Compatible circuits in colored eulerian digraphs

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October 1, 2013
Seven Bridges of Königsberg

**Question:** Can you find a walk that crosses each bridge once and exactly once?

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Eulerian graphs

**Def.** An *eulerian graph* is a graph $G$ that contains a closed walk that visits each edge exactly once. Such a walk is called an *eulerian circuit*.

**Thm.** A graph $G$ is eulerian if and only if the degree of each vertex is even and $G$ is connected.
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**Def.** An *eulerian digraph* $G$ is a directed graph or digraph (edges have directions like one way streets) that contains a closed walk that visits each edge exactly once.

**Thm.** A digraph $G$ is eulerian if and only if $\text{deg}^- (v) = \text{deg}^+ (v)$ for all vertices $v$ and $G$ is strongly (weakly) connected.
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**Thm.** A digraph $G$ is eulerian if and only if $\deg^-(v) = \deg^+(v)$ for all vertices $v$ and $G$ is strongly (weakly) connected.
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**Question:** Can you find an eulerian circuit that avoid U-turns in the following digraph?
Applications

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**Question:** Can you find an eulerian circuit that avoid U-turns in the following digraph?

How can we determine if there is such a circuit?
Compatible circuits

**Def.** A *colored eulerian digraph* $G$ is an eulerian digraph with a fixed edge coloring (not necessarily proper).

A *compatible circuit* is an eulerian circuit of $G$ such that no two consecutive edges in the tour have the same color (i.e. no monochromatic transitions).
Compatible circuits

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![Diagram of compatible circuits](image)

- **Good:** The circuit does not have two consecutive edges of the same color.
- **Bad:** The circuit does have two consecutive edges of the same color.
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**Examples**

**Big Question:** When does an colored eulerian digraph have a compatible circuit?

Not all graphs have compatible circuits.
Simple necessary condition

Let $\gamma(\nu)$ be the size of the largest color class incident to $\nu$. 
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**Prop.** If there exists a vertex $\nu$ where $\gamma(\nu) > \deg^+(\nu)$, then $G$ does not have a compatible circuit.
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**Ex.**
Simple necessary condition

Let $\gamma(v)$ be the size of the largest color class incident to $v$.

**Prop.** If there exists a vertex $v$ where $\gamma(v) > \deg^+(v)$, then $G$ does not have a compatible circuit.

**Ex.**
Undirected eulerian graphs

**Thm.** [Kotzig 1968] Let $G$ be a colored eulerian *undirected graph*. The graph $G$ has a compatible circuit if and only if $\gamma(v) \leq \deg(v)/2$ for all vertices $v$. 
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Splitting vertices

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The graph $G$ has a compatible circuit if and only if the graph $G'$ after splitting has a compatible circuit.

Henceforth, we may assume that $\gamma(v) < \text{deg}^+(v)$ for all $v$. 
**Def.** Let $T$ be an eulerian circuit of $G$ and $v$ a vertex of $G$. An *excursion* in $T$ is the walk between consecutive visits to $v$. The *excursion graph* $L_T(v)$ tracks the entering and exiting edges of the excursions at $v$. 
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We want to remove monochromatic transitions of $T$ at $v$ by rearranging the excursions at $v$. 

Let $M$ be any matching between $E^+$ and $E^-$, and let $L_M(v)$ be the implied excursion graph. A vertex $v$ is fixable if $L_M(v)$ has a compatible circuit for any matching $M$ between $E^+$ and $E^-$. 

![Diagram](image_url)
Fixable vertices

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**Prop.** If every vertex is fixable, then $G$ has a compatible circuit.

**Proof.** Pick a (not necessarily compatible) eulerian circuit $T$ of $G$. Iteratively fix fixable vertices. The resulting circuit is compatible. ■
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**Prop.** A vertex is fixable unless \( \gamma(v) = \deg^+(v) - 1 \) and there are 2 color classes of size \( \gamma(v) \) with both in and out edges, and the other two edges are one incoming and one outgoing.
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**Ex.** The excursion graph $L_M(v)$ has no compatible circuit.
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Proof sketch when $\gamma(v) < \deg^+(v) - 1$.
Fix a matching $M$ from $E^+(v)$ to $E^-(v)$.
Construct a digraph $F$ whose vertices are excursions.
An edge $u$ to $v$ in $F$ indicates the last edge of $u$ has a different color than the first edge of $v$. 

![Diagram](image-url)
**Prop.** A vertex is fixable unless $\gamma(v) = \deg^+(v) - 1$ and there are 2 color classes of size $\gamma(v)$ with both in and out edges, and the other two edges are one incoming and one outgoing.

**Proof sketch when $\gamma(v) < \deg^+(v) - 1$.**

Then a hamiltonian cycle in $F$ corresponds to a compatible circuit in $L_M(v)$. Apply Meyniel’s Theorem to show $F$ is hamiltonian.

*Meyniel’s Thm is the analogue of Ore’s Thm for digraphs.*
Nonfixable vertices

Let $S$ be the set of vertices that are not fixable. Let $S_3$ be the subset of $S$ with vertices of outdegree three.

We will consider colored eulerian digraphs with no nonfixable vertices of outdegree three.
Splitting nonfixable vertices

We form a new graph $G_S$ by splitting each of the nonfixable vertices into three new vertices.

Compatible circuits through $G$ can insert $v_1$ into $v_2$ or $v_3$, but $v_2$ and $v_3$ cannot be combined.
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We form a new graph $G_S$ by splitting each of the nonfixable vertices into three new vertices.

A compatible circuit through $v$ can insert $v_1$ into $v_2$ or $v_3$, but $v_2$ and $v_3$ cannot be combined.

Can we glue vertices so that the whole graph is connected?
The component graph $H_G$ has components of $G$ as vertices. For each $\nu_1 \in S$, put an edge in $H_G$ between $D_1 \ni \nu_1$ and $D_2 \ni \nu_2$ and an edge between $D_1 \ni \nu_1$ and $D_3 \ni \nu_3$. 
The component graph \( H_G \) has components of \( G_S \) as vertices. For each \( v \in S \), put an edge in \( H_G \) between \( D_1 \ni v_1 \) and \( D_2 \ni v_2 \) and an edge between \( D_1 \ni v_1 \) and \( D_3 \ni v_3 \).
Rainbow spanning trees

**Prob.** The edge set of $H_G$ is the disjoint union of 2-trails. Does there exist a subset $E'$ of the edges such that

1. $E'$ contains at most one edge from each 2-trail, and
2. the spanning subgraph with edge set $E'$ is connected?

If so, then $H_G$ contains a *rainbow spanning tree*.
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**Thm.** Let $G$ be a colored eulerian digraph with no nonfixable vertices of outdegree three. 
Then $G$ has a compatible circuit if and only if the component graph $H_G$ contains a rainbow spanning tree.
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Rainbow spanning trees

**Prop.** [Broersma and Li 1997; Schrijver 2003; Suzuki 2006] A multigraph $H$ has a *rainbow spanning tree* if and only if for any partition $\pi$ of $V(H)$,

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(#\text{colors between the parts}) \geq (#\text{parts in } \pi) - 1.
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$$(\text{#colors between the parts}) \geq (\text{#parts in } \pi) - 1.$$ 

One proof uses the Matroid Intersection Theorem.

There is a polynomial-time algorithm to determine if a multigraph $H$ contains a rainbow spanning tree.
Thm. Let $G$ be a colored eulerian digraph with no nonfixable vertices of outdegree three.
Then $G$ has a compatible circuit if and only if the component graph $H_G$ contains a rainbow spanning tree.

Thm. Let $G$ be a colored eulerian digraph with no nonfixable vertices of outdegree three.
There is a polynomial-time algorithm to determine if $G$ has a compatible circuit, and to produce a compatible circuit if one exists.
Algorithm

\[ G \]

\[ H_G \]

\[ G' \]

\[ G_S \]

\[ D_1 \]

\[ D_2 \]

\[ D_3 \]
Difficulty with $S_3$ vertices

The difficulty with nonfixable vertices of outdegree 3.

$G$

$G_S$

$G'_S$
We checked many eulerian digraphs where all the vertices are in $S_3$ to find patterns and examples. For many of the graphs we looked at exactly half the colorings have a compatible circuit and the other half do not. Some graphs have less than half the colorings give a compatible circuit.

In some special cases we can characterize when a graph has a compatible circuit.
Consider an edge-coloring of a graph $G$ where the head and tail can receive different colors. Throughout the rest of this talk $G$ only has vertices in $S_3$. 

Def. A pseudocompatible circuit of $G$ is an eulerian circuit, where the transitions at each vertex are either all monochromatic or none of them are.
Consider an edge-coloring of a graph $G$ where the head and tail can receive different colors. Throughout the rest of this talk $G$ only has vertices in $S_3$. 

![Graph with colored edges](image-url)
Variation

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Pseudocompatible Circuits

**Lemma** If $G$ has a pseudocompatible circuit, then $G$ has a compatible circuit.
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Think about the excursion graph at a vertex \( L_T(v) \), where \( T \) is the pseudocompatible circuit.
**Lemma** If $G$ has a pseudocompatible circuit, then $G$ has a compatible circuit.

Think about the excursion graph at a vertex $L_T(v)$, where $T$ is the pseudocompatible circuit.

If $v$ has 3 monochromatic transitions, then $L_T(v)$ is one of the two top right graphs.
Rotations and Reflections

Let $G$ be an colored eulerian digraph, where all the vertices are in $S_3$. We describe two simple ways of changing the edge-coloring of $G$. 
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A \textit{reflection} at $v$:

![Reflections diagram]

\begin{align*}
\text{Reflections} & \quad R & R & G \\
& \quad B & G & B \\
& \quad G & B & R
\end{align*}
Rotations and Reflections

Let $G$ be an colored eulerian digraph, where all the vertices are in $S_3$. We describe two simple ways of changing the edge-coloring of $G$.

A *reflection* at $v$:
Prop. If $G$ has a compatible circuit then applying a rotation to a vertex $v$ gives a new edge-coloring that also has a compatible circuit.
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A *spanning cactus* is a spanning subgraph of the component graph such that it has one edge of each dashed triangle and no cycles besides the 3-cycles from the solid triangles.
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Example

$G$

$H_G$
Other Questions

- Can be completely characterize when a graph has a compatible circuit?
- Can you find a spanning cactus in $H_G$ in polynomial time?
- Edge-colored Chinese Postman Problem: For a noneulerian graph, minimize both total length of a walk and the number of monochromatic transitions.
- Can you characterize other generalizations or variations of this problem?

Thank You