On extremal graphs with a fixed number of perfect matchings

Stephen G. Hartke  Derrick Stolee¹  Douglas B. West
Matthew Yancey

January 8, 2011

¹The speaker is supported by NSF grants CCF-0916525 and DMS-0914815. This work began at the Combinatorics REGS at the University of Illinois, Summer 2010.
Outline

1. Background
2. Structure of Extremal Graphs
3. Size of Extremal Graphs
4. Characterization for Small $p$
A **perfect matching** is a set of edges which cover each vertex exactly once.
A **perfect matching** is a set of edges which cover each vertex exactly once.
A **perfect matching** is a set of edges which cover each vertex exactly once.

\[ \Phi(G) \] is the number of perfect matchings in the graph \( G \).
A perfect matching is a set of edges which cover each vertex exactly once.

\[ \Phi(G) \] is the number of perfect matchings in the graph \( G \).
A **perfect matching** is a set of edges which cover each vertex exactly once.

\[ \Phi(G) \] is the number of perfect matchings in the graph \( G \).
Maximization Problem

Problem

Which graphs with a fixed number of vertices and edges maximize $\Phi(G)$?

Work of Gross, Kahl, and Sacco\-man found that such graphs are threshold graphs.
Dudek and Schmitt reversed this problem:
Dudek and Schmitt reversed this problem:

**Problem**

Over graphs with a **fixed** number of vertices and **fixed** \( \Phi(G) \), what is the **maximum** number of edges?
Dudek and Schmitt reversed this problem:

Problem

*Over graphs with a *fixed* number of vertices and *fixed* $\Phi(G)$, what is the *maximum* number of edges?*

Definition

*Let $n$ be an even number and fix $p \geq 1$.*

$$f(n, p) = \max \{|E(G)| : |V(G)| = n, \Phi(G) = p\}.$$  

*The extremal graphs form a set $\mathcal{F}_p$:*

$$\mathcal{F}_p = \{G : \Phi(G) = p, |E(G)| = f(|V(G)|, p)\}.$$
Hetyei’s Theorem

**Theorem (Hetyei’s Theorem)**

For all even $n \geq 2$, $f(n, 1) = \frac{n^2}{4}$. Moreover, each graph in $\mathcal{F}_1$ is built as below.
Hetyei’s Theorem

**Theorem (Hetyei’s Theorem)**

For all even \( n \geq 2 \), \( f(n, 1) = \frac{n^2}{4} \). Moreover, each graph in \( F_1 \) is built as below.
Hetyei’s Theorem

Theorem (Hetyei’s Theorem)

For all even \( n \geq 2 \), \( f(n, 1) = \frac{n^2}{4} \). Moreover, each graph in \( \mathcal{F}_1 \) is built as below.
Theorem (Hetyei’s Theorem)

For all even \( n \geq 2 \), \( f(n, 1) = \frac{n^2}{4} \). Moreover, each graph in \( \mathcal{F}_1 \) is built as below.
“On the Size and Structure of Graphs with a Constant Number of 1-Factors”
The Form of $f(n, p)$

**Theorem (Dudek & Schmitt)**

For each $p$, there exist constants $n_p, c_p$ so that for all $n \geq n_p$,

$$f(n, p) = \frac{n^2}{4} + c_p.$$

Take $G$ with $\frac{n^2}{4} + c$ edges.
The Form of $f(n, p)$

**Theorem (Dudek & Schmitt)**

For each $p$, there exist constants $n_p, c_p$ so that for all $n \geq n_p$,

$$f(n, p) = \frac{n^2}{4} + c_p.$$
The Form of $f(n, p)$

**Theorem (Dudek & Schmitt)**

For each $p$, there exist constants $n_p, c_p$ so that for all $n \geq n_p$,

$$f(n, p) = \frac{n^2}{4} + c_p.$$
Achieving the Constant

Definition

Let $\Phi(G) > 0$. The **achieved constant** $c(G)$ is

$$c(G) = |E(G)| - \frac{|V(G)|^2}{4}.$$
Achieving the Constant

Definition

Let $\Phi(G) > 0$. The achieved constant $c(G)$ is

$$c(G) = |E(G)| - \frac{|V(G)|^2}{4}.$$

In this sense, lower bounds on $c_p$ are “easy” (just give a $G$ with $\Phi(G) = p, c(G) \leq c_p$).

Upper bounds are hard: must prove NO graph achieves a higher constant!
More Specifics on $c_p$

In addition to the formula $f(n, p) = \frac{n^2}{4} + c_p$, D&S found:
More Specifics on $c_p$

In addition to the formula $f(n, p) = \frac{n^2}{4} + c_p$, D&S found:

$$-(p - 1)(p - 2) \leq c_p \leq p - 2.$$
More Specifics on $c_p$

In addition to the formula $f(n, p) = \frac{n^2}{4} + c_p$, D&S found:

- $-(p - 1)(p - 2) \leq c_p \leq p - 2$.

- Exact values of $c_p$ for $p \in \{2, 3, 4, 5, 6\}$.
In addition to the formula $f(n, p) = \frac{n^2}{4} + c_p$, D&S found:

- $-(p - 1)(p - 2) \leq c_p \leq p - 2$.

- Exact values of $c_p$ for $p \in \{2, 3, 4, 5, 6\}$.

- Exact structure of graphs for $p \in \{2, 3\}$. 
$c_p$ for small $p$

<table>
<thead>
<tr>
<th>$p$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_p$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$n_p$</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
“On extremal graphs with a given number of perfect matchings”
Our Results

We make the following progress on the problem:
Our Results

We make the following progress on the problem:

Show $c_p \geq 1$ for all $p \geq 2$. 
Our Results

We make the following progress on the problem:

- Show $c_p \geq 1$ for all $p \geq 2$.
- Conjecture an $O\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ upper bound on $c_p$. 

See Matthew Yancey’s talk, Sunday morning.
Our Results

We make the following progress on the problem:

- Show $c_p \geq 1$ for all $p \geq 2$.
- Conjecture an $O\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ upper bound on $c_p$.
- General structure of graphs in $F_p$ for all $p$. 
Our Results

We make the following progress on the problem:

- Show $c_p \geq 1$ for all $p \geq 2$.
- Conjecture an $O\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ upper bound on $c_p$.
- General structure of graphs in $\mathcal{F}_p$ for all $p$.
- Bound $n_p = O(\sqrt{p})$. 

Computational verification of $c_p$ for $p \in \{7, 8, 9, 10\}$.

Exact characterization of $\mathcal{F}_p$ for $p \in \{4, 5, 6, 7, 8, 9, 10\}$.

See Matthew Yancey’s talk, Sunday morning.
Our Results

We make the following progress on the problem:

- Show $c_p \geq 1$ for all $p \geq 2$.

- Conjecture an $O\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ upper bound on $c_p$.

- General structure of graphs in $\mathcal{F}_p$ for all $p$.

- Bound $n_p = O(\sqrt{p})$.

- Computational verification of $c_p$ for $p \in \{7, 8, 9, 10\}$. 
Our Results

We make the following progress on the problem:

- Show $c_p \geq 1$ for all $p \geq 2$.

- Conjecture an $O\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ upper bound on $c_p$.

- General structure of graphs in $\mathcal{F}_p$ for all $p$.

- Bound $n_p = O(\sqrt{p})$.

- Computational verification of $c_p$ for $p \in \{7, 8, 9, 10\}$.

- Exact characterization of $\mathcal{F}_p$ for $p \in \{4, 5, 6, 7, 8, 9, 10\}$.
Our Results

We make the following progress on the problem:

- Show $c_p \geq 1$ for all $p \geq 2$.
- Conjecture an $O\left(\left(\frac{\ln p}{\ln \ln p}\right)^2\right)$ upper bound on $c_p$.
- General structure of graphs in $\mathcal{F}_p$ for all $p$.
- Bound $n_p = O(\sqrt{p})$.
- Computational verification of $c_p$ for $p \in \{7, 8, 9, 10\}$.
- Exact characterization of $\mathcal{F}_p$ for $p \in \{4, 5, 6, 7, 8, 9, 10\}$.

See Matthew Yancey’s talk, Sunday morning.
Outline

1. Background
2. Structure of Extremal Graphs
3. Size of Extremal Graphs
4. Characterization for Small $p$
Matching Theory

Graph Factors and Matching Extensions
Edge Types

**Definition**

Let $\Phi(G) > 0$ and $e \in E(G)$.

- $e$ is **allowable** if there exists a perfect matching containing $e$.
- $e$ is **forbidden** otherwise.
Let $G$ be connected with $\Phi(G) > 0$. 

**Definition**

$G$ is called $1$-extendable if all edges are allowable. $G$ is elementary if the set of allowable edges forms a connected subgraph. $G$ is saturated if adding any missing edge increases $\Phi(G)$. $G$ is extremal if $G \in F(\Phi(G))$. 

**Graph Types**
Graph Types

Definition

Let $G$ be connected with $\Phi(G) > 0$.

- $G$ is **1-extendable** if all edges are allowable.
Graph Types

Definition

Let $G$ be connected with $\Phi(G) > 0$.

- $G$ is 1-extendable if all edges are allowable.
- $G$ is elementary if the set of allowable edges forms a connected subgraph.
Graph Types

Definition

Let $G$ be connected with $\Phi(G) > 0$.

- $G$ is 1-extendable if all edges are allowable.
- $G$ is elementary if the set of allowable edges forms a connected subgraph.
- $G$ is saturated if adding any missing edge increases $\Phi(G)$.
Graph Types

Definition

Let $G$ be connected with $\Phi(G) > 0$.

- $G$ is **1-extendable** if all edges are allowable.
- $G$ is **elementary** if the set of allowable edges forms a connected subgraph.
- $G$ is **saturated** if adding any missing edge increases $\Phi(G)$.
- $G$ is **extremal** if $G \in \mathcal{F}_\Phi(G)$.
Let $G$ be connected with $\Phi(G) > 0$.

- $G$ is **1-extendable** if all edges are allowable.
- $G$ is **elementary** if the set of allowable edges forms a connected subgraph.
- $G$ is **saturated** if adding any missing edge increases $\Phi(G)$.
- $G$ is **extremal** if $G \in \mathcal{F}_{\Phi(G)}$.

$1$-extendable $\Rightarrow$ elementary

extremal $\Rightarrow$ saturated
Let $G$ be connected with $\Phi(G) > 0$.

- $G$ is **1-extendable** if all edges are allowable.
- $G$ is **elementary** if the set of allowable edges forms a connected subgraph.
- $G$ is **saturated** if adding any missing edge increases $\Phi(G)$.
- $G$ is **extremal** if $G \in F_{\Phi(G)}$.
Let $\Phi(G) > 0$. A set $X \subset V(G)$ is a barrier if

$$|X| = \# \text{ of odd connected components in } G - X$$
Let $\Phi(G) > 0$. A set $X \subset V(G)$ is a **barrier** if

$$|X| = \# \text{ of odd connected components in } G - X$$
Let $\Phi(G) > 0$. A set $X \subset V(G)$ is a barrier if

$$|X| = \# \text{ of odd connected components in } G - X$$
Let $\Phi(G) > 0$. A set $X \subset V(G)$ is a **barrier** if

$$|X| = \# \text{ of odd connected components in } G - X$$
The Lovász Cathedral Theorem characterizes saturated graphs. Extremal graphs only use a special case: towers.
Let $G_1, \ldots, G_k$ be elementary graphs with $X_1, \ldots, X_k$ barriers where $X_i$ is of maximum size in $G_i$. 

\[ G_4 \]

\[ G_3 \]

\[ G_2 \]

\[ G_1 \]
Let $G_1, \ldots, G_k$ be elementary graphs with $X_1, \ldots, X_k$ barriers where $X_i$ is of maximum size in $G_i$. 

\begin{align*}
&G_1 \\
&\quad X_1 \\
&G_2 \\
&\quad X_2 \\
&G_3 \\
&\quad X_3 \\
&G_4 \\
&\quad X_4
\end{align*}
Let $G_1, \ldots, G_k$ be elementary graphs with $X_1, \ldots, X_k$ barriers where $X_i$ is of maximum size in $G_i$. 

![Diagram of graphs $G_1$ to $G_4$ with barriers $X_1$ to $X_4$.]
Let $G_1, \ldots, G_k$ be elementary graphs with $X_1, \ldots, X_k$ barriers where $X_i$ is of maximum size in $G_i$. 
Let $G_1, \ldots, G_k$ be elementary graphs with $X_1, \ldots, X_k$ barriers where $X_i$ is of maximum size in $G_i$. 
Extremal Graphs are Towers

Theorem

If $G \in \mathcal{F}_p$, then $G$ is a tower of elementary components $G_1, \ldots, G_k$, where the barrier chosen in each component is of maximum size.
Extremal Graphs are Towers

Theorem

If \( G \in \mathcal{F}_p \), then \( G \) is a tower of elementary components \( G_1, \ldots, G_k \), where the barrier chosen in each component is of maximum size.

If \( G \) is a tower of elementary graphs \( G_1, \ldots, G_k \), then

\[
\Phi(G) = \prod_{i=1}^{k} \Phi(G_i).
\]
Extremal Graphs are Towers

Theorem

If \( G \in \mathcal{F}_p \), then \( G \) is a tower of elementary components \( G_1, \ldots, G_k \), where the barrier chosen in each component is of maximum size.

If \( G \) is a tower of elementary graphs \( G_1, \ldots, G_k \), then

\[
\Phi(G) = \prod_{i=1}^{k} \Phi(G_i).
\]

\[
c(G) \leq \sum_{i=1}^{k} c(G_i) \text{ with equality if and only if } \frac{|X_i|}{|V(G_i)|} = \frac{1}{2} \text{ for all } i < k.
\]
Order of Elementary Graphs

\[ \frac{|X_4|}{|V(G_4)|} \]
\[ \frac{|X_3|}{|V(G_3)|} \]
\[ \frac{|X_2|}{|V(G_2)|} \]
\[ \frac{|X_1|}{|V(G_1)|} \]
Order of Elementary Graphs

\[ \frac{|X_4|}{|V(G_4)|} \leq \frac{|X_3|}{|V(G_3)|} \leq \frac{|X_2|}{|V(G_2)|} \leq \frac{|X_1|}{|V(G_1)|} \]
Order of Elementary Graphs

\[
\begin{align*}
|X_4| & \leq |V(G_4)| \\
|X_2| & \leq |V(G_2)| \\
|X_3| & \leq |V(G_3)| \\
|X_1| & \leq |V(G_1)|
\end{align*}
\]
Outline

1. Background

2. Structure of Extremal Graphs

3. Size of Extremal Graphs

4. Characterization for Small $p$
Elementary Graphs: Forbidden Edges

Lemma

If $G$ is an elementary graph with $n$ vertices, then $G$ has at most

$$\frac{n^2}{8} - \frac{n}{4}$$

forbidden edges.
1-Extendable Graphs: Number of Edges

The **Lovász Two Ears Theorem** gives a structure of 1-extendable graphs using ear decompositions.
1-Extendable Graphs: Number of Edges

The **Lovász Two Ears Theorem** gives a structure of 1-extendable graphs using ear decompositions.

**Lemma**

If $G$ is 1-extendable with $n$ vertices and $p$ perfect matchings, then $G$ has at most $2p + n - 4$ edges.
1-Extendable Graphs: Number of Edges

The Lovász Two Ears Theorem gives a structure of 1-extendable graphs using ear decompositions.

**Lemma**

If $G$ is 1-extendable with $n$ vertices and $p$ perfect matchings, then $G$ has at most $2p + n - 4$ edges.

**Theorem**

An elementary graph with $n$ vertices and $p$ perfect matchings has at most

$$\frac{1}{8}n^2 + \frac{3}{4}n + 2p - 4$$

edges.
Elementary Graphs: Number of Vertices

Theorem

If $G$ is an extremal elementary graph with $n$ vertices and $p$ perfect matchings, then

$$\frac{1}{8}n^2 + \frac{3}{4}n + 2p - 4 \geq \frac{1}{4}n^2 + c_p$$

and

$$n \leq 3 + \sqrt{16p - 8c_p - 23}.$$

Define $N_p$ to be the largest even integer at most $3 + \sqrt{16p - 8c_p - 23}$. 
To characterize extremal graphs, we must know:

Which factorizations \( p = p_1 p_2 \cdots p_k \) give \( c_{p_1} + c_{p_2} + \cdots + c_{p_k} \geq c_p \)?

Which elementary graphs achieve these constants, \( c_{p_i} \)?

Which elementary graphs achieve these constants with \( s_{i n_i} = 1 \)?

Once the possible elementary graphs are decided, the rest of the components are copies of \( K_2 \).

The elementary graphs can be found by searching all graphs of order \( N_p \) with \( N_p + c_p \) edges.
To characterize extremal graphs, we must know:

- Which factorizations $p = p_1 p_2 \cdots p_k$ give $c_{p_1} + c_{p_2} + \cdots c_{p_k} \geq c_p$?
Extremal Graphs: Number of Vertices

To characterize extremal graphs, we must know:

- Which factorizations $p = p_1 p_2 \cdots p_k$ give $c_{p_1} + c_{p_2} + \cdots c_{p_k} \geq c_p$?
- Which elementary graphs achieve these constants, $c_{p_i}$?
Extremal Graphs: Number of Vertices

To characterize extremal graphs, we must know:

- Which factorizations $p = p_1 p_2 \cdots p_k$ give $c_{p_1} + c_{p_2} + \cdots + c_{p_k} \geq c_p$?
- Which elementary graphs achieve these constants, $c_{p_i}$?
- Which elementary graphs achieve these constants with $\frac{s_i}{n_i} = \frac{1}{2}$?
Extremal Graphs: Number of Vertices

To characterize extremal graphs, we must know:

- Which factorizations $p = p_1 p_2 \cdots p_k$ give $c_{p_1} + c_{p_2} + \cdots c_{p_k} \geq c_p$?
- Which elementary graphs achieve these constants, $c_{p_i}$?
- Which elementary graphs achieve these constants with $\frac{s_i}{n_i} = \frac{1}{2}$?
- Once the possible elementary graphs are decided, the rest of the components are copies of $K_2$. 
Extremal Graphs: Number of Vertices

To characterize extremal graphs, we must know:

- Which factorizations \( p = p_1 p_2 \cdots p_k \) give \( c_{p_1} + c_{p_2} + \cdots + c_{p_k} \geq c_p \)?
- Which elementary graphs achieve these constants, \( c_{p_i} \)?
- Which elementary graphs achieve these constants with \( \frac{s_i}{n_i} = \frac{1}{2} \)?
- Once the possible elementary graphs are decided, the rest of the components are copies of \( K_2 \).

The elementary graphs can be found by searching all graphs of order \( N_{p_i} \) with \( \frac{N_{p_i}^2}{4} + c_{p_i} \) edges.
<table>
<thead>
<tr>
<th>Outline</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>
Structure for $p = 7$

7 is prime!

Figure: The unique extremal elementary graph, at the top of the tower.
Structure for $p = 8$

$8 = 2 \cdot 4 = 2 \cdot 2 \cdot 2$ gives $3 = c_8 = c_2 + c_4 = c_2 + c_2 + c_2$ and all have barriers of size $\frac{1}{2}$.

(a) At top
(b) At top
(c) Anywhere

(d) Any order
(e) Any order
Structure for $p = 9$

$9 = 3 \cdot 3$ gives $4 = c_9 = c_3 + c_3 = 2 + 2$, but the elementary graph for $p = 3$ has barrier size $\frac{1}{2}$.

**Figure:** The unique extremal elementary graph, at the top of the tower.
Structure for $p = 10$

$10 = 2 \cdot 5$ gives $4 = c_{10} > c_2 + c_5 = 1 + 2$.

**Figure:** The unique extremal elementary graph, at the top of the tower.
[DS10+] Found $c_p$ for $p \in \{2, 3, 4, 5, 6\}$.

[HSWY11+] Found $c_p$ for $p \in \{7, 8, 9, 10\}$ (until $N_p$ is too big).

[Sto11+] Found $c_p$ for $p \in \{11, 12, \ldots, 20\}$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_p$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$n_p$</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$N_p$</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

[DS10+] [HSWY11+] [Sto11+]


[Sto11+] D. Stolee, “Isomorph-free generation of 2-connected graphs with applications,” (under preparation)
Matthew Price Yancey, “On extremal graphs with a given number of perfect matchings,” AMS Session on Combinatorics and Graph Theory, VII. 8:30 a.m. Sunday, Southdown Room, 4th Floor, Sheraton

Derrick Stolee, “Isomorph-free generation of 2-connected graphs with applications,” Special Session on Graph Theory, AMS Sectional Meeting, Iowa City, IA. March 18-20, 2011.