): Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is called **complete** is whenever $E \in \mathcal{M}$ is a null set, we also find $F \in \mathcal{M}$ for each $F \subseteq E$, i.e. μ is complete if all subsets of null sets are measurable.

Theorem (37): Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Set $\mathcal{N} := \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} := \{E \bigcup F : E \in \mathcal{M} \text{ and } F \subseteq N \in \mathcal{N}\}$. Then, $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension of μ to a complete measure $\overline{\mu}$ on $\overline{\mathcal{M}}$.

Definition (38): The measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ is called the **completion** of $(X, \mathcal{M}, \mu)$.

Theorem (39): (Basic Properties of Measures) Let $(X, \mathcal{M}, \mu)$ be a measure space. Then μ has the following properties:

1. (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
2. (Countable Subadditivity) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$, then $\mu\left(\bigcup_{j=1}^{\infty}\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$.
3. (Continuity from Below) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ satisfy $E_1 \subseteq E_2 \subseteq \ldots$, then $\mu\left(\bigcup_{j=1}^{\infty}\right) = \lim_{j\to\infty} \mu(E_j)$.
4. (Continuity from Above) If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ satisfy $E_1 \supseteq E_2 \supseteq \ldots$ and $\mu(E_1) < \infty$, then $\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$.

2 Abstract Integration

Notation: Given a measurable space $(X, \mathcal{M})$, set $L^+ := \{f : X \to [0, \infty] : f \text{ is measurable}\}$. Let $(X, \mathcal{M}, \mu)$ be a measure space.

Definition (40):

1. Let $\varphi \in L^+$ be a simple function. Let $\sum_{j=1}^{n} a_j \chi_{E_j}$ be the standard representative for φ. The **Lebesgue integral of φ with respect to μ** is

$$\int_X \varphi \, d\mu := \sum_{j=1}^{n} a_j \mu(E_j).$$

2. Let $f \in L^+$ be given. Then, the **Lebesgue integral of f with respect to μ** is
$$\int_X f\,d\mu := \sup\{\int_X \varphi\,d\mu : \varphi \in L^+ \text{ is simple and } 0 \le \varphi \le f\}$$

3. Let $f \in L^+$ and $A \in \mathcal{M}$ be given. The **Lebesgue integral of f with respect to μ over A** is
$$\int_A f\,d\mu = \int_X f\chi_A\,d\mu.$$

Proposition (41): Let $f \in L^+$ be given

1. If $A \in \mathcal{M}$ and $\mu(A) = 0$, then $\int_A f\,d\mu = 0$.
2. If $f(x) = 0$ for a.e. $x \in X$, then $\int_X f\,d\mu = 0$.

(Simple Function Technique) To prove property P for the Lebesgue integral:

1. Prove P for integral of simple functions in L^+.
2. Use definition or convergence theorem to get P for integrals of $f \in L^+$
3. Use (2) on positive and negative parts, f^+ and f^-, to get P for integrals of integrable functions.
4. Use (3) for real and imaginary pares of integrable complex functions to obtain the result.

Proposition (42): Let $\varphi \in L^+$ be a simple function. Define $\lambda : \mathcal{M} \to [0, \infty]$ by
$$\lambda(E) := \int_E \varphi\,d\mu.$$
Then, λ is a measure.

Proposition (43): Let $\varphi, \Psi \in L^+$ be simple functions and $c \in [0, \infty]$ be given. Then the following hold:

1. $\int_X (c\varphi)\ d\mu = c\int_X \varphi\,d\mu$
2. $\int_X (\varphi + \Psi)\ d\mu = \int_X \varphi\,d\mu + \int_X \Psi\,d\mu$
3. If $\varphi(x) \le \Psi(x)$ for a.e. $x \in X$, then $\int_X \varphi\,d\mu \le \int_X \Psi\,d\mu$

Theorem (45): (Monotone Convergence Theorem: MCT) Let $\{f_j\}_{j=1}^{\infty} \subseteq L^+$ be given. Suppose $0 \le f_1 \le f_2 \le \dots \le f_j \le \dots$ Define $f(x) = \lim_{j\to\infty} f_j(x)$ for $x \in X$. Then, $f \in L^+$ and
$$\int_X f\,d\mu = \lim_{j\to\infty} \int_X f_j\,d\mu.$$

Corollary (44): If $f, g \in L^+$ and $f(x) \le g(x)$ for a.e. $x \in X$, then
$$\int_X f\,d\mu \le \int_X g\,d\mu.$$

Corollary (46): Suppose that $\{a_{j,k}\}_{j,k=1}^{\infty} \subset [0,\infty]$ satisfies $0 \leq a_{j,1} \leq a_{j,2} \leq ... \leq a_{j,k} \leq ...$ for each $j \in \mathbb{N}$. Then,

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{j,k} = \sum_{j=1}^{\infty} \lim_{k\to\infty} a_{j,k}.$$

Theorem (47): If $\{f_j\}_{j=1}^{\infty} \subseteq L^+$ and $f(x) = \sum_{j=1}^{\infty} f_j(x)$ for each $x \in X$, then

$$\int_X f \, d\mu = \sum_{j=1}^{\infty} \int_X f_j \, d\mu.$$

Proposition (48): If $f \in L^+$, then $\int_X f \, d\mu = 0$ if and only if $f = 0$ a.e. $x \in X$.

Corollary (49): If $\{f_j\}_{j=1}^{\infty} \subseteq L^+$ and $f \in L^+$ satisfy $f_j(x)$ increases to $f(x)$ for a.e. $x \in X$, then

$$\int_X f \, d\mu = \lim_{j\to\infty} \int_X f_j \, d\mu.$$

Lemma (50): (Fatou's Lemma) If $\{f_j\}_{j=1}^{\infty} \subseteq L^+$, then

$$\int_X (\liminf f_j) \, d\mu \leq \liminf_{j\to\infty} \int_X f_j \, d\mu.$$

Corollary (51): If $\{f_j\}_{j=1}^{\infty} \subseteq L^+$ and $\lim_{j\to\infty} f_j(x) = f(x)$ for a.e. $x \in X$, then

$$\int_X f \, d\mu \leq \liminf_{j\to\infty} \int_X f_j \, d\mu.$$

Proposition (52): If $f \in L^+$ and $\int_X f \, d\mu < \infty$, then $\{x \in X : f(x) < +\infty\}$ is a null set and $\{x \in X : f(x) > 0\}$ is a σ-finite set.

Define $\tilde{L}^1(X, \mathcal{M}, \mu)$ to be the collection of measurable functions $f : X \to \overline{\mathbb{R}}$ such that

$$\int_X |f| \, d\mu.$$

We say that f is **integrable**, or μ**-integrable**, if $f \in \tilde{L}^1$. If f is integrable, we define

$$\int_X f \, d\mu = \int_x f^+ \, d\mu - \int_X f^- \, d\mu.$$

Proposition (54): If $f \in \tilde{L}^1$, then

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

Proposition (55): Let $f, g \in \tilde{L}^1$ be given. The following are equivalent:

1. $\int_E f \, d\mu = \int_E g \, d\mu$ for all $E \in \mathcal{M}$.
2. $\int_X |f - g| \, d\mu = 0$
3. $f = g\mu$ a.e.

Definition (56): We define $L^1(X,\mu)$**, or just** $L^1(X)$**,** $L^1(\mu)$**, or** L^1 to be the collection of all equivalence classes of integrable functions with respect to the equivalence relation just described.

Proposition (57): Suppose that μ is a complete measure

1. If f is measurable and $f = g\mu$ a.e., then g is also measurable.
2. If $\{f_j\}_{j=1}^{\infty}$ is a sequence of measurable functions and $\lim_{j\to\infty} f_j(x) = f(x)\mu$ a.e. $x \in X$. Then f is measurable. (Compare with prop. 26)

Proposition (58): Let $(X, \mathcal{M}, \mu)$ be the completiong of $(X, \mathcal{M}, \mu)$. If $f : X \to \overline{\mathbb{R}}$ is $\mathcal{M}$-measurable, then there is a $g : X \to \overline{\mathbb{R}}$ that is $\mathcal{M}$-measurable such that $g = f\mu$ a.e.

(Convention) We do not distinguish between $L^1(\mu)$ and $L^1(\overline{\mu})$.

Definition (59): Define $\rho_1 : L^1(\mu) \times L^1(\mu) \to [0,\infty]$ by $\rho_1(f,g) := \int\limits_X |f-g|\, d\mu$.

Proposition (60): The function ρ_1 is a metric on $L^1(\mu)$.

Theorem (61): (Dominated Convergence Theorem DCT) Let $\{f_j\}_{j=1}^{\infty} \subseteq L^1(\mu)$ be a sequence satisfying

1. $\lim_{j\to\infty} f_j(x) = f(x)\mu$ a.e. $x \in X$
2. There is $g \in L^1(\mu)$ such that for all $j \in \mathbb{N}|f_j(x)| \leq g(x)\mu$ a.e. $x \in X$

Then, $f \in L^1(\mu)$ and

$$\int\limits_X f\, d\mu = \lim_{j\to\infty} \int\limits_X f_j\, d\mu.$$

Definition (62): If $\{f_j\}_{j=1}^{\infty} \subseteq L^1(\mu)$ satisfying $\lim_{j\to\infty} \rho_1(f_j, f) = 0$. Then we write $f_j \to f$ in L^1 and say that f_j **converges strongly to f in** L^1.

Corollary (63): Under the hypthesis in DCT, we actually have that $f_j \to f$ in L^1.

Definition (64): A **seminorm** on X is a function $\|\cdot\| : X \to [0,\infty)$ such that

1. $\|x+y\| \leq \|x\| + \|y\|$ for each $x, y \in X$
2. $\|\lambda x\| = |\lambda|\, \|x\|$ for each $\lambda \in \mathbb{R}$, or $\lambda \in \mathbb{C}$, and $x \in X$.

If $\|x\|$ also satisfies

1. $\|x\| = 0$ if and only if $x = 0$,

then we call this a norm on X. A pair $(X, \|\cdot\|)$, with $\|\cdot\|$ a norm, is called a **normed vector space**.

Example (65):

1. $\mathbb{R}$ is a vector space. The function $\|\cdot\|_p : \mathbb{R}^n \to [0,\infty)$ defined by

   $$\|x\|_p := \left(\sum_{j=1}^{n} |x_i|^p\right)^{1/p}, \text{ for all } p \in [1,\infty),$$

   is a norm. Also $\|\cdot\|_\infty : \mathbb{R}^n \to [0,\infty)$ defined by

   $$\|x\|_\infty = \max\{|x_1|, |x_2|, ..., |x_n|\}$$

   is also a norm.

2. Define $\|\cdot\|_1 : L^1(\mu) \to [0,\infty)$ by

$$\|f\|_1 := \int_X |f| \, d\mu$$

It is easy to show that $(L^1(\mu), \|\cdot\|_1)$ is a vector space. The topology induced by $\rho_{\|\cdot\|}$ is called the **norm (strong) topology**.

Definition (66): Two norms $\|\cdot\|$ and $\|\cdot\|_\#$ on X are **equivalent** if there exists constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \|x\| \le \|x\|_\# \le c_2 \|x\|, \text{ for all } x \in X.$$

Equivalents norms induce equivalent topologies.
Example (67):

1. In $\mathbb{R}^n$, each of the norms $\|\cdot\|_p$ for $p \in [0,\infty]$ are called equivalent.
2. In $\mathbb{R}^{\mathbb{N}}$ (the space of sequences of real numbers) the norms

$$\|\cdot\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p},$$

where $x = \{x_j\}_{j=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ are not equivalent.
3. As p increases, the topology gets bigger.
4. If $\{F\}$ converges for p, then $\{F_j\}$ converges for $k \le p$.

Definition (68): If $(X, \|X\|)$ is complete normed vector space with respect to the metric $\rho_{\|\cdot\|}$, then we call $(X, \|\cdot\|)$, or just X, a Banach Space.
Definition (69): Let $(X, \|\cdot\|)$ be a normed vector space and let $\{x_j\}_{j=1}^{\infty} \subseteq X$ be given. The series $\sum_{j=1}^{\infty} x_j$ **converges to** $x \in X$ if and only if $\lim_{N\to\infty} \sum_{j=1}^{N} x_j = x$ with respect to the norm, i.e.

$$\lim_{N\to\infty} \left\| \sum_{j=1}^{n} x_j - x \right\| = 0.$$

We say that $\sum_{j=1}^{N} x_j$ is **absolutely convergent** if $\sum_{j=1}^{\infty} \|x_j\| < \infty$.
Theorem (70): A normed vector space $(X, \|\cdot\|)$ is complete if and only if all absolute convergent sequences converge in X.
Theorem (71): The normed vector space $(L^1(\mu), \|\cdot\|_1)$ is a Banach Space. (Use Theorem 70 and absolutely convergent series to prove.)
Proposition (72): The set of simple functions in $L^1(\mu)$ is dense in $L^1(\mu)$ with respect to the strong topology.
Definition (73): For each $p \in (0,\infty)$, define $L^p(X,\mu)$, or $L^p(\mu)$, or L^p by

$$L^p(X,\mu) := \{f \text{ is measurable } : \int_X |f|^p \, d\mu < \infty\}.$$

Define

$$L^{\infty}(X,\mu) := \{f \text{ is measurable } : \text{ess sup}_{x\in X}|f(x)| < \infty\}.$$

Hence, for $f : X \to \overline{\mathbb{R}}$ measurable, then

$$\text{ess sup}_{x\in X} f(x) := \inf\{a \in \overline{\mathbb{R}} : \mu(\{x \in X : f(x) > a\}) = 0\}.$$

Definition (74): For each $p \in (0,\infty)$ define $\|\cdot\|_p : L^p(X,\mu) \to [0,\infty)$ by

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}.$$

Define $vnorm\cdot_\infty : L^{\infty}(X,\mu) \to [0,\infty)$ by

$$\|f\|_\infty := \text{ess sup}_{x\in X}|f(x)|.$$

Theorem (75): (Minkowski's Inequality) Suppose that $p \in [0,\infty)$. Let $f, g \in L^+$ be given. Then,

$$\left(\int_X (f+g)^p \, d\mu\right)^{1/p} \leq \left(\int_X f^p \, d\mu\right)^{1/p} + \left(\int_X g^p \, d\mu\right)^{1/p}.$$

Theorem (76): (Hölder's Inequality) Suppose that $p, q \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g \in L^+$ be given. Then

$$\int_X fg \, d\mu \leq \left(\int_X f^p \, d\mu\right)^{1/p} \left(\int_X g^q \, d\mu\right)^{\frac{1}{q}}.$$

Theorem (77): (Young's Inequality) Suppose that $p, q \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $a, b \in \mathbb{R}$ be given. Then,

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Theorem (78): The norm vector space $(L^p(\mu); \|\cdot\|_p)$ is a Banach space for all $p \in [1,\infty]$.
Theorem (79): For each $p \in [1,\infty]$, the simple functions in L^p are dense in L^p with respect to the norm topology.
Proposition (80): If $1 \leq p \leq q \leq r \leq \infty$, then $L^p \bigcap L^r \subseteq L^q$ and $\|f\|_q \leq \|f\|_p^{\lambda} \|f\|_r^{1-\lambda}$ where $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$.
Proposition (81): If μ is finite, and $1 \leq p \leq q \leq \infty$, then $L^q \subseteq L^p$ and $\|f\|_p \leq \|f\|_q \, \mu(X)^{\frac{1}{p}-\frac{1}{q}}$.

3 Modes of Convergence

Let $\{f_j\}_{j=1}^{\infty}$ and f be measurable functions.

(Uniform Convergence (Strongest Form)) We say $f_j \to f$ uniformly in X if for each $\epsilon > 0$, there is a $N(\epsilon)$ such that for all $j \geq N(\epsilon)$

$$|f_j(x) - f(x)| < \epsilon.$$

(Pointwise Convergence) We say $f_j \to f$ pointwise if for each $x \in X$ and each $\epsilon > 0$, there is $N(\epsilon, x)$ such that for $j \geq N(x,\epsilon)$

$$|f_j(x) - f(x)| < \epsilon.$$

(Convergence in L^p ($p \in [1,\infty]$)) We say $f_j \to f$ in $L^p(X,\mu)$ if for each $\epsilon > 0$, there is $N(\epsilon)$ such that for all $j \geq N(\epsilon)$

$$\|f_j - f\|_p < \epsilon.$$

(Convergence Almost Everywhere) We say $f_j \to f$ a.e. in X, if there is a μ-null set E such that $f_j \to f$ pointwise in $X \backslash E$.

(Convergence in Measure) We say $f_j \to f$ is measure if for each $\epsilon > 0$

$$\lim_{j\to\infty} \mu(\{x \in X : |f_j(x) - f(x)| \geq \epsilon\}) = 0.$$

Theorem (82): (Egoroff's Theorem) Suppose that μ is finite and $\{f_j\}_{j=1}^{\infty}$ and f are measurable functions such that $f_j \to f$ a.e. in X. Then for each $\alpha > 0$, there is a measurable $E \subseteq X$ such that $\mu(E) < \alpha$ and $f_j \to f$ uniformly on $X \backslash E$.

(Cauchy in Measure) We say that measurable functions $\{f_j\}_{j=1}^{\infty}$ is Cauchy in measure if for each $\epsilon > 0$,

$$\lim_{j,k\to\infty} \mu(\{x \in X : |f_j(x) - f_k(x)| \geq \epsilon\}) = 0.$$

Theorem (83): Suppose that $\{f_j\}_{j=1}^{\infty}$ is a sequence of measurable functions that are Cauchy is measure. Then there is a measurable f such that $f_j \to f$ in measure.

Theorem (84): If $\{f_j\}_{j=1}^{\infty}$ is a sequence of measurable functions such that $f_j \to f$ in measure, with f measurable, then there is a subsequence $\{f_{j_k}\}_{j=1}^{\infty} \subseteq \{f_j\}_{j=1}^{\infty}$ such that $f_{j_k} \to f \, \mu$ a.e. in X.

Theorem (85): If $\{f_j\}_{j=1}^{\infty}$ is a sequence of measurable functions such that $f_j \to f$ in measure and $f_j \to g$ in measure, with f and g measurable, then $f = g \, \mu$ a.e. in X.

Theorem (86): Let $p \in [1,\infty]$ be given. If $\{f_j\}_{j=1}^{\infty} \subseteq L^P(\mu)$ is such that $f_j \to f$ in L^p for some $f \in L^p$, then there is a subsequence $\{f_{j_k}\}_{k=1}^{\infty} \subseteq \{f_j\}_{j=1}^{\infty}$ such that $f_{j_k} \to f \, \mu$ a.e. in X.

4 Constructing Measures

Theorem (87): (Ulam) Suppose that μ is a measure on $\mathbb{R}$ such that every subset $E \subseteq \mathbb{R}$ is measurable. If $\mu([n, n+1)) < \infty$, for each $n \in \mathbb{Z}$ and $\mu(\{x\}) = 0$ for each $x \in \mathbb{R}$, then $\mu(E) = 0$ for all $E \subseteq \mathbb{R}$.

Definition (88): A family of sets $\xi \subseteq \mathcal{P}(X)$ is a **semi-algebra (elementary family)** on X if if has the following properties:

1. $\emptyset, X \in \xi$
2. If $E_1, E_2 \in \xi$, then $E_1 \bigcap E_2 \in \xi$.
3. If $E \in \xi$, then there is a finite disjoint sequence $\{E_j\}_{j=1}^{n} \subset \xi$ such that $E^c = \bigcup_{j=1}^{n} E_j$.

Example (89):

1. $\mathcal{I} := \{$open, half-open, and closed intervals in $\mathbb{R}\} \subseteq \mathcal{P}(\mathbb{R})$
2. $\mathcal{I}^n := \{$Cartesian Products of n members of $\mathcal{I}\} \subseteq \mathcal{P}(\mathbb{R}^n)$

Notation:

$I(a,b)$ will be used to denote any one of the following: $(a,b),[a,b],(a,b],[a,b)$.

Similarly define $I(-\infty,b)$ and $I(a,\infty)$.

Then, $\mathcal{I}=\{I(a,b): a\leq b \text{ and } a,b\in\mathbb{R}\}$ and $\mathcal{I}^n=\{\prod_{j=1}^n I(a_j,b_j): I(a_j,b_j)\in\mathcal{I} \text{ for all } 1\leq j\leq n\}$.

Definition (90): Given $n\in\mathbb{N}$, define $m^n:\mathcal{I}\to[0,\infty]$ by

$$m^n\left(\prod_{j=1}^n I(a_j,b_j)\right)=\prod j=1^n(b_j-a_j).$$

When n is understood, we write m for m^n.

Definition (91): Let $\xi\subseteq\mathcal{P}(X)$ be a semi-algebra. A textbfset function $\mu:\xi\to[0,\infty]$ is called:

1. **Monotone** if $\mu(E)\leq\mu(F)$ whenever $E,F\in\xi$ and $E\subseteq F$.
2. **Finitely Additive** if $\mu\left(\bigcup_{j=1}^n E_j\right)=\sum_{j=1}^n\mu(E_j)$ whenever $\{E_j\}_{j=1}^n\subseteq\xi$ are disjoint and $\bigcup_{j=1}^n E_j\in\xi$.
3. **Countably Additive** if $\mu\left(\bigcup_{j=1}^\infty E_j\right)=\sum_{j=1}^\infty\mu(E_j)$ whenever $\{E_j\}_{j=1}^\infty\subseteq\xi$ are disjoint and $\bigcup_{j=1}^\infty E_j\in\xi$.
4. **Countably Subadditive** if $\mu\left(\bigcup_{j=1}^\infty E_j\right)\leq\sum_{j=1}^\infty\mu(E_j)$ whenever $\{E_j\}_{j=1}^\infty\subseteq\xi$ and $\bigcup_{j=1}^\infty E_j\in\xi$.

Example (92): It is possible to show that m^n is monotone, finitely additive, and countably subadditive. See notes!

Definition (93): A family of sets $\mathcal{A}\subseteq\mathcal{P}(X)$ is called an **algebra** if the following hold:

1. $\emptyset,X\in\mathcal{A}$;
2. If $E_1,E_2\in\mathcal{A}$, then $E_1\bigcap E_2\in\mathcal{A}$;
3. If $E\in\mathcal{A}$, then $E^c\in\mathcal{A}$.

Proposition (94): Let $\mathcal{C}\subset\mathcal{P}(X)$ be given. There is a unique smallest algebra $\mathcal{A}(\mathcal{C})\subseteq\mathcal{P}(X)$ such that $\mathcal{C}\subseteq\mathcal{A}(\mathcal{C})$ and if $\mathcal{F}\subset\mathcal{P}(X)$ is an algebra containing $\mathcal{C}$, then $\mathcal{A}(\mathcal{C})\subseteq\mathcal{F}$.

Definition (95): The algebra $\mathcal{A}(\mathcal{C})$ provided by proposition 94 is the **algebra generated** by $\mathcal{C}$.

Proposition (96): Suppose that $\xi\subset\mathcal{P}(X)$ is a semi-algebra. Then,

$$\mathcal{A}(\xi)=\{E\subseteq X:\ E=\bigcup_{j=1}^n E_j \text{ for some disjoint sequence } \{E_j\}_{j=1}^n\subseteq\xi\}$$

Example (97):

$$\mathcal{A}(\mathcal{I})=\{\text{Finite unions of intervals in } \mathbb{R}\}$$
$$\mathcal{A}(\mathcal{I}^n)=\{\text{Finite unions of boxes in } \mathbb{R}^n\}$$

Theorem (98): Suppose that μ is a finitely additive, countably subadditive set function on a semi-algebra ξ such that $\mu(\emptyset)=0$. There is a unique countably additive set function $\tilde{\mu}$ on $\mathcal{A}(\xi)$ such that $\tilde{\mu}(E)=\mu(E)$ for all $E\in\xi$.

Theorem (99): Let $\mathcal{A}$ be an algebra of sets and let $\tilde{\mu}:\mathcal{A}\to[0,\infty]$ such that $\tilde{\mu}(\emptyset)=0$ be given. Then, $\tilde{\mu}$ is countably additive if and only if it is both finitely additive and countably subadditive.

Definition (100): Suppose that $\mathcal{A}\subset\mathcal{P}(X)$ is an algebra. A function $\tilde{\mu}:\mathcal{A}\to[0,\infty]$ is a **premeasure** on $\mathcal{A}$ if the following hold:

1. $\tilde{\mu}(\emptyset)=0$

2. $\tilde{\mu}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \tilde{\mu}(E_j)$, where $\{E_j\}_{j=1}^{\infty}$ are each disjoint and $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.

Notation:
Define $\mathcal{H}$ by

$$\mathcal{H} = \{(a,b] : (a,b] \subseteq \mathbb{R}\} \bigcup \{(a,\infty) : a \in \overline{\mathbb{R}}\}.$$

Then, $\mathcal{H}$ is a semialgebra and $\mathcal{B}_{\mathbb{R}}$ is generated by $\mathcal{H}$.
Given a function $F : \mathbb{R} \to \mathbb{R}$, define

$$F(\infty) = \lim_{x\to\infty} F(x)$$
$$F(-\infty) = \lim_{x\to-\infty} F(x),$$

provided the limits exist in $\overline{\mathbb{R}}$.
Proposition (101): Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Define $\mu_F : \mathcal{H} \to [0,\infty]$ by

$$\mu_F((a,b]) := F(b) - F(a) \text{ if } (a,b] \in \mathbb{R}.$$

and

$$\mu_F((a,\infty)) : F(\infty) - F(a) \text{ if } a \in [-\infty,\infty).$$

Then, μ_f is well-defined. Moreover μ_F is finitely additive and monotone. If F is right continuous, then μ_F is countably subadditive.
Proposition (102): Let $F : \mathbb{R} \to \mathbb{R}$ (finite for finite values of $\mathbb{R}$) be non-decreasing and right-continuous. Define $\mu_F : \mathcal{H} \to [0,\infty]$ as in proposition 101. Then, $\tilde{\mu}_F : \mathcal{A}(\mathcal{H}) \to [0\infty]$ defined by

$$\tilde{\mu}_F\left(\bigcup_{j=1}^{n} \mathcal{I}_j\right) = \sum_{j=1}^{n} \mu_F(\mathcal{I}_j),$$

whenever $\{\mathcal{I}_j\}_{j=1}^{n} \subseteq \mathcal{H}$ is mutually disjoint, is a premeasure on $\mathcal{A}(\mathcal{H})$.
Proposition (103): Suppose that $\mu : \mathcal{H} \to [0,\infty]$ is finitely additive and $\mu((a,b]) < \infty$ for each $a, b \in \mathbb{R}$. Then there is an $F : \mathbb{R} \to \mathbb{R}$ that is non-decreasing such that

$$\mu((a,b]) = F(b) - F(a), \text{ for } a, b \in \mathbb{R}.$$

If μ is also countably additive, then F is right-continuous and $\mu = \mu_F$, as defined in proposition 101.
Definition (104): An **outer measure** on X is a monotone countably subadditive set function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ and $\mu^*(\emptyset) = 0$.
Proposition (105): Let $\mathcal{C} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{C} \to [0,\infty]$ such that $\emptyset, X \in \mathcal{C}$ and $\rho(\emptyset) = 0$. For each $E \subseteq X$ define

$$\mu^*(E) := \inf\{\sum_{j=1}^{\infty} \rho(E_j) : \{E_j\}_{j=1}^{\infty} \subseteq \mathcal{C} \text{ and } E \subseteq \bigcup_{j=1}^{\infty} E_j\}.$$

Then, μ^* is an outer measure.
Definition (106): We call μ^* in proposition 105 the **outer measure induced by** ρ.
Definition (107): If μ^* is an outer measure on X, then a set $A \subseteq X$ is μ^*-measurable if

$$\mu^*(E) = \mu^*\left(E \bigcap A\right) + \mu^*(E \bigcap A^c), \text{ for all } E \subseteq X.$$

Remark:
In order to show that $A \subseteq X$ is μ^*-measurable, it is sufficient to show that

$$\mu^*(E) \geq \mu^*(E \bigcap A) + \mu^*(E \bigcap A^c) \text{ for all } E \subseteq X.$$

Theorem (108): (Caratheodory's Theorem) If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra. Moreover, the restriction of μ^* to $\mathcal{M}$ is a complete measure.
Proposition (109): If $\tilde{\mu}$ is a premeasure on an algebra $\mathcal{A} \subset \mathcal{P}(X)$ and μ^* is the outer measure induced by $\tilde{\mu}$, then

1. $\mu^* \mid_{\mathcal{A}} = \tilde{\mu}$
2. Every set in $\mathcal{A}$ is μ^*-measurable.

Theorem (110): Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let $\tilde{\mu}$ be a premeasure on $\mathcal{A}$ If $\mathcal{M}$ is the σ-algebra generated y $\mathcal{A}$, then

1. There is a measure μ on $\mathcal{M}(\mathcal{A})$ such that $\mu \mid_{\mathcal{A}} = \tilde{\mu}$. (In fact, $\tilde{\mu} = \mu^*|_{\mathcal{M}}$.)
2. If ν is an extension of $\tilde{\mu}$ to $\mathcal{M}$ and ν is a measure, then $\nu(t) \leq \mu(E)$ for all $E \in \mathcal{M}$ and $\nu(E) = \mu(E)$ whenever $\mu(E) < \infty$.
3. If $\tilde{\mu}$ is *sigma*-finte, then μ is the unique extension of $\tilde{\mu}$ to a measure on $\mathcal{M}$.

Remark:
Actually Caratheodory's Theorem gives a complete measure on the σ-measurable sets. Usually we use μ_F for both the measure on $\mathcal{B}_{\mathbb{R}}$ and the measure on $\mathcal{M}_{\mu_F}$ is called the **Lebesgue-Stieltjes measure associated with** F, and F is called the **Distribution function for** μ_F. If $E \in \mathcal{M}_{\mu_F}$, then

$$\mu_F(E) = \inf\{\sum_{j=1}^{\infty}[F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j]\}$$

In fact, it can be shown that

$$\mu_F(E) = \inf\{\sum_{j=1}^{\infty}[F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)\}.$$

Definition (111): Let μ_F be a Lebesgue-Stieltjes measure on $\mathcal{M}_{\mu_F}$. if $g \in L^1(\mu_F)$ then

$$\int_{\mathbb{R}} g \, d\mu$$

is called the **Lebesgue-Stieltjes integral of** g with repsect to μ_F.
Theorem (112): If $F : \mathbb{R} \to \mathbb{R}$ is non-decreasing and continuously differentiable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$\int_{(a,b]} g \, d\mu_F = \int_a^b gF' \, dx.$$

Remark:
Since F is continuous above, it turns out that $\mu_F(\{x\}) = 0$ for all $x \in \mathbb{R}$, and

$$\int_{(a,b]} g \, d\mu = \int_{[a,b)} g \, d\mu = \text{ etc...}$$

Corollary (113): If $F(x) = x$, then $\mu_F = m$ and

$$\int_{(a,b]} g\,dm = \int_a^b g\,dx.$$

Definition (114): Let μ be a Borel measure on X and let $E \in \mathcal{B}_X$ be given. The measure μ is **outer regular** on E if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ is open}\}.$$

The measure μ is **inner regular** on E if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}.$$

μ is **regular** if it is inner regular and outer regular on all $E \in \mathcal{B}_X$.
Definition (115): A Borel measure μ on X is called a **Radon measure** if μ is

1. Finite on all compact sets in X.
2. Outer regular on all Borel sets.
3. Inner regular on all open sets.

Theorem (116): Let μ_F be a Lebesgue-Stieltjes measure on $\mathbb{R}$. Then μ_F is inner and outer regular on all $E \in \mathcal{B}_{\mathbb{R}}$. In fact, μ_F is "inner and outer regular on: all $E \in \mathcal{M}_{\mu_F}$. (Note: Strictly speaking, regularity was only defined on Borel sets, but analogue for generalized measurable sets is obvious.)
Corollary (117): If μ_F is a Lebesgue-Stieltjes measure, then it is a Radon measure. In fact, it is a regular measure.
Theorem (118): Let μ_F be a Lebesgue-Stieltjes measure. If $E \subseteq \mathbb{R}$, then the following are equivalent:

1. $E \in \mathcal{M}_{\mu_F}$
2. $E = V \backslash N_1$ for some G_δ set V and μ_F-null set N_1.
3. $E = H \bigcup N_2$ for some F_σ set H and μ_F-null set N_2.

Theorem (119): If $E \in \mathcal{L}^n$, then $E + s \in \mathcal{L}^n$ and $TE \in \mathcal{L}^n$, for each $s \in \mathbb{R}^n$ and $T \in \mathbb{R}^{n\times n}$. Moreover,

$$m(E+s) = m(E) \quad \text{and} \quad m(TE) = |\det T| m(E).$$

Corollary (120): Of $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$. Moreover,

$$m(E+s) = m(E) \quad \text{and} \quad m(rE) = |r|m(E) \text{ for all } s, r \in \mathbb{R}.$$

Remark:
If μ is another Radon measure in $\mathbb{R}^n$ such that $\mu(E+s) = \mu(E)$ for all $s \in \mathbb{R}^n$ and all $E \in \mathcal{B}_{\mathbb{R}^n}$, then there is a $c \geq 0$ such that $\mu = cm$.
Definition (121): Let $p \geq 0$ and $\delta > 0$. Define $H_{p,\delta}, H_p : \mathcal{P}(X) \to [0, \infty]$ by

$$H_{p,\delta}(E) := \inf\{\sum_{j=1}^{\infty} (\text{diam} E_j)^p : E \subseteq \bigcup_{j=1}^{\infty} E_j \text{ and diam} E_j \leq \delta\},$$

and

$$H_p(E) := \lim_{\delta\to 0^+} H_{p,\delta}(E).$$

The function H_p is called the **p-dimensional Hausdorff (outer) measure**.

Definition (122): An outer measure μ^* on (X,ρ) is called a metric outer measure if

$$\mu^*(A\bigcup B) = \mu^*(A) + \mu^*(B),$$

whenever $\rho(A,B) > 0$. Recall that $\rho(A,B) := \inf\{\rho(x,y) : x \in A, y \in B\}$. (This is almost finite additivity, but requires that the boundaries cannot touch.)

Proposition (123): If μ^* is a metric outer measure, then every Borel set is μ^*-measurable.

Proposition (124): Let A be a Borel set. If $H_p(A) < \infty$, then $H_q(A) = 0$ for all $q > p$. If $H_p(A) > 0$, then $H_q(A) = \infty$ for all $q < p$. It follows that

$$\inf\{p \geq 0 : H_p(A) = 0\} = \sup\{p \geq 0 : H_p(A) = \infty\}.$$

This value is called the Hausdorff dimension of A.

5 Product Spaces

Definition (125): Let $\{(X_a, \mathcal{M}_a)\}_{\alpha\in\mathcal{A}}$ be measurable spaces. The **product σ-algebra** on $X = \prod_{\alpha\in\mathcal{A}} X_\alpha$ is the σ-algebra generated by

$$\{\prod_{\alpha}^{-1} : E_\alpha \in \mathcal{M}_\alpha, \alpha \in \mathcal{A}\}.$$

Here $\prod_\alpha : X \to X_\alpha$ is the α^{th}-coordinate map.

Notation: This product σ-algebra is denoted by $\otimes_{\alpha\in\mathcal{A}} \mathcal{M}_\alpha$.

Proposition (126): If $\mathcal{A}$ is countable, then $\otimes_{\alpha\in\mathcal{A}} \mathcal{M}_\alpha$ is the σ-algebra generated by

$$\{\prod_{\alpha\in\mathcal{A}} E_\alpha : E_\alpha \in \mathcal{M}_{alpha}\}.$$

Proposition (127): Let $X_1, ..., X_n$ be metric spaces and let $X = \prod_{j=1}^n X_j$ be equipped with the product metric. Then $\otimes_{j=1}^n \mathcal{B}_{X_j} \subseteq \mathcal{B}_X$. If each X_j is separable, then $\otimes_{j=1}^n \mathcal{B}_{X_j} = \mathcal{B}_X$.

Corollary (128): $\mathcal{B}_{\mathbb{R}^n} = \otimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$.

Notation: Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces for a while.

Definition (129): If $E \in \mathcal{M}$ and $F \in \mathcal{N}$, then $E \times F$ is a measurable rectangle.

Note:

$\mathcal{M} \times \mathcal{N}$ is the σ-algebra generated by $\mathcal{R}$ by proposition 126. Also observe that $\mathcal{R}$ is a semi-algebra.

Theorem (130): Let $\pi : \mathcal{R} \to [0,\infty]$ be defined by

$$\pi(E \times F) = \mu(E)\nu(F).$$

Then π is well-defined, countably additive, and $\pi(\emptyset) = 0$.

Theorem (131): There is a unique extension of π to a premeasure on $\mathcal{A}(\mathcal{R})$.

Theorem (132): The premeasure $\tilde{\pi}$ generates an outer measure π^* on $X \times Y$. The restriction of π^* to $\mathcal{M} \times \mathcal{N}$ is a measure extending π. Moreover, if μ and ν are σ-finite then so is the measure
Notation:
We denote the measure
Definition (133): If $E \subseteq X \times Y$, then for each $x \in X$, the **x-section of** E is $E_x := \{y \in Y : (x, y) \in E\}$. For each $y \in Y$, the **y-section of** E is $E^y := \{x \in X : (x, y) \in E\}$. If $f : X \times Y \to \overline{\mathbb{R}}$, then the **$x$-section** f_x and **y-section** f^y of f are $f_x(y) = f(x, y)$ and $f_y(x) = f(x, y)$.
Proposition (134):

1. If $E \in \mathcal{M} \times \mathcal{N}$, then $E_x \in \mathcal{M}$ for all $x \in X$ and $E^y \in \mathcal{N}$ for all $y \in Y$.
2. If f is $\mathcal{M} \times \mathcal{N}$ measurable, then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{N}$-measurable for each $x \in X$ and $y \in Y$ respectively.

Definition (135): A collection $\mathcal{C} \subseteq \mathcal{P}(X)$ is a **monotone class** if it has the following properties:

1. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq ...$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{C}$.
2. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq ...$, then $\bigcap_{j=1}^{\infty} E_j \in \mathcal{C}$.

Note:
All σ-algebras are monotone classes. Given a subset $\xi \subseteq \mathcal{P}(X)$, there is a unique smallest monotone class $\mathcal{C}(\xi)$ such that $\xi \subseteq \mathcal{C}(\xi)$. This is called the **monotone class generated by** ξ.
Theorem (136): $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ-algebra if and only if it is an algebra and a monotone class.
Lemma (137): If $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra, then the monotone class $\mathcal{C}(\mathcal{A})$ and the σ-algebra $\mathcal{M}(\mathcal{A})$ are equal.
Theorem (138): Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measurable spaces. Let $E \in \mathcal{M} \times \mathcal{N}$ be given. Then the functions

$$x \mapsto \nu(E_x) \quad \text{and} \quad y \mapsto \mu(E^y)$$

are measurable, and

$$\mu \times \nu(E) = \int_X \nu(E_x)\, d\mu(x) = \int_X \mu(E^y)\, d\nu(y).$$

Theorem (139): (Fubini-Tonelli Theorem) Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite.

1. (Tonelli) If $f \in L^+(X \times Y)$, then

   $$g(x) = \int_Y f_x\, d\nu(y)$$

   and

   $$h(y) = \int_X f^y\, d\mu(x)$$

   are in $L^+(X)$ and $L^+(Y)$, respectively, and

   $$\int_{X \times Y} f(x, y)\, d(\mu \times \nu) = \int_X \int_Y f\, d\nu d\mu = \int_Y \int_X f\, d\mu d\nu. \tag{1}$$

2. (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ and $f^y \in L^1(\mu)$ for almost every $x \in X$ and almost every $y \in Y$. Moreover, the functions g and h from (a) are in $L^1(\mu)$ and $L^1(\nu)$ respectively and (1) holds.

6 Decomposition of Measures

Note: In this section, let $(X, \mathcal{M})$ be a measurable space.
Definition (140): Let μ and ν be measures on $\mathcal{M}$.

1. μ and ν are **singular (mutually)** and we write $\mu \perp \nu$, if there are disjoint sets $X_\mu, X_\nu \in \mathcal{M}$ such that $X = X_\mu \bigcup X_\nu$ and $\mu(E) = \mu(E \bigcap X_\mu)$ and $\nu(E) = \nu(E \bigcap X_\nu)$.

2. ν is **absolutely continuous with respect to** μ, and we write $\nu \ll \mu$, if for every $E \in \mathcal{M}$ we have $\nu(E) = 0$ whenever $\mu(E) = 0$.

3. ν is **diffuse with respect to** μ if for each $E \in \mathcal{M}$, we have $\nu(E) = 0$ whenever $\mu(E) < \infty$.

Note: Diffuse $\Rightarrow$ absolutely continuous.
Remarks:

1. If $\mu \perp \nu$, then X_ν is a μ-null set and X_μ is a ν-null set.

2. If ν is diffuse with respect to μ, then ν is absolutely continuous with respect to μ.

3. If $f \in L^+$, then $\nu : \mathcal{M} \to [0, \infty]$ defined by

$$\nu(E) := \int_E f \, d\mu \tag{2}$$

is a measure. By proposition 41, ν is absolutely continuous with respect to μ. It turns out that (2) characterizes all measures that are absolutely continuous with respect to μ if μ is σ-finite, i.e. $\nu \ll \mu$ if and only if (2) holds for some $f \in L^+$.

Theorem (141): Let μ and ν be measures on $\mathcal{M}$ with ν finite. Then, $\nu \ll \mu$ if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that $\nu(E) < \epsilon$ whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$.
Corollary (142): If $f \in L^1(\mu)$, then for each $\epsilon > 0$, there is $\delta > 0$ such that if $\mu(E) < \delta$, then $\left|\int_E f \, d\mu\right| < \epsilon$.
Notation: If $\nu(E) = \int_E f \, d\mu$, with $f \in L^+(\mu)$, then we may write $\frac{d\nu}{d\mu}$ for f, or we may write $d\nu$ for $f d\mu$. Sometimes we may refer to $d\nu$ as a measure, when really we need to integrate over a set to get a measure.
Theorem (143): (Radon-Nikodym Theorem) Let μ and ν be measures on $\mathcal{M}$. Suppose μ is σ-finite and $\nu \ll \mu$. Then there is an $\in L^+$ such that

$$\nu(E) = \int_E f \, d\mu.$$

For each $E \in \mathcal{M}$, the function f is unique upto a μ-null set. The function f is called the **Radon-Nikodym derivative of ν with respect to μ.**
Lemma (144): Let μ, ν be measures on $\mathcal{M}$. Define $\nu_{\text{ac}} : \mathcal{M} \to [0, \infty]$ by

$$\nu_{\text{ac}}(E) := \sup\{\int_E f \, d\mu : \, f \in L^+ \text{ and } \int_F f \, d\mu \le \nu(F) \text{ for all } F \subseteq E \text{ with } F \in \mathcal{M}\}.$$

Then,

1. ν_{ac} is a measure.

2. $\nu_{\text{ac}} \ll \mu$

3. For each $E \in \mathcal{M}$, there is an admissible f such that $\nu_{\text{ac}}(E) = \int_E f \, d\mu$. Moreover, if ν_{ac} if σ-finite, then f can be chosen independent of E.

Lemma (145): Let μ, ν be finite measures on $\mathcal{M}$. For each $E \in \mathcal{M}$, define

$$(\nu - \mu)^+(E) := \sup\{\nu(F) - \mu(F) : F \subseteq E \text{ and } F \in \mathcal{M}\}.$$

This is a measure, and for each $E \in \mathcal{M}$

$$(\nu - \mu)^+(E) := \sup\{\nu(F) - \mu(F) : F \subseteq E, F \in \mathcal{M}, \text{ and } (\mu - \nu)^+(F) = 0\}.$$

Definition (146): Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of measurable functions $f_\alpha : X \to \overline{\mathbb{R}}$. We call $f : X \to \overline{\mathbb{R}}$ an **essential supremum** of $\mathcal{F}$ if

1. f is measurable

2. $f(x) \geq f_\alpha(x)$ for each $\alpha \in \mathcal{A}$ and for μ-almost everywhere $x \in X$.

3. If $f(x) \leq g(x)$ for μ-almost everywhere $x \in X$, whenever g satisfies (a) and (b).

Definition (147): Let $\{E_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{M}$ be given. A set $E \in \mathcal{M}$ is called an **essential union** of $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ if χ_E is an essential supremum of $\{\chi_{E_\alpha}\}$.

Remark: It can be shown that if $\{\chi_{E_\alpha}\}_{\alpha \in \mathcal{A}}$ has an essential supremum, then $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ has an essential union.

Definition (148): A measure μ on $\mathcal{M}$ is **localizable** if any family of measurable sets has an essential union. Any σ-finite set is localizable.

Theorem (149): Let μ be a σ-finite measure on $\mathcal{M}$. Let $\mathcal{F} = \{f_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of measurable functions $f_\alpha : X \to \overline{\mathbb{R}}$. Then there is a countable set $A \subseteq \mathcal{A}$ such that the function $x \mapsto \sup_{\alpha \in I} f_\alpha(x)$ is an essential supremum.

Theorem (150): (Radon-Nikodym Theorem II) Let μ, ν be measures on $\mathcal{M}$. Suppose that mu is localizable, $\nu \ll \mu$ and that

$$\nu(E) = \sup\{\nu(E \bigcap F) : F \in \mathcal{M} \text{ and } \mu(F) < \infty\}$$

Moreover, if μ is semi-finite, then f is unique up to a μ-null set.

Remark: Some hypotheses on μ must be assumed to get some version of the Radon-Nikodym Theorem.

Theorem (151): (Lebesgue Decomposition Theorem) Let μ, ν be measures on $\mathcal{M}$ with μ and ν σ-finite. Then there exists measures λ, ρ on $\mathcal{M}$ such that

$$\rho \ll \mu, \lambda \perp \mu, \text{ and } \nu = \rho + \lambda.$$

Moreover, this decomposition is unique; i.e. if

$$\nu = \overline{\rho} + \overline{\lambda} \text{ with } \overline{\rho} \ll \mu \text{ and } \overline{\lambda} \perp \mu,$$

then $\overline{\rho} = \rho$ and $\overline{\lambda} = \lambda$.

Theorem (152): Let μ, ν be measures on $\mathcal{M}$, with μ σ-finite. Then, $\nu = \nu_{\text{ac}} + \nu_s$, with $\nu_{\text{ac}} \ll \mu$ given in Lemma 144 and

$$\nu_s(E) := \{\nu(F) : F \in \mathcal{M}, F \subseteq E, \text{ and } \mu(F) = 0\}.$$

7 Signed Measures

Definition (153): A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

1. $\nu(\emptyset) = 0$
2. ν takes at most one of the values $+\infty$ or $-\infty$, i.e.

$$\nu : \mathcal{M} \to (-\infty, \infty] \quad \text{or} \quad \nu : \mathcal{M} \to [-\infty, \infty).$$

3. If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ are mutually disjoint, then

$$\nu\left(\bigcup_{j=1}^{\infty} E_n\right) = \sum_{j=1}^{\infty} \nu(E_n)$$

and the sum converges absolutely if $\left|\nu\left(\bigcup_{j=1}^{\infty} E_n\right)\right| < \infty$.

Remarks: Condition (c) can be simplified to just countable additivity by using a Rearrangement argument.
Example (154):

1. If μ_1 and μ_2 are positive measures with at least one of these finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.
2. If f is a measurable function such that either

$$\int_X f^+ \, d\mu \quad \text{or} \quad \int_X f^- \, d\mu$$

is finite, then

$$\nu(E) = \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

is a signed measure. Such f are called **extended μ-integrable**.

Proposition (155): Let ν be a signed measure on $\mathcal{M}$. Then the following hold:

1. (Continuity from Below) If $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq ...$, then $\lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right)$.
2. (Continuity from Above) If $\{E_n\}_{n=1}^{\infty} \supseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq ...$, then $\lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcap_{n=1}^{\infty} E_n\right)$.

Definition (156): Let ν be a signed measure on $\mathcal{M}$. A set $E \in \mathcal{M}$ is called

1. **positive** if $\nu(F) \geq 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$.
2. **negative** if $\nu(F) \leq 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$.
3. **null** if $\nu(E) = 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$.

Example (157): Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function in $L^1(m)$ with m the Lebesgue measure on $\mathcal{B}_{\mathbb{R}}$; e.g. $f(x) = xe^{-x^2}$, with $f(x) > 0$ for $x > 0$. Define ν by

$$\nu(E) = \int_E f \, d\mu.$$

1. E is a positive set if $E \subseteq [0,\infty)$ except for possibly some m-null set.

2. E is a negative set if $E \subseteq (-\infty,0]$ except for possibly some m-null set.

3. E is a null set if $m(E)=0$.

Proposition (158): Let ν be a signed measure on $\mathcal{M}$. Let $E \in \mathcal{M}$ be such that $0 < \nu(E) < \infty$. Then there ecists a positive set $F \in \mathcal{M}$ such that $F \subseteq E$ and $\nu(F) > 0$.

Theorem (159): (Hahn Decomposition Theorem) Let ν be a signed measure on $\mathcal{M}$. Then there ecists $\mathscr{P}, \mathscr{N} \in \mathcal{M}$ such that $\mathscr{P} \bigcap \mathscr{N} = \emptyset, \mathscr{P} \bigcup \mathscr{N} = X, \mathscr{P}$ is positive, and $\mathscr{N}$ is negative.

Definition (160): A pair of sets $\mathscr{P}$ and $\mathscr{N}$ satisfying the conclusion of Theorem 159 is called a **Hahn Decomposition**.

Theorem (161): (Jordan Decomposition Theorem) Let ν be a signed measure on $\mathcal{M}$. There are unique measures ν^+ and ν^- on $\mathcal{M}$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Definition (162): Let ν be a signed measure on $\mathcal{M}$. The decomposition in Theorem 161 is called the **Jordan Decomposition of** ν. The measure ν^+ is called the **positive variation**, or **upper variation**. The measure ν^- is called the **negative or lower variation**. The positive measure $|\nu| = \nu^+ + \nu^-$ is the **total variation of** ν.

Definition (163): Let ν and ω be signed measures on $\mathcal{M}$.

1. $E \in \mathcal{M}$ is **σ-finite** (with respect to ν) if it is σ-finite for $|\nu|$.

2. ν is **σ-finite** if $|\nu|$ is σ-finite.

3. ν and ω are **mutually singular** if $|\nu| \perp |\omega|$.

4. ν is **absolutely continuous with respect to** ω if $|\nu| \ll |\omega|$.

Definition (164): Suppose that ν is a signed measure on $\mathcal{M}$. Set $L^1(\nu) = L^1(\nu^+) \bigcap L^1(\nu^-)$ and define

$$\int_X f\,d\nu = \int_x f\,d\nu^+ - \int_X f\,d\nu^-, \text{ for all } f \in L^1(\nu).$$

Theorem (165): (Lebesgue-Radon-Nikodym Theorem) Let ν be a signed measure (σ-finite) on $\mathcal{M}$ and let μ be a positive σ-finite measure on $\mathcal{M}$. Then there are unique σ-finite signed measures λ, ρ on $\mathcal{M}$ such that $\lambda \perp \mu, \rho \ll \mu$, and $\nu = \rho + \lambda$. Moreover, there is an extended μ-integrable function f such that $d\rho = f\,d\mu$ ($\rho(E) = \int_E f\,d\mu$). The function f is unique up to a μ-null set and the decomposition of ν into λ and ρ is unique.

Example (166):

1. Let ν be a σ-finite measure on $\mathcal{M}$. Let $\mathscr{P}$ and $\mathscr{N}$ be a Hahn Decomposition for ν. Then,

$$\begin{aligned} \nu(E) &= \nu^+(E) - \nu^-(E) \\ &= \int_E \chi_{\mathscr{P}}\,d\nu^+ - \int_E \chi_{\mathscr{N}}\,d\nu^- \\ &= \int_E [\chi_{\mathscr{P}} - \chi_{\mathscr{N}}]\,d|\nu|. \end{aligned}$$

   Thus, $\nu \ll |\nu|$ and $\chi_{\mathscr{P}} - \chi_{\mathscr{N}} = \frac{d\nu}{d|\nu|}$.

2. See notes from class.

Example (167): See notes from class. (Cantor Vitalli Function)

8 Differentiation

Definition (168): A family $\{E_r\}_{r>0} \subset \mathcal{B}_{\mathbb{R}^n}$ is said to **shrink nicely** to $x \in \mathbb{R}^n$ if

1. $E_r \subseteq B(r,x)$ for each $r > 0$.
2. There is an $\alpha > 0$ such that $\mu(E_r) \geq \alpha m\left(B(r,x)\right)$ for all $r > 0$.

Definition (169): A measurable function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is **locally integrable** with respect to m if

$$\int_K |f| \, dm < \infty$$

for every compact set $K \subseteq \mathcal{B}_{\mathbb{R}^n}$. (Bounded in Folland) The space of locally integrable functions on $\mathbb{R}^n$ is $L^1_{loc}(\mathbb{R}^n)$ or just L^1_{loc}.

Definition (170): Let $f \in L^1_{loc}$ be given. For each bounded set $E \in \mathcal{B}_{\mathbb{R}^n}$ with $m(E) > 0$, we define the **mean value** of f on E by

$$⨍_E f \, dm = \frac{1}{m(E)} \int_E f \, dm.$$

Theorem (171): (Lebesgue Differentiation Theorem) Suppose that $f \in L^1_{loc}$. For each **Lebesgue Point of** f, $x \in \mathbb{R}^n$, we have

$$\lim_{r \to 0^+} ⨍_{E_r} |f(y) - f(x)| \, dm(y) = 0$$

and

$$\lim_{r \to 0^+} ⨍_{E_r} f(y) \, dm(y) = f(x)$$

for each $\{E_r\}_{r>0} \subseteq \mathcal{B}_{\mathbb{R}^n}$ that shrink nicely to x.

Theorem (172): Let $f \in L^1_{loc}$ be given and let g be any representative for L^1_{loc}. Then for almost every $x \in \mathbb{R}^n$

$$\lim_{r \to 0^+} ⨍_{B(r,x)} g \, dm = f(x).$$

Aside: The function

$$f^*(x) := \begin{cases} \lim_{r \to 0^+} ⨍_{B(r,x)} f \, dm, & \text{if limit exists} \\ 0, & \text{otherwise} \end{cases}$$

is called the precise representative for the equivalence class $f \in L^1_{loc}$.

Definition (173): For each $f \in L^1_{loc}$ (f is a representative), the set

$$L_f := \{x \in \mathbb{R}^n : \lim_{r \to 0^+} ⨍_{B(r,x)} |f(y) - f(x)| \, dm(y) = 0\}$$

is called the **Lebesgue set** of f. The points in L_f are called **Lebesgue points**.

Theorem (174): If $f \in L^1_{loc}$, then $m(\mathbb{R}^n \backslash L_f) = 0$.

Definition (176): Let $f \in L^1_{loc}$ be given. The **Hardy Littlewood Maximal function** $Hf : \mathbb{R}^n \to [0, \infty]$ is given by

$$Hf(x) = \sup_{r>0} ⨍_{B(r,x)} |f| \, dm.$$

Proposition (177): Let $f, g \in L^1_{loc}$ be given. Then

1. $0 \leq Hf(x) \leq +\infty$ for all $x \in \mathbb{R}^n$
2. $H(f+g) \leq Hf + Hg$
3. $H(cf) = |c|H(f)$
4. Hf is lower semicontinuous.
5. Hf is $\mathcal{B}_{\mathbb{R}^n}$-measurable.

Example (178): See notes.

Theorem (179): (Chebychev's Inequality) Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^p(\mu)$, for some $p \in [1, \infty)$, then for each $\alpha > 0$,

$$\mu(\{x \in X : |f(x)| > \alpha\}) \leq \frac{1}{\alpha^p} \|f\|_{L^p}^p .$$

Definition (180): Let $(X, \mathcal{M}, \mu)$ be a measure space. For a measurable function $f : X \to \overline{\mathbb{R}}$, define for $p \in [1, \infty)$

$$[f_p] := \sup_{\alpha > 0} \{\alpha^p \mu(\{x \in X : |f(x)| > \alpha\})\}.$$

We say that f is in weak-L^p if $[f_p] < \infty$.
Remarks:

- $[f_p]$ is not a norm. It doesn't satisfy the triangle inequality.
- $L^p \subseteq$ weak-L^p and $[f_p] \leq \|f\|_{L^p}$.
- Although $Hf \notin L^1$ in general, it will belong to weak-L^1 if $f \in L^1$.

Theorem (181): There is a constant $c > 0$ (3^n) such that for each $f \in L^1$ and all $\alpha > 0$

$$m(\{x \in X : Hf(x) > \alpha\}) \leq \frac{c}{\alpha} \|f\|_{L^1} .$$

Lemma (182): (Simple Vitalli Covering Lemma) Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$ and set $U := \bigcup_{B \in \mathcal{C}} B$. If $\tau < \mu(U)$, then there exists disjoint balls $\{B_j\}_{j=1}^k \subseteq \mathcal{C}$ such that

$$\sum_{j=1}^k m(B_j) > \frac{\tau}{3^n}.$$

(Baire Category Theorem) Let X be a complete metric space.

1. If $\{U_n\}_{n=1}^\infty$ are open dense sets in X, then $\bigcap_{n=1}^\infty U_n$ is dense in X.
2. X is not the countable union of nowhere dense sets.

Recall: $E \subseteq X$ is nowhere dense if $\overline{E}$ has empty interior.

Definition (I): f X is a topological space, then $E \subseteq X$ is **of the first category**, or **meager** if E is the countable union of nowhere dense sets. All other sets are called **of the second category**. The complements of meager sets are called **residual**.

See notes for implications of above results. The two above results and their implications were part of the extra lectures.

Definition (183): A sgned measure ν on $\mathcal{B}_{\mathbb{R}^n}$ is **regular** if

1. $|\nu|(K) < \infty$ for each compact $K \subseteq \mathbb{R}^n$.
2. $|\nu|(E) = \inf\{|\nu|(U) : U \text{ is open and } E \subseteq U\}$ for all $E \in \mathcal{B}_{\mathbb{R}^n}$

Remark: If ν is positive, then definition 183 is equivalent to definition 114.

- If ν is regular, then $|\nu|$ is a Radon measure.
- If ν is regular, then ν is σ-finite.
- If $d\nu = f\,dm$, then ν is regular if and only if $|f|$ is locally integrable. (Argument in Folland)

Theorem (184): Let ν be a regular signed measure on $\mathcal{B}_{\mathbb{R}^n}$. Let $d\nu = d\lambda + f dm$ be its Lebesgue-Radon-Nikodym Decomposition with respect to m. Then for m-almost everywhere $x \in \mathbb{R}^n$

$$\lim_{r\to 0^+} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

for every family $\{E_r\}_{r>0} \subset \mathcal{B}_{\mathbb{R}^n}$ that shrink nicely to x. (Here ν regular$\Rightarrow$ ν is σ-finite $\Rightarrow$ Leb-Rad-Nik Thm. can be used.)

Remark: Example 166 continued in notes here.

Theorem (185): (Lusin's Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a Borel-measurable function. Suppose that there is $A \in \mathcal{B}_{\mathbb{R}^n}$ such that $m(A) < \infty$ and $f(x) = 0$ for all $x \in A^c$. Fir each $\epsilon > 0$ there is a continuous function such that $g : \mathbb{R}^n \to \mathbb{R}$.

1. $\sup_{x\in\mathbb{R}^n} |g(x)| \leq \sup_{x\in\mathbb{R}^n} |f(x)|$
2. $m(\{x \in \mathbb{R}^n : g(x) \neq f(x)\}) < \epsilon$.

Theorem (186): (Differentiability of Monotone Functions) Let $F : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function and define$G(x) = F(x^+)$.

1. The points of discontinuity for F constitute an m-null set. In fact, the set is countable.
2. F is differentiable almost everywhere in $\mathbb{R}$ and $F' = G'$ almost everywhere in $\mathbb{R}$.

Definition (187): Let $F : \mathbb{R} \to \mathbb{R}$ be given. Define $T_F : \mathbb{R} \to [0, \infty]$ by

$$T_F(x) := \sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N} \text{ and } -\infty < x_0 < x_1 < .. < x_n = x\}$$

We call T_F the **total variation function** of F. If $[a, b] \subset \mathbb{R}$, then $T_F(b) - T_F(a)$ is the **total variation** of F over $[a, b]$.

Definition (188): Let $F : \mathbb{R} \to \mathbb{R}$ be given. If $T_F(\infty) < \infty$, then F is of **bounded variation** on $\mathbb{R}$. We set

$$BV(\mathbb{R}) := \{F : \mathbb{R} \to \mathbb{R} : T_F(\infty) < \infty\}.$$

Remarks:

- T_F is increasing.

- If F is in $BV(\mathbb{R})$, then

$$T_F(b) - T_F(a) = \sup\{\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N} \text{ and } a = x_0 < x_1 < ... < x_n = b\}.$$

- $BV(\mathbb{R})$ and $BV([a,b])$ are vector spaces. They can be made into Banach spaces.
- If $F \in BV([a,b])$, then

$$\overline{F} := \begin{cases} F(a), & x < a \\ F(x), & x \in [a,b] \\ F(b), & x > b \end{cases}$$

is in $BV(\mathbb{R})$.

Example (190):

- All bounded monotone functions are in BV.
- The function $\sin(x)$ is not in $BV(\mathbb{R})$, but the function $\sin(x)\chi_K$ is in BV for any compact $K \subseteq \mathbb{R}$.
- The function

$$F(x) = \begin{cases} x\sin(\frac{1}{x}), & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is not in $BV(\mathbb{R})$.

Lemma (191): If $F \in BV(\mathbb{R})$, then $x \mapsto T_F(x) + F(x)$ and $x \mapsto T_F(x) - F(x)$ are both nondecreasing.
Theorem (192): If $F \in BV(\mathbb{R})$, the $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ and $\frac{1}{2}(T_F + F)$ and $\frac{1}{2}(T_F - F)$ are nondecreasing. Also, if F is the difference of two non-decreasing functions, then $F \in BV(\mathbb{R})$.
Theorem (193):

1. If $F \in BV(\mathbb{R})$, then $F(x^+)$ and $F(x^-)$ exist for all $x \in \mathbb{R}$, $F(-\infty)$ exists, and $F(\infty)$ exists in $\mathbb{R}$.
2. If $F \in BV(\mathbb{R})$, then F is discontinuous on a set of at most countably many points.
3. If $F \in BV(\mathbb{R})$ and $G : \mathbb{R} \to \mathbb{R}$ is defined by $G(x) := F(x^+)$ for all $x \in \mathbb{R}$, then F' exists almost everywhere in $\mathbb{R}$, and $F' = G'$ almost everywhere in $\mathbb{R}$.

Definition (194): If $F \in BV(\mathbb{R})$, then the decomposition

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$$

is called the **Jordan Decomposition** of F, $\frac{1}{2}(T_F + F)$ is called the **positive variation** of F, and $\frac{1}{2}(T_F - F)$ is called the **negative variation** of F.
Definition (195): Set

$$NBV(\mathbb{R}) := \{F \in BV(\mathbb{R}) : F(-\infty) = 0 \text{ and } F \text{ is right continuous}\}.$$

Lemma (196): If $F \in BV(\mathbb{R})$, then $T_F(-\infty) = 0$. If $F \in BV(\mathbb{R})$ is right-continuous, then $T_F \in NBV(\mathbb{R})$.

Theorem (197): If μ is a finite signed Borel measure and $F : \mathbb{R} \to \mathbb{R}$ is defined by $F(x) = \mu((-\infty, x])$ for all $x \in \mathbb{R}$, then $F \in NBV(\mathbb{R})$. If $F \in NBV(\mathbb{R})$, then there is a unique signed Borel measure μ_F such that

$$\mu_F((-\infty, x]) = F(x) \text{ for all } x \in \mathbb{R}.$$

Moreover, $|\mu_F| = \mu_{T_F}$; i.e.

$$|\mu_F|((-\infty, x]) = T_F(x) \text{ for all } x \in \mathbb{R}.$$

Proposition (198): Let $F \in BV(\mathbb{R})$ be right-continuous. Then $F' \in L^1$. Moreover if $F \in NBV(\mathbb{R})$, then

1. $\mu_F \perp m$ if and only if $F' = 0$ m almost everywhere in $\mathbb{R}$.
2. $\mu_F \ll m$ if and only if

$$F(x) = \int_{(-\infty,x]} F' \, dm \text{ for all } x \in \mathbb{R}.$$

Definition (199): We say that $F : \mathbb{R} \to \mathbb{R}$ is **absolutely continuous** if for each $\epsilon > 0$, there is a $\delta > 0$ such that whenever $\{(a_j, b_j)\}_{j=1}^N$ are disjoint intervals satisfying

$$\sum_{j=1}^N b_j - a_j < \delta \text{ we have } \sum_{j=1}^N |F(b_j) - F(a_j)| < \epsilon.$$

We say F is **absolutely continuous on** $[a, b]$ if the above property holds with intervals taken from $[a, b]$.
Proposition (200): If $F \in NBV$, then F is absolutely continuous if and only if $\mu_F \ll m$.
Corollary (201): If $f \in L^1(m)$, then

$$\int_{(-\infty,x]} f \, dm \in NBV$$

is absolutely continuous and $F' = f$ almost everywhere. Also, if $F \in NBV$ is absolutely continuous, then $F' \in L^1$ and

$$F(x) = \int_{(-\infty,x]} F' \, dm.$$

Theorem (202): (Fundamental Theorem of Calculus for Lebesgue Integrals) Let $[a, b] \in \mathbb{R}$ and $F : [a, b] \to \mathbb{R}$ be given. The following are equivalent:

1. F is absolutely continuous.
2. $F(x) = F(a) + \int_{(a,x]} f \, dm$ for some $f \in L^1([a, b], m)$.
3. F' exists almost everywhere in $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) = F(a) + \int_{(a,x]} F' \, dm$.

Definition (203): If $F \in NBV$, then the integral of a Borel measurable function with respect to μ_F is called a **Lebesgue-Stieltjes integral**. (If it makes sense by definition 164.)

Theorem (204): (Integration by Parts) Suppose that $F, G \in NBV$ and either F or G is continuous. Then for any $[a,b] \subset \mathbb{R}$, we have

$$\int_{(a,b]} F\,dG = (F(b)G(b) - F(a)G(a)) - \int_{(a,b]} G\,dF.$$

Proposition (205): Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \mathcal{N})$ a measurable space. Let $T : X \to Y$ be measurable and define $\mu \circ T^{-1} : \mathcal{N} \to [0, \infty]$ by

$$\left(\mu \circ T^{-1}\right)(E) = \mu(T^{-1}(E)).$$

Then, $\mu \circ T^{-1}$ is a measure on $\mathcal{N}$.

Definition (206): The measure $\mu \circ T^{-1}$ is called the **push forward** of μ through T, or the **measure induced by μ and T**.

Example (207): See notes.

Theorem (208): Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(Y, \mathcal{N})$ be a measurable space. Suppose that $T : X \to Y$ is a measurable function. Then for any $\mathcal{N}$-measurable function $F : X \to \mathbb{R}$, we have

$$\int_X f(T(x))\,d\mu = \int_Y f(y)\,d\mu \circ T^{-1}$$

provided one of the integrals makes sense.

Corollary (209): Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let $T : X \to Y$ be measurable. If $\nu \ll \mu \circ T^{-1}$, then there is a measurable $\varphi : Y \to \mathbb{R}$ such that for each $f \in L^1(\nu)$ satisfying $f \circ T \in L^1(\mu)$, we have

$$\int_E f(y)\,d\nu(y) = \int_{T^{-1}(E)} f(T(x))\varphi(T(x))\,d\mu(x).$$

Theorem (210): Suppose that $T : [a,b] \to [c,d]$ is an increasing absolutely continuous bijection from$[a,b]$ to $[c,d]$. Let $f \in L^1([c,d], m)$ be given such that $f \circ T \in L^1([a,b])$. Then,

$$\int_{[a,b]} f(T(x))T'(x)\,dm(x) = \int_{[c,d]} f(y)\,dm(y).$$

Theorem (211): Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear transformation. If $f \in L^1(m)$, then $f \circ Y \in L^1(m)$ and

$$\int_{\mathbb{R}^n} f\,dm = |\text{det}T| \int_{\mathbb{R}^n} f \circ T\,dm.$$

Theorem (212): (Change of Variables) Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and that $G : \Omega \to \mathbb{R}^n$ is a C^1-diffeomorphism.

1. If $f \in L^1(G(\Omega), m)$, then

$$\int_{G(\Omega)} f(y)\,dm(y) = \int_\Omega f(G(x))|\text{det }D_x G(x)|\,dm(x).$$

2. If $E \subseteq \Omega$ and $E \in \mathcal{L}^n$, then $G(E) \in \mathcal{L}^n$ and

$$m(G(E)) = \int_E |\text{det }D_x(G(x))\,dm(x).$$

Theorem (213): There is a unique Borel measure σ_{n-1} on S^{n-1} such that $m_* = \rho_n \times \sigma_{n-1}$.
Some Formulae in Polar Coordinates:

- If $f \in L^1(\mathbb{R}^n, m)$, then
$$\int\limits_{\mathbb{R}^n} f(x)\, dm(x) = \int\limits_{(0,\infty)} \int\limits_{S^{n-1}} f(r\theta)\, d\sigma_{n-1}(\theta)\, d\rho_n(r).$$
- If $E \in \mathcal{B}_{S^{n-1}}$, it can be shown that $\sigma_{n-1}(E) = \mathscr{H}_{n-1}(E)$ and $r^{n-1}\sigma_{n-1}(E) = \mathscr{H}_{n-1}(rE)$. So
$$\int\limits_{\mathbb{R}^n} f(x)\, dm(x) = \int\limits_{(0,\infty)} \int\limits_{\partial B(r,0)} f(y)\, d\mathscr{H}_{n-1}(y)\, dm(r).$$
Moreover
$$\int\limits_{(0,s)} f(x)\, dm(x) = \int\limits_{(0,s)} \int\limits_{\partial B(r,0)} f(y)\, d\mathscr{H}_{n-1}(y)\, dm(r).$$
is absolutely continuous. So the FTC implies
$$F'(s) = \frac{d}{ds}\left[\int\limits_{(0,s)} f(x)\, dm\right] = \int\limits_{\partial B(r,0)} f(x)\, d\mathscr{H}_{n-1}(x).$$

9 Duality and Weak Convergence

Notation: For the sequel $(X, \mathcal{M}, \mu)$ is a measure space. Recall that for $f : X \to \mathbb{R}$ measurable, we define
$$\|f\|_{L^p} := \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}}, \quad \text{for } p \in [1, \infty),$$
and
$$\begin{aligned} \|f\|_{L^\infty} &:= \text{esssup}_{x\in X}|f(x)| \\ &:= \inf\{\alpha \geq 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}. \end{aligned}$$
For $p \in [1, \infty]$,
$$L^p = L^p(\mu) = L^P(X, \mu) := \{f : X \to \mathbb{R} : f \text{ is measurable } \|f\|_{L^p} < \infty\}.$$
For each $p \in [1, \infty]$, the space $(L^p, \|\cdot\|_{L^p})$ is a Banach space.
(Hölder's Inequality): Suppose $p, q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then
$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$
Definition (214): If $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ is a normed vector space. The space of continuous linear functionals $(L(\mathscr{X}, \mathbb{R}), |\!|\!|\cdot|\!|\!|)$ is called the **continuous dual space** of $\mathscr{X}$. It is denoted by $\mathscr{X}^*$ or $(\mathscr{X}^*, \|\cdot\|_{\mathscr{X}^*})$.
Remark: $\mathscr{X}^*$ is always a Banach space.
Example (215): See Notes.

Theorem (216): (Riesz Representation Theorem) Suppose that $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then for each $\varphi \in (L^p)^*$, there is $g \in L^q$ such that

$$\varphi(f) = \int_X fg\, dm, \quad \text{for all } f \in L^p.$$

If μ is σ-finite, then the same result also holds for $p = 1$ and $q = \infty$.
Remark: For each $g \in L^q$, clearly $\varphi_g : L^p \to \mathbb{R}$ defined by

$$\varphi_g(f) = \int_X fg\, d\mu$$

is a linear functional. By Hölder's Inequality

$$\begin{aligned} |\varphi_g(f)| &\leq \int_X |fg|\, d\mu \\ &\leq \|f\|_{L^p} \|g\|_{L^q} < \infty. \end{aligned}$$

Thus, $\varphi_g \in (L^p)^*$ and $\|\varphi\|_{(L^p)^*} \leq \|g\|_{L^q}$. Thus, L^q can be indentified with a subset of $(L^p)^*$. We want to show that $(L^p)^* \cong L^q$.
Theorem (217): Suppose that $p, q \in [1, \infty)$ and satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and that $g : X \to \mathbb{R}$ is measurable and satisfies

1. $fg \in L^1$ for all $f \in \Sigma$, where

$$\Sigma := \{h \in L^1 : h \text{ is simple.}\},$$

2. $M_q(g) := \sup \left\{ \left| \int_X fg\, d\mu \right| : f \in \Sigma \bigcap L^p \text{ and } \|F\|_{L^p} \leq 1 \right\} < \infty,$

3. Either $S_g := \{x \in X : |g(x)| \neq 0\}$ is σ-finite or μ is semi-finite,

then $g \in L^q$ and $\|g\|_{L^q} = M_q(g)$.
Remark: From the proof and Hölder's Inequality, we see that g is unique up to a μ-null set and $\|\varphi\|_{(L^p)^*} = \|g\|_{L^q}$.
Corollary (218): If $p \in (1, \infty)$, then L^p is reflexive; i.e. $(L^p)^{**}$ can be identified with L^p. (Using the remark we get an isometry.)
Definition (219): A **directed set** is a non-empty set A with a relation $\precsim$ such that

- $\alpha \precsim \alpha$ for all $\alpha \in A$.
- $\alpha \precsim \beta, \beta \precsim \gamma \Rightarrow \alpha \precsim \gamma$,
- If $\alpha, \beta \in A$, then there is $\gamma \in A$ such that $\alpha \precsim \gamma$ and $\beta \precsim \gamma$.

An element of a directed set is an **index**.
Example (220): See notes.
Definition (221): A **net** in X is a dunction or mapping $x : A \to X$ with A a directed set, we denote the function by $< x_\alpha >_{\alpha \in A}$. The set A is called the index set.
Definition (222): Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$ be given. Let $< x_\alpha >_{\alpha \in A}$ be a net.

- $< x_\alpha >$ is **eventually in E** if there is an $\alpha_0 \in A$ such that $x_\alpha \in E$ whenever $\alpha_0 \precsim \alpha$.

- $< x_\alpha >$ is **frequently in E**, if for every $\alpha \in A$ there is $\beta \in A$ such that $\alpha \precsim \beta$ and $x_\beta \in E$.
- $< x_\alpha >$ **converges** to a point $x \in X$ if for every open neighborhood U of x, $< x_\alpha >$ is eventually in U . We write $x_\alpha \to x$.

Proposition (223): If $(X, \mathcal{T})$ is a topological space and $(Y, \mathcal{S})$ is a topological space and $f : X \to Y$, then f is continuous at $x \in X$ if and only if for each net $< x_\alpha >$ converging to X, we have $< f(x_\alpha) >$ converges to $f(x)$; i.e.

$$x_\alpha \to x \Rightarrow f(x_\alpha) \to f(x).$$

Definition (224): Let X be set and $\{(Y_\alpha, \mathcal{T}_\alpha)$ is a family of topological spaces. Given a family $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$, there is a unique weakest topology on X that makes each f_α continuous. This topology is called the **weak topology** generated by $\{f_\alpha\}_{\alpha \in A}$.

Definition (225): Let $(\mathscr{X}, \|\cdot\|)$ be a normed vector space. The **weak topology** on $\mathscr{X}$ is the weak topology generated by $\mathcal{X}^*$. Convergence in the weak topology is called **weak convergence**.

Remark:

- If $< x_\alpha > \subseteq \mathcal{X}$ is a net, $x_\alpha \to x$ strongly if and only if for each $\epsilon > 0$, eventually $\|x_\alpha - x\|_{\mathscr{X}} < \epsilon$.
- On the other hand, $x_\alpha \to x$ weakly ($x_\alpha \rightharpoonup x$ in $\mathscr{X}$) if and only if $F(x_\alpha) \to f(x)$ for all $f \in \mathscr{X}^*$.
- Recall that x can be identified as a subset of $\mathscr{X}^{**}$. So each $x \in \mathscr{X}$ can be identified with a linear functional on $\mathscr{X}^*$.

Definition (226): Let $(X, \|\cdot\|)$ be a normed vector space. The **weak* topology** on $\mathscr{X}^*$ is the weak topology generated by $\mathscr{X}$. Convergence in the weak* topology is called **weak* convergence**.

Remark: If $< f_\alpha > \subseteq \mathscr{X}^*$ is a net, then $f_\alpha \to f$ weak* if and only if $f_\alpha(x) \to f(x)$ for all $x \in \mathscr{X}$.

(Alaoglu's Theorem) The unit ball in $\mathscr{X}^*$ is compact in weak* topology.

Definition (227): A **subnet** of a net $< x_\alpha >_{\alpha \in A}$ is a net $< y_\beta >_{\beta \in B}$ together with a map $\beta \mapsto \alpha_\beta$ from B into A such that

- For each $\alpha_0 \in A$ there is a $\beta_0 \in B$ such that $\alpha_\beta \succsim \alpha_0$ whenever $\beta \succsim \beta_0$.
- $y_\beta = x_{\alpha_\beta}$.

Theorem: If $\mathscr{X}$ is reflexive then something holds for weak topology on $\mathscr{X}^*$.

Corollary (228): (To Riesz's Theorem) Suppose μ is σ-finite and $< f_\alpha >$ is a bounded net in L^∞. Then there is a subnet $< g_\beta >$ of $< f_\alpha >$ and a function $g \in L^\infty$ such that $g_\beta \overset{*}{\rightharpoonup} g$ in L^∞; i.e.

$$\int_X g_\beta f \, d\mu \to \int_X g f \, d\mu, \qquad \text{for all } f \in L^1.$$

Corollary (229): If $p \in (1, \infty)$ and $< f_\alpha >$ is a bounded net in L^p, then there is a subnet $< g_\beta >$ of $< f_\alpha >$ and a $g \in L^p$ such that $g_\beta \rightharpoonup g$ in L^p; i.e.

$$\int_X g_\beta f \, d\mu \to \int_X g f \, d\mu, \qquad \text{for all } f \in L^q. \quad (q = \frac{p}{p-1})$$

Corollary (230): If $p \in (1, \infty)$ and $\{f_n\}_{n=1}^{\infty}$ is bounded in L^p, then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \rightharpoonup f$ in L^p for some $f \in L^p$.

Remark: The following is a result of the uniform boundedness principle (5.13) in Folland.

Corollary (231): If $p \in (1, \infty)$ and $\{f_n\}_{n=1}^{\infty}$ in L^p such that $f_n \rightharpoonup f$ in L^p, then $\sup_{n \in \mathbb{N}} \|f_n\|_{L^p} < \infty$.

Theorem (232): (Dunford-Pettis) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n=1}^{\infty} \subset L^1$ be given.

1. $f_n \rightharpoonup f$ in L^1 implies $\{f_n\}_{n=1}^{\infty}$ is bounded in L^1 and are equiintegrable.

2. If $\{f_n\}_{n=1}^{\infty} \subseteq L^1$ is bounded and equiintegrable, then there is a subsequence and an $f \in L^1$ such that $f_{n_k} \rightharpoonup f$ in L^1.

Definition (233): A linear functional $I : C_c(\mathbb{R}^n) \to \mathbb{R}$ is called **positive** if $I(f) \geq 0$ whenever $f \geq 0$.

Theorem (234): Suppose that I is a positive linear functional on $C_c(\mathbb{R}^n)$. Then there is a unique radon measure μ on $\mathbb{R}^n$ such that $I(f) = \int_{\mathbb{R}^n} f\, d\mu$ for all $f \in C_c(\mathbb{R}^n)$. Moreover μ satisfies

1. $\mu(U) = \sup\{I(f) : f \in C_c(\mathbb{R}^n), 0 \leq f \leq 1 \text{ and } \operatorname{supp}(f) \leq U\}$ for all open sets $U \subseteq \mathbb{R}^n$.

2. $\mu(K) = \int\{I(f) : f \in C_c(\mathbb{R}^n) \text{ and } f \geq \chi_K\}$ for all compact $K \subseteq \mathbb{R}^n$.

Definition (235): A **signed radon measure** ν is a signed measure such that ν^+ and ν^- are both Radon measures.

Theorem (236): (Riesz Representation Theorem for C_0)

1. Let $I \in C_0(\mathbb{R}^n)^+$ be given. Then there is a unique $\mu \in M(\mathbb{R}^n)$ such that $I(f) := \int_{\mathbb{R}^n} f\, d\mu$ for all $f \in C_0(\mathbb{R}^n)$.

2. For each $\mu \in \mathbb{R}^n$ and $f \in C_0(\mathbb{R}^n)$, put

$$I_\mu(f) := \int_{\mathbb{R}^n} f\, d\mu.$$

   Then $I_\mu \in (C_0(\mathbb{R}^n))^*$ and the map $\mu \mapsto I_\mu$ is an isometric isomorphism $M(\mathbb{R}^n)$ to $C_0(\mathbb{R}^n)^*$.

Corollary (237): If $\{\mu_j\}_{j=1}^{\infty}$ is bounded in $M(\mathbb{R}^n)$, there is a subsequence $\{\mu_{j_k}\}_{k=1}^{\infty}$ such that $\mu_{j_k} \overset{*}{\rightharpoonup} \mu$ for some $\mu \in M(\mathbb{R}^n)$, i.e.

$$\int_{\mathbb{R}^n} f\, d\mu_{j_k} \to \int_{\mathbb{R}^n} f\, d\mu, \quad \text{for all } f \in C_0.$$

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