On the hardness of recognizing triangular line graphs

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Line Graph \( L(G) \): Dynamics
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Line Graph $L(G)$: Reversibility
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There is a polynomial-time algorithm to
  ▶ detect if a graph $G$ is a line graph.
  ▶ reconstruct the unique* graph $H$ so that $G \cong L(H)$. 
Triangular Line Graph $T(G)$
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The triangular line graph is also known as:

- 2-anti-Gallai operator [Gal67].
- 2-in-3 operator $\Phi_{2,3}$ [Pri95].
- $K_3$-line graph [CGS95].
- Link Graph [AEGM10].
Characterization

Theorem (Van Bang Le, 1996)

A graph $G$ is a triangular line graph if and only if there is a family of induced subgraphs $\mathcal{F} = \{G_i : i \in I\}$ of $G$ such that:

1. Every vertex in $G$ appears in exactly two members of $\mathcal{F}$.
2. Every edge in $G$ appears in exactly one member of $\mathcal{F}$.
3. $|G_i \cap G_j| \leq 1$, for every $i \neq j$.
4. For any distinct $i, j, k \in I$, if $\{v_i\} = G_i \cap G_k$ and $\{v_j\} = G_j \cap G_k$ and $v_i \neq v_j$, then $(v_i, v_j) \in E(G)$ if and only if $G_i \cap G_j \neq \emptyset$. 
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Algorithmic Pursuits

Problem

Compute $T^{-1}(G)$.
Algorithmic Pursuits

Problem

Compute $T^{-1}(G)$.

Problem

Is $G$ a triangular line graph?
Pranav Anand
UCSC

Henry Escuadro
Juniata

Ralucca Gera
NPGS

Stephen G. Hartke
UNL

“On the hardness of recognizing triangular line graphs”
“On the hardness of recognizing triangular line graphs”
Algorithmic Failure

Theorem (AEGHS10)

*Deciding if a given graph is a triangular line graph is NP-Complete.*
3-Conjunctive Normal Form (3CNF)

\[ x = x_1 x_2 \ldots x_n \in \{0, 1\}^n \]
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\[ \hat{x}_i \in \{x_i, \overline{x}_i\} \]
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\[ \phi(x) = \bigwedge_{j=1}^{m} (\hat{x}_{ij,1} \lor \hat{x}_{ij,2} \lor \hat{x}_{ij,3}) \]

\[ j^{\text{th}} \text{ clause } C_j(x) \]
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\[ \phi(x) = \bigwedge_{j=1}^{m} (\hat{x}_{i_j,1} \lor \hat{x}_{i_j,2} \lor \hat{x}_{i_j,3}) \]

\[ \phi(x) = 1 \text{ iff } C_j(x) = 1 \text{ for all } j. \]
Proof Outline

Method: 3-SAT $\rightarrow T^{-1}(G)$. 

$\varphi(x) \rightarrow G \varphi \exists x, \varphi(x) = 1 \iff \exists H, T(H) = G$. 

Variables: 7-suns with exactly two preimages. 

Logic: NOT and EQUALS gates. 

Clauses: Tying suns together.
Method: 3-SAT $\rightarrow T^{-1}(G)$. 
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- $\phi(x) \mapsto G_\phi$
Proof Outline

**Method:** $3$-SAT $\rightarrow T^{-1}(G)$.

- $\phi(x) \leftrightarrow G_\phi$
- $\exists x, \phi(x) = 1 \iff \exists H, T(H) = G$. 

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**Variables:** 7-suns with exactly two preimages.

**Logic:** NOT and EQUALS gates.

**Clauses:** Tying suns together.
Triangle Trails
Triangle Trails
Triangle Trails
Triangle Trails
Triangle Trails
Suns
Wheels
Wheels
Wheels
Squared Cycles
Squared Cycles
Squared Cycles
Squared Cycles
Proposition

$W_7$ and $C_7^2$ are the only two graphs with triangular line graph $S_7$. 
Variables

Each $x_i$ is assigned a 7-sun $H_{x_i}$. 
Each $x_i$ is assigned a 7-sun $H_{x_i}$.

$x_i = 1 \iff T^{-1}(H_{x_i}) = C^2_7 = \quad \quad$ 

$x_i = 0 \iff T^{-1}(H_{x_i}) = W_7 = \quad \quad$
Variables

Each $x_i$ is assigned a 7-sun $H_{x_i}$.

$x_i = 1 \iff T^{-1}(H_{x_i}) = C_7^2 = \begin{array}{c}
\end{array}$

$x_i = 0 \iff T^{-1}(H_{x_i}) = W_7 = \begin{array}{c}
\end{array}$

Next: Logic!
Logic “Gates”
Logic “Gates”
Logic “Gates”
Logic “Gates”

EQUAL

NOT EQUAL
Labeled 7-Sun

![Graph Diagram]

- **ROOT**
- **NOT**
- **EQUAL**
Labeled 7-Sun

\[ H_{x_i} \]
Labeled 7-Sun

$H_{x_i}$ $H_{x_i,1}$
Labeled 7-Sun
Labeled 7-Sun

$H_{x_i}$  $H_{x_i, 1}$  $H_{x_i, 2}$

$H'_{x_i, 1}$
Labeled 7-Sun

\[ H_{x_i}, H_{x_i,1}, H_{x_i,2}, H'_{x_i,1}, H'_{x_i,2} \]
Labeled 7-Sun

$H_{x_i}$  $H_{x_i,1}$  $H_{x_i,2}$

$H'_{x_i,1}$  $H'_{x_i,2}$
Larger Suns

\[ T(W_n) = S_n \]

\[ T(C_n^2) = S_n \]
Larger Suns

\[ T(W_n) = S_n \]

\[ T(C_n^2) = S_n \]

For \( n \geq 8 \), these are not the only preimages of \( S_n \)!
Binary-Enforced Suns
Binary-Enforced Suns
Binary-Enforced Suns
Binary-Enforced Suns
Variable Gadgets

$X_i$

$H_{x_i}$
Variable Gadgets

\[ x_i \quad \text{and} \quad \overline{x}_i \]

\[ H_{x_i} \quad \text{and} \quad H_{x_i,1} \]
Variable Gadgets
Variable Gadgets

\[ x_i \quad \overline{x_i} \quad x_{i,1} \quad H_{x_{i,1}} \quad x_{i,2} \quad H_{x_{i,2}} \quad x_{i,3} \quad H_{x_{i,3}} \]
Variable Gadgets
Variable Gadgets
Clause Gadgets

**Given:** Clause $C_j(x) = \hat{x}_{i,j,1} \lor \hat{x}_{i,j,2} \lor \hat{x}_{i,j,3}$.

**Need:** A gadget which has a pre-image iff

(a) At least one of three variables is set to 1.

(b) At least one of three suns has a $C_2^n$ preimage.
Clause Gadgets

1

2

3
Clause Gadgets
Clause Gadgets

1

2

3

\[ a \quad b \]

\[ b \quad a \]

\[ a \quad b \]
Clause Gadgets

1

2

3

a_{1,1} b_{1,2} a_{1,3} \quad a_{2,1} b_{2,2} a_{2,3}

b_{1,3} a_{1,2} b_{1,1}

b_{2,3} a_{2,2} b_{2,1}

a_{3,1} b_{3,2} a_{3,3}

b_{3,1} a_{3,2} b_{3,3}
Clause Gadgets
Clause Gadgets

Proposition

If $G$ is a clause gadget, then $G = T(H)$ if and only if $H$ is a combination of three binary-enforced 12-sun preimages where at least one is a $C_{12}^2$. 
For each number of cycles $\geq 1$, check.

### Table 4: The edge-labeled preimage of a clause with two wheels

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Adjacencies (Labels)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1 \ (S_{0,12,12})$</td>
</tr>
<tr>
<td>1</td>
<td>$0 \ (S_{1,10,12})$</td>
</tr>
<tr>
<td>2</td>
<td>$1 \ (S_{1,12,12})$</td>
</tr>
<tr>
<td>3</td>
<td>$0 \ (S_{2,2,12})$</td>
</tr>
<tr>
<td>4</td>
<td>$0 \ (S_{3,2,12})$</td>
</tr>
<tr>
<td>5</td>
<td>$0 \ (S_{4,2,12})$</td>
</tr>
<tr>
<td>6</td>
<td>$0 \ (S_{5,2,12})$</td>
</tr>
</tbody>
</table>

### Table 3: The edge-labeled preimage of a clause with one wheel

<table>
<thead>
<tr>
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<tbody>
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</tr>
<tr>
<td>3</td>
<td>$0 \ (S_{2,2,12})$</td>
</tr>
<tr>
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<td>6</td>
<td>$0 \ (S_{5,2,12})$</td>
</tr>
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</table>

Table 4 and Table 3 demonstrate the edge-labeled preimage of clauses with a single and a double number of wheels, respectively.
Clause Gadgets

(⇒) If all three wheels, $T(H) \neq G$. 
Clause Gadgets

$(\Rightarrow)$
Clause Gadgets

(⇒)
Clause Gadgets

(⇒)
Clause Gadgets

\[
(\Rightarrow)
\]
Clause Gadgets

(⇒)
Clause Gadgets

$(\Rightarrow)$
Big Picture
Big Picture
QED.
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Triangle-Closed Subgraphs

Definition

A subgraph $H_1 \subseteq H_2$ is a triangle-closed subgraph (denoted $H_1 \triangleleft H_2$) if for every triangle $xyz$ in $H_2$ with the edge $xy$ in $H_1$, then $xyz$ is in $H_1$.

Lemma

Let $H_2 = T(G_2)$, and $H_1 \triangleleft H_2$ where $H_1$ is a triangular line graph. Then, there exists $G_1 \subseteq G_2$ so that $H_1 = T(G_1)$. 