1. (10 pts) Let a tournament be an orientation of the complete graph on \( n \) vertices: that is, every pair \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) has exactly one of the edges \( i \rightarrow j \) or \( j \rightarrow i \). Let the \( \binom{n}{2} \) variables \( x_{i,j} \) be given by these pairs with \( i < j \) with

\[
x_{i,j} = \begin{cases} 
1 & \text{if } i \rightarrow j \text{ is an edge} \\
0 & \text{if } j \rightarrow i \text{ is an edge} 
\end{cases}.
\]

Let \( N = \binom{n}{2} \) and \( f : \{0,1\}^N \rightarrow \{0,1\} \) be the function

\[
f(x) = \begin{cases} 
1 & \text{there is a vertex } i \text{ where } i \rightarrow j \text{ is an edge for all } j \neq i \\
0 & \text{otherwise}
\end{cases}.
\]

(Tournaments can be thought of describing a round-robin competition among \( n \) teams, where every team plays every other team exactly once and no ties are allowed. Then, \( i \rightarrow j \) implies that team \( i \) beat team \( j \). The function \( f \) has value 1 if and only if there is some team that beat all the other teams [i.e. an all-around champion]. Feel free to use this description instead of vertices and edges.)

Prove that \( D(f) \leq 2n \approx \frac{1}{4} \sqrt{N} \).

(Bonus 5pts) Prove that \( D(f) \leq 2n - \lceil \log n \rceil \).

2. (10 pts) Let \( f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\} \) be a function on \( 2n \) variables. Prove that if \( f \) has a fooling set of size \( M \) then the communication complexity of \( f \) is at least \( \log M \).

(Hint: Bound the number of possible exchanges of information and use the pigeonhole principle.)