

1 Section 8.1: Sequences

1.1 Limit Rules

When: Given convergent sequences $\{a_n\}, \{b_n\}$ with $\lim a_n = A, \lim b_n = B$.

Sum Rule: $\lim(a_n + b_n) = A + B$,

Difference Rule: $\lim(a_n - b_n) = A - B$,

Product Rule: $\lim a_n b_n = AB$,

Constant Multiple: $\lim k a_n = k A$,

Quotient Rule: $\lim \frac{a_n}{b_n} = \frac{A}{B}$ when $B \neq 0$.

Sandwich Thm: If $a_n \leq c_n \leq b_n$ for some sequence $\{c_n\}$ and $A = B$, then $A = \lim c_n = B$.

Check: This does not hold for divergent sequences.

Why? Sequences are easier when considering these rules.

Link: http://en.wikipedia.org/wiki/Limit_of_a_sequence

1.2 Common Sequence Limits

- $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$.

- $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

- $\lim_{n \rightarrow \infty} x^{1/n} = 1$ ($x > 0$).

- $\lim_{n \rightarrow \infty} x^n = 0$ ($|x| < 1$).

- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$.

- $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$.

Link: http://en.wikipedia.org/wiki/Limit_of_a_sequence

2 Section 8.2: Infinite Series

2.1 Definition

When: Given an infinite sum, $\sum_{n=0}^{\infty} a_n$.

How: $\sum_{n=0}^{\infty} a_n := \lim_{m \rightarrow \infty} \sum_{n=0}^m a_n$.

Why? There are lots of convergence tests, but this definition will be important for telescoping sums.

Link: http://en.wikipedia.org/wiki/Infinite_series

2.2 nth Term Test

When: A sum $\sum a_n$, but $a_n \not\rightarrow 0$.

How: If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum a_n$ diverges.

Check: This only works to show divergence, not convergence!

Why? Lots of series can be eliminated quickly this way! Always try this first!

Link: http://en.wikipedia.org/wiki/Term_test

2.3 Geometric Series

When: Given a sum in the form $\sum_{n=0}^{\infty} ar^n$.

Necessary: Be careful of index requirements, and $|r| < 1$.

How: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Check: Try a few partial sums to see if it converges to the calculated result.

Why? This is a common example.

Link: http://en.wikipedia.org/wiki/Geometric_series

2.4 Series Combinations

When: Given two convergent series $\sum a_n = A$, $\sum b_n = B$, then

- How:
1. Sum Rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$.
 2. Difference Rule: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$.
 3. Constant Multiple Rule: $\sum ka_n = k \sum a_n = kA$.

Why? Knowing parts of sums first may help occasionally. Note that this does NOT work when the either $\sum a_n$ or $\sum b_n$ diverge!

3 Section 8.3: Integral Test

3.1 Integral Test

When: Consider the sum $\sum_{n=A}^{\infty} a_n$ and there exists a function $f(x)$ with $f(n) = a_n$ for $n \geq A$.

Necessary: $f(x) \geq 0$, decreasing.

How: $\sum_{n=A}^{\infty} a_n$ and $\int_A^{\infty} f(x)dx$ behave the same. Use convergence tricks from 7.7.

Why? Some series can be checked using improper integrals easier than other tests.

Link: http://en.wikipedia.org/wiki/Integral_test

3.2 p -Series

When: A sum can be organized as $\sum_{n=1}^{\infty} \frac{a}{n^p}$.

- How:
1. If $p > 1$, converges.
 2. If $p \leq 1$, diverges.

Why? If $p \leq 1$, then direct comparison test shows $\frac{1}{n} \leq \frac{1}{n^p}$, so diverges. It turns out that any larger power will converge.

Link: [http://en.wikipedia.org/wiki/Harmonic_series_\(mathematics\)#P-series](http://en.wikipedia.org/wiki/Harmonic_series_(mathematics)#P-series)

4 Section 8.4: Comparison Tests

4.1 Direct Comparison Test (DCT)

When: Showing convergence or divergence of a sum $\sum a_n$ when a_n can be compared to b_n and $\sum b_n$ is known.

Necessary: $a_n \geq 0$.

- How:
1. If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
 2. If $b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Link: http://en.wikipedia.org/wiki/Comparison_test

4.2 Limit Comparison Test (LCT)

When: Show convergence or divergence of a sum $\sum_{n=0}^{\infty} a_n$ when a_n behaves similarly to b_n and $\sum_{n=0}^{\infty} b_n$ is known to converge or diverge.

- How:
1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ behave the same.
 2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ converges, then $\sum a_n$ converges.
 3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = +\infty$, and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Why? Sometimes this is an easier comparison than the direct route.

Link: http://en.wikipedia.org/wiki/Limit_comparison_test

5 Section 8.5: Ratio and Root Tests

5.1 Ratio Test

- When: Consider a sum $\sum_{n=0}^{\infty} a_n$ where consecutive terms a_{n+1} and a_n have many shared terms (n th powers, factorials, etc).
- Necessary: Let $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$.
 Then, (a) converges if $\rho < 1$,
 (b) diverges if $\rho > 1$,
 (c) inconclusive when $\rho = 1$.
- Why? This is particularly useful for power series.
- Link: http://en.wikipedia.org/wiki/Ratio_test

5.2 Root Test

- When: Consider a sum $\sum_{n=0}^{\infty} a_n$ where a_n has n th-exponents around complicated functions.
- Necessary: 1. $a_n \geq 0$ for $n \geq N$,
 2. $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \rho$,
 Then, (a) converges if $\rho < 1$,
 (b) diverges if $\rho > 1$,
 (c) inconclusive when $\rho = 1$.
- Why? Some series can only be solved using Root Test
- Link: http://en.wikipedia.org/wiki/Root_test

6 Section 8.6 Alternating Series, Absolute and Conditional Convergence

6.1 Alternating Series Test

- When: Given a series $\sum_{n=0}^{\infty} (-1)^n a_n$
- Necessary: 1. $a_n \geq 0$.
 2. $a_n \geq a_{n+1}$.
 3. $\lim_{n \rightarrow \infty} a_n = 0$.
 When all three are true, then the sum converges.
- Why? Several series converge conditionally, but not absolutely. See alternating harmonic series.
- Link: http://en.wikipedia.org/wiki/Alternating_series

6.2 Absolute/Conditional Convergence

- When: If a series has positive and negative terms, it can converge absolutely (when all the terms are taken with absolute value) or conditionally (convergent only when the terms keep their sign) or always diverge (never a convergent series, no matter what the signs do).
- Necessary: Check conditional convergence with the Alternating Series Test,
- Link: http://en.wikipedia.org/wiki/Absolute_convergence

7 Section 8.7: Power Series

7.1 Power Series

- What: A power series about a has the form $\sum_{n=0}^{\infty} c_n (x - a)^n$ for a sequence c_n .
- How: Power series absolutely converge for values of x given by the Ratio Test. Check boundary conditions for conditional convergence.
- Why? Power series are important to understand before trying Taylor series.
- Link: http://en.wikipedia.org/wiki/Power_series
http://en.wikipedia.org/wiki/Radius_of_convergence

7.2 Term-by-Term Integration and Derivation

When: Approximate a function by knowing a power series for derivative or integral.

How: $\frac{d}{dx} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$.
 $\int [\sum_{n=0}^{\infty} c_n (x-a)^n] dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} + C$.

Why? Try proving $\frac{d}{dx} e^x = e^x$ using its Taylor series.

Link: http://en.wikipedia.org/wiki/Power_series#Differentiation_and_integration

8 Section 8.8: Taylor and Maclaurin Series

8.1 Taylor/Maclaurin Series

When: Approximating or simplifying expressions for a function $f(x)$ by using a polynomial expression.

Necessary: $f(x)$ infinitely differentiable on an interval $(a-R, a+R)$, and knowing the values of each derivative at a .

How: $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$ is the n th Taylor Polynomial. The limit of this polynomial as n approaches infinity is the Taylor Series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$.
 The Maclaurin series is the Taylor series with $a=0$.

Check: Carefully watch the derivatives and the pattern created.

Why? Polynomials are easier to do a lot of things, including derivate and integrate.

Link: http://en.wikipedia.org/wiki/Taylor_series

9 Section 8.9: Convergence of Taylor Series

9.1 Taylor's Theorem and Formula

Necessary: A function $f(x)$ continuous and $(n+1)$ -differentiable on an interval $(a-R, a+R)$.

How: The Taylor polynomial $P_n(x)$ and the remainder $R_n(x)$ sum to exactly the function: $f(x) = P_n(x) + R_n(x)$ where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$ for some c strictly between x and a .

Why? An elaborate use of the Intermediate Value Theorem, paired with the remainder estimation theorem proves convergence of Taylor series.

9.2 Remainder Estimation Theorem

When: Computing how close a Taylor polynomial of degree n will estimate the function.

Necessary: Knowing $P_n(x)$ and the $(n+1)$ th derivative.

How: Given a Taylor polynomial $P_n(x)$ centered at a , to be calculated at x , let M be an upper bound for $|f^{(n+1)}(t)|$ for $x < t < a$. Then, the remainder term $R_n(x)$ satisfies $|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$.

Check: Calculate the polynomial and the function values at x , and see if they are as close as this estimate.

Why? Computers can calculate polynomials better than general functions, but it would be good to know how many computations are required to achieve a certain degree of accuracy.

Link: http://en.wikipedia.org/wiki/Taylor_polynomial#Estimates_of_the_remainder

9.3 Frequently Used Taylor Series

When:	Asking to approximate an answer using “known” Taylor series.
Necessary:	Calculating a value usually hard to compute by hand.
Geometric:	$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, x < 1$
Alternating Geometric:	$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-x)^n, x < 1$
Exponential:	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x < \infty$
Sine:	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, x < \infty$
Cosine:	$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, x < \infty$
Natural Log:	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, -1 < x \leq 1$
Natural Log:	$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1} \frac{(x-1)^n}{n} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}, 0 < x \leq 2$
Natural Log:	$\ln \frac{1+x}{1-x} = 2 \tanh^{-1} x = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots \right) = 2 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, x < 1$
Inverse Tangent:	$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, x < 1$
Check:	Compute the first few derivatives and check their value. Make sure this agrees with your first few values. Note that zero values are removed from these sums.
Why?	It is easier to memorize a few examples than to repeatedly generate them.
Link:	http://en.wikipedia.org/wiki/Taylor_series#List_of_Taylor_series_of_some_common_functions

10 Section 8.10: Binomial Series

10.1 Definition

When:	Looking for Taylor series representation of $f(x) = (1+x)^m$
How:	Note that $f^{(k)}(x) = \binom{m}{k} x^k$ $\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$ Then $f(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$.
Why?	Knowing this expansion makes polynomial expansion easier.
Link:	http://en.wikipedia.org/wiki/Binomial_series