0.1 Substitution

When: Given a function $f(x)$ inside a different function $g(x)$, with $f'(x)$ outside $g(x)$.
Necessary: An integral $\int f'(x)g(f(x))dx$.
How: Let $u = f(x), du = f'(x)dx$
$\int f'(x)g(f(x))dx = \int g(u)du = G(u) + C = G(f(x)) + C$.
Check: After substitution, there should no longer be any x's around.
Why? This is essentially the chain rule for differentiation in reverse.
Link: http://en.wikipedia.org/wiki/Integration_by_substitution

1 Chapter 7: Techniques of Integration

1.1 7.1: Integration by Parts

When: Given two functions $f(x)$, $g'(x)$ that don't cooperate through integration or differentiation. Essentially, when substitution for $f(x)$ or $g'(x)$ won't work. Also requires that $f(x)$ is differentiable and $g'(x)$ can be integrated.
Necessary: An integral $\int f(x)g'(x)dx$.
How: Let $u = f(x), du = f'(x)dx$
$dv = g'(x)dx, v = \int g'(x)dx = g(x)$.
$\int udv = uv - \int vdu$.
Check: Differentiate v to see that it gets dv.
Some integrations by parts takes several attempts to remove the integral part. Some will never actually remove the integral through normal integration. What is the trick when integrating $\int \cos xe^x dx$?
Why? This is the product rule for differentiation in reverse, with a bit of term-shifting.
Link: http://en.wikipedia.org/wiki/Integration_by_parts

1.2 7.2: Trigonometric Integrals

When: Given an integral in the form of powers and multiples of trigonometric functions.
How: Use trigonometric identities. Big ones are:
$\cos^2\theta + \sin^2\theta = 1$
$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$
$\sin^2\theta = \frac{1}{2} + \frac{1}{2}\sin 2\theta$
See full list in Section 1.3, pp. 26-27.
Reduce the integral into mostly one function (For example, all in $\sin x$) and a single representation of that function's derivative ($\cos x$ for this example). Now, substitute $u = \sin x$ and integrate.
Check: Differentiate v to see that it gets dv.
Why? Trigonometric integrals are more complicated after using substitutions in 7.3.
Link: http://en.wikipedia.org/wiki/Trigonometric_identity

1.3 7.3: Trigonometric Substitutions

When: The integral has $\sqrt{x^2+a^2}, \sqrt{x^2-a^2}$, or $\sqrt{a^2-x^2}$ somewhere in the function. This entity will be referred to as S from now on.
How: Consider a, x, and S as side lengths for a right triangle with an angle θ (see Figure 7.2 on p. 461). Use Pythagorean Theorem to determine which are legs and which is hypotenuse. Then, use trig functions to substitute in place of x, dx, a, and S.
After this new integral is solved, use the triangle diagram to replace trigonometric functions in θ with appropriate ratios of x, a, and S.
Check: Differentiate, or check algebra and trig identities really well.
Why? Applying geometry to calculus recalls the importance of high school math.
Link: http://en.wikipedia.org/wiki/Trigonometric_substitution

1.4 7.4: Integration by Partial Fraction Decomposition

When: When given a rational function $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials and $g(x)$ factors.

How: First, the degree of $f(x)$ must be less than degree of $g(x)$. If not, use long division of polynomials to get $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$ where the degree of $r(x)$ is less than the degree of $g(x)$. Also, multiply out by a constant so the first term of $g(x)$ is x^d, or so the first coefficient is 1.

Now, split $g(x)$ into linear and quadratic factors:
$g(x) = (x - a_1)^{m_1} \cdots (x - a_k)^{m_k}(x^2 + b_1 x + c_1)^{n_1} \cdots (x^2 + b_\ell x + c_\ell)^{n_\ell}$.

Set up an equation with $\frac{f(x)}{g(x)}$ on the left.

For each linear term $(x - a)^m$ in $g(x)$, add $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}$ to the right.

For each quadratic term $(x^2 + bx + c)^n$ in $g(x)$, add
$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n}$ to the right.

Now, multiply by $g(x)$ on both sides to get an equality of polynomials. By combining like terms on the right, we get a system of equations between the coefficients on the left and the coefficients on the right. Solve for the unknown A's, B's, and C's to result in the partial fraction expansion.

Integrate each partial fraction individually, and combine the resulting antiderivatives.

Check: Add the expansion together by finding a common denominator. You should get the original rational function back. Also, combine logarithms! See properties of logarithms on p. 41.

Why? Polynomials are easy, but rational functions are more difficult. Solve by breaking the problem into several smaller problems.

Trick: Sometimes a substitution will turn a complicated function into a rational function. See problems 35-38 in Section 7.4, p. 470.

Links: http://en.wikipedia.org/wiki/Partial_fraction_decomposition_over_the_reals
http://en.wikipedia.org/wiki/Polynomial_long_division

1.5 7.5: Integral Tables and Computer Algebra Systems

When: The integral is a familiar function, such as a polynomial, trigonometric function, or the derivative of a well-known function.

How: Memorize the integrals and derivatives that show up a lot. I recommend knowing how to integrate and differentiate all the trig functions.

To show your work, write $\frac{d}{dx}F(x) = f(x)$ to show $F(x) = \int f(x)dx$.

Be careful to make sure to have an exact substitution! If you have constant multiples, make sure they are cancelled properly.

Why? Certain integrals show up frequently near the end of big problems, and can be solved quickly to save time for reverse-substitution or doing other problems.

Link: http://en.wikipedia.org/wiki/Table_of_integrals

1.6 7.6: Numerical Integration

When: In order to approximate integrals that cannot be solved, we need a minimum number of intervals to use for computation in order to reach a certain error bound.

How: For the trapezoid rule, we need $f''(x)$ to be continuous.

Let M be an upper bound for $f''(x)$ for $x \in [a, b]$.

Now, solve for n in the equation $\frac{M(b-a)^3}{12n^2} \leq E$, where E is the maximum allowed error.

For Simpson's rule, we need $f^{(4)}(x)$ (the fourth derivative of $f(x)$) to be continuous.

Let M be an upper bound for $f^{(4)}(x)$ for $x \in [a, b]$.

Now, solve for n in the equation $\frac{M(b-a)^5}{180n^4} \leq E$, where E is the maximum allowed error.

Why? Not all functions have integrals, so a computer would be used to compute an approximate answer, but the number of intervals required can be predicted.

Link: http://en.wikipedia.org/wiki/Simpson's_rule

1.7 7.7: Improper Integrals

When: If given an integral with infinite bounds, or a definite integral where $f(x)$ is discontinuous at or within the bounds.

How: Take the limit!

Type I.a: $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$.

Type I.b: $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$.

Type I.c: $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$.

Type II.a: $f(x)$ discontinuous at b, $\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$.

Type II.b: $f(x)$ discontinuous at a, $\int_a^b f(x)dx = \lim_{c\to a^+} \int_a^c f(x)dx$.

Type II.c: $f(x)$ discontinuous at $c \in (a,b)$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Why? Infinite regions may have finite area if a function converges to 0 fast enough. Also, functions may be point-wise discontinuous but still have useful integrals.

Link: http://en.wikipedia.org/wiki/Improper_integral

2 Chapter 6: Applications of Definite Integrals

2.1 6.1: Volumes by Slicing and Rotation About an Axis

When: Given a shape defined by a function with independent variable the same as the axis of rotation.

How: Disk method: $\int_a^b \pi(r(x))^2 dx$.

Washer method: $\int_a^b \pi \left[(R(x))^2 - (r(x))^2\right] dx$

See diagrams in textbook.

Why? Volumes are too difficult to put into regular formulas, so integrals give an exact answer for almost any function.

Link: http://en.wikipedia.org/wiki/Disk_integration

2.2 6.2: Volumes by Cylindrical Shells

When: Given a shape defined by a function with independent variable different from the axis of rotation.

How: Shell method: When rotating about $x = L$, with height function $y = h(x)$,

Compute $\int_a^b 2\pi|x - L|h(x)dx$

See diagrams in textbook.

Why? The washer method doesn't always work, especially if the bound function is not well-defined in terms of the axis of rotation variable.

Link: http://en.wikipedia.org/wiki/Shell_integration

2.3 6.3: Lengths of Plane Curves

When: Given a path defined parametrically or as a function and asked for its length based on bounds of parameters.

How: Shell method: When rotating about $x = L$, with height function $y = h(x)$,

Compute $\int_a^b 2\pi|x - L|h(x)dx$

See diagrams in textbook.

Why? The washer method doesn't always work, especially if the bound function is not well-defined in terms of the axis of rotation variable.

Link: http://en.wikipedia.org/wiki/Arc_length

2.4 6.4: Areas of Surfaces of Revolution

When: Given a surface in three dimensions, calculate the area by integration.

How: For a curve $y = f(x)$ over $x \in [a, b]$ revolving around the x-axis, evaluate

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2} dx.$$

For a curve $x = g(y)$ over $y \in [a, b]$ revolving around the y-axis, evaluate

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_a^b 2\pi g(x)\sqrt{1 + (g'(x))^2} dy.$$

For a parametric curve $x = f(t), y = g(t)$ for $a \leq t \leq b$ revolved around y-axis, evaluate

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

For a parametric curve $x = f(t), y = g(t)$ for $a \leq t \leq b$ revolved around x-axis, evaluate

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Why? Not all surfaces have nice formulas, so integrals are needed.

Link: http://en.wikipedia.org/wiki/Surface_of_revolution

3 Problems

(*) denotes a difficult problem.

Evaluate:

$$\int \frac{1}{a^2-u^2}du$$
$$\int \frac{\sec^2 x}{(1+\tan x)^2}dx$$
$$\frac{d}{dx}\int_1^x \frac{1}{t}dt, \text{ for } x>0$$
$$\int x\sec x\tan x dx$$
$$\int \tan^{-1} x dx$$
$$\int \frac{(4+x^2)^2}{x^3}dx$$
$$\int x(2x+3)^{99}dx$$
$$\int \frac{1}{\sqrt{4+x^2}}dx$$
$$\int e^{4x}\sin(5x)dx$$

$$\int \frac{1}{x^2-4x+8}dx$$
$$\int \cos(5x)\cos(3x)dx$$
$$\int \frac{\sqrt{x^2-9}}{x}dx$$
$$\int \frac{(\sqrt{x}+3)^2}{\sqrt{x}}dx$$
$$\int_2^6 \frac{6x-11}{(x-1)^2}dx$$
$$\int \cos^3 x\sin^4 x dx$$
$$\int x\sqrt[3]{7-6x^2}dx$$
$$\int \frac{1}{x^2\sqrt{a^2-x^2}}dx, \text{ for some } a>0$$
$$\int (2x^3+1)^7x^2dx$$

$$\int \frac{2x-1}{x^2-6x+13}dx$$
$$\int \tan^2 x\sec^4 x dx$$
$$\int_0^{\pi/2}\cos^2 x dx$$
$$\int_2^{10}\frac{3}{\sqrt{5x-1}}dx$$
$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}dx$$
$$\int \sin^5 x dx$$
$$\int \frac{5x^2-10x-8}{x^3-4x}dx$$
$$\frac{d}{dx}\int_0^x (t^3-4\sqrt{t}+5)dt \text{ for } x>0.$$
$$\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}d\theta$$

State whether the integral converges or diverges. If it converges, evaluate.

$$\int_{-\infty}^0 \frac{1}{x^2-3x+2}dx$$
$$\int_0^\infty e^{-2x}dx$$
$$\int_0^\infty \cos x dx$$
$$\int_2^\infty \frac{1}{(x-1)^2}dx$$
$$\int_1^\infty \frac{\ln x}{x}dx$$
$$\int_{-\infty}^\infty \frac{1}{1+x^2}dx$$

For the following integrals, find the minimal number of intervals required for (a) the Trapezoidal rule; and (b) Simpson's rule. Then, compute the minimum error guaranteed when $n = 100$.

$$\int_0^2 \frac{1}{4+x^2}dx \qquad \int_1^4 \frac{1}{x}dx \qquad \int_{-2}^2 \sqrt{16-x^2}dx$$

Consider the following problems:

Compute the area between the x-axis and the curve $f(x) = 4 - 2\sin^2 x$.

Find the area between the curves $y = e^{-x}$ and $y = x^2$ where $1 \leq x \leq 4$.

*Find the volume of a "donut." Let $y_1 = \sqrt{1+(x-2)^2}$ be the top of the circle of radius 1 around the point $(2, 0)$ and let $y_2 = -\sqrt{1+(x-2)^2}$ be the bottom of the same circle. Let the region between y_2 and y_1 be revolved around the y-axis. Compute the volume.

Find the general form for y if $\frac{dy}{dx} = \frac{3x}{y+1}$

If $f(x) = \sin\sqrt{x}$, find the area of the region between the x-axis and $f(x)$ from 0 to π^4.

A spherical tank of radius 10 feet is filled with water. Find the work done in pumping all of the water out through the top of the tank. Recall that the density of water is $\approx 62.4\text{lb}/\text{ft}^3$.

A cup of fast food coffee is 180°F when freshly poured. After 2 minutes in a room at 70°F, the coffee has coooled to 165°F. Find the temperature at any time t and find the time at which the coffee has cooled to 120°F.

Does the integral $\int_1^\infty \frac{2+\cos x}{x}dx$ converge or diverge?

Suppose the line segment $y = 1 + \frac{x}{3}$ from $x = 0$ to $x = 12$ is revolved about the x-axis. The resulting solid looks like a megaphone. Compute the volume of this solid.

*Construct a volume by rectangular cross-sections for $0 \leq x \leq 1$: the base extends from $y = 0$ to $y = 1 - x^3$, and the height is x^2.

Let R be the region bounded by $y = 4 - x^2$ and $y = 0$. Find the volume of the solids obtained by revolving R about each of the following: (a) the y-axis, (b) the line $y = -3$, and (c) the line $y = 7$.

*Find the volume of the solid given by rotating $y = (x^3 - 2x^2 + x)\cos x$ around the x axis for $0 \leq x \leq \frac{3\pi}{2}$.