1 SEQUENCES

DEF: An ordered field is a field \( F \) and total order \( < \) (for all \( x, y, z \in F \)):
- (i) \( x < y, y < x \) or \( x = y \),
- (ii) \( x < y, y < z \Rightarrow x < z \),
- (iii) \( x < y \Rightarrow x + z < y + z \),
- (iv) \( 0 < y, x \Rightarrow 0 < xy \).  

DEF: The Archimedean property on an ordered field \( F \) is \( \forall x, y \in F, x, y > 0 \), there exists \( N \in \mathbb{N} \) such that \( n \cdot x > y \).  

FACT: \( \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{ra + sd}{rb + sd} = \frac{a}{b} \) for all \( a, b, c, d, r, s \in \mathbb{Z} \) with \( rb + sd \neq 0 \).  

DEF: A real number \( L \) is the limit of a sequence of real numbers \( (a_n)_{n=1}^{\infty} \) if for every \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \) such that \( |a_n - L| < \varepsilon \) for all \( n \geq N \). Then \( (a_n) \) converges to \( L \).  

THM: “Squeeze Theorem” Suppose three sequences \( (a_n), (b_n), (c_n) \) satisfy \( a_n \leq b_n \leq c_n \) and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \). Then \( \lim_{n \to \infty} b_n = L \).  

PROP: If \( (a_n)_{n=1}^{\infty} \) converges, then the set \( \{a_n | n \in \mathbb{N}\} \) is bounded.  

THM: If \( \lim_{n \to \infty} a_n = L \), \( \lim_{n \to \infty} b_n = M \), and \( \alpha \in \mathbb{R} \), then
- (i) \( \lim_{n \to \infty} \alpha a_n = \alpha L \),
- (ii) \( \lim_{n \to \infty} \alpha a_n = \alpha M \),
- (iii) \( \lim_{n \to \infty} a_n b_n = LM \),
- (iv) \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M} \) if \( M \neq 0 \).  

THM: For a sequence \( a_n \geq 0 \), we have \( \lim_{n \to \infty} a_n = +\infty \) if an only if \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).  

THM: “Least Upper Bound Principle” Every nonempty subset \( S \) of \( \mathbb{R} \) that is bounded above has a supremum. Similarly, every nonempty subset \( S \) of \( \mathbb{R} \) that is bounded below has an infimum.  

THM: “Monotone Convergence Theorem” A monotone increasing sequence that is bounded above converges. A monotone decreasing sequence that is bounded below converges.  

THM: Let \( (a_n) \) be a sequence.
- (i) If \( \lim a_n \) is defined, then \( \lim \inf a_n = \lim \sup a_n = \lim a_n \).
- (ii) If \( \lim \inf a_n = \lim \sup a_n = L \), then \( \lim a_n = L \).  

DEF: A subsequence \( (a_{n_k})_{k=1}^{\infty} \) is a new sequence \( (a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, \ldots) \) where \( n_1 < n_2 < \cdots \).  

THM: If the sequence \( (a_n) \) converges, then every subsequence converges to the same limit.  

THM: Every sequence \( (a_n) \) has a monotonic subsequence.  

COR: Let \( (a_n) \) be a sequence. There exists a monotonic subsequence whose limit is \( \limsup a_n \) and there exists a monotonic subsequence shows limit is \( \liminf a_n \).  

DEF: Let \( (a_n) \) be a sequence in \( \mathbb{R} \). A subsequential limit is any real number (or symbol +\(\infty\), \(-\infty\)) that is the limit of some subsequence \( (a_{n_k}) \).  

THM: Let \( (a_n) \) be any sequence in \( \mathbb{R} \), and let \( S \) denote the set of subsequential limits of \( (a_n) \).
- (i) \( S \) is nonempty.
- (ii) \( \sup S = \lim \sup a_n \) and \( \inf S = \lim \inf a_n \).
- (iii) \( \lim a_n \) exists if an only if \( S \) has a single element, namely \( \lim a_n \).  

THM: Let \( S \) denote the set of subsequential limits of a sequence \( (a_n) \). Suppose \( (b_n) \) is a sequence in \( S \cap \mathbb{R} \) and that \( t = \lim b_n \). Then \( t \in S \).  

LMA: “Nested Intervals Lemma” Suppose that \( I_n = [a_n, b_n] = \{ x \in \mathbb{R} | a_n \leq x \leq b_n \} \) are nonempty closed intervals such that \( I_{n+1} \subseteq I_n \) for each \( n \geq 1 \). Then the intersection \( \bigcap_{n \geq 1} I_n \) is nonempty.  

THM: “Bolzano-Weierstrass Theorem” Every bounded sequence of real numbers has a convergent subsequence.  

FACT: For a bounded sequence \( (a_n) \), \( \limsup a_n \) (\( \liminf a_n \)) is the largest (smallest) possible value for a convergent subsequence.  

DEF: A sequence \( (a_n)_{n=1}^{\infty} \) is called a Cauchy sequence if for every \( \varepsilon > 0 \), there is an integer \( N \) such that \( |a_m - a_n| < \varepsilon \) for all \( m, n \geq N \).  

DEF: A subset \( S \) of \( \mathbb{R} \) is said to be complete if every Cauchy sequence converges to a point in \( S \).  

THM: “Completeness Theorem” A sequence of real numbers converges if and only if it is a Cauchy sequence. In particular, \( \mathbb{R} \) is complete.
2 Series

DEF: Given a sequence \((a_n)_{n=1}^{\infty}\), the infinite series \(\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k\) converges if the limit exists, diverges otherwise.\(^{25}\)

FACT: Some series can be solved using a telescoping sum, by cancelling elements between sequence terms.\(^{26}\)

THM: “nth term test” If \(\lim a_n \neq 0\), then \(\sum a_n\) diverges.\(^{27}\)

THM: “Cauchy Criterion for Series” The following are equivalent for a series \(\sum_{n=1}^{\infty} a_n\).

(i) The series converges.
(ii) For every \(\varepsilon > 0\), there is an \(N \in \mathbb{N}\) so that \(\left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon\) for all \(n \geq N\).
(iii) For every \(\varepsilon > 0\), there is an \(N \in \mathbb{N}\) so that \(\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon\) if \(n, m \geq N\).\(^{28}\)

PROP: If \(a_k \geq 0\) for \(k \geq 1\) and \(s_n = \sum_{k=1}^{n} a_k\), then either

(i) \((s_n)_{n=1}^{\infty}\) is unbounded, in which case \(\sum_{n=1}^{\infty} a_n\) diverges.
(ii) \((s_n)_{n=1}^{\infty}\) is bounded above, in which case \(\sum_{n=1}^{\infty} a_n\) converges.

THM: “Convergence of Geometric Series” A geometric series \(\sum_{n=0}^{\infty} ar^n\) converges to \(\frac{a}{1-r}\) if \(|r| < 1\).\(^{30}\)

THM: “Convergence of p-series” A \(p\)-series \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges if and only if \(p > 1\).\(^{31}\)

THM: “Comparison Test” Consider two sequences \((a_n), (b_n)\) with \(a_n \leq b_n\) for all \(n \geq 1\). If \((b_n)\) is summable, then \((a_n)\) is summable and \(\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n\). If \((a_n)\) is not summable, then \((b_n)\) is not summable.\(^{32}\)

THM: “Ratio Test” A series \(\sum a_n\) of nonzero terms

(i) Converges absolutely if \(\limsup |a_{n+1}/a_n| < 1\),
(ii) Diverges if \(\liminf |a_{n+1}/a_n| > 1\).\(^{33}\)

THM: “Root Test” Suppose that \(a_n \geq 0\) for all \(n\) and let \(\ell = \limsup \sqrt[n]{a_n}\). If \(\ell < 1\), then \(\sum_{n=1}^{\infty} a_n\) converges absolutely, and if \(\ell > 1\), the series diverges.\(^{34}\)

THM: “Integral Test” If \(f: \mathbb{R} \to \mathbb{R}\) is positive and decreasing, then \(\sum_{n=1}^{\infty} f(n)\) converges if and only if \(\int_{1}^{\infty} f(x)dx\) exists (and is finite).\(^{35}\)

THM: “Limit Comparison Test” If \(\sum_{n=1}^{\infty} b_n\) is a convergent series of positive numbers \(b_n\) and \(\lim_{n \to \infty} \frac{|a_n|}{b_n} < \infty\) then \(\sum_{n=1}^{\infty} a_n\) converges.\(^{36}\)

THM: A series \(\sum_{n=1}^{\infty} a_n\) is called absolutely convergent if the series \(\sum_{n=1}^{\infty} |a_n|\) converges. A series that converges but is not absolutely convergent is called conditionally convergent.\(^{40}\)

THM: A rearrangement of a series \(\sum_{n=1}^{\infty} a_n\) is another series with the same terms in a different order. This can be described by a permutation \(\pi\) of the natural numbers \(\mathbb{N}\) determining the series \(\sum_{n=1}^{\infty} a_{\pi(n)}\).\(^{41}\)

THM: For an absolutely convergent series, every rearrangement converges to the same limit.

THM: “Leibniz Alternating Series Test” Suppose that \((a_n)_{n=1}^{\infty}\) is a monotone decreasing sequence \(\sum_{n=1}^{\infty} a_n\) converges.\(^{39}\)

COR: Suppose that \((a_n)_{n=1}^{\infty}\) is a monotone increasing sequence and that \(\lim_{n \to \infty} a_n = 0\). Then the difference between the sum of the alternating series \(\sum_{n=1}^{\infty} (-1)^n a_n\) and the \(N\)th partial sum is at most \(|a_N|\).

THM: A series \(\sum_{n=1}^{\infty} a_n\) is called absolutely convergent if the series \(\sum_{n=1}^{\infty} |a_n|\) converges. A series that converges but is not absolutely convergent is called conditionally convergent.\(^{40}\)

THM: A rearrangement of a series \(\sum_{n=1}^{\infty} a_n\) is another series with the same terms in a different order. This can be described by a permutation \(\pi\) of the natural numbers \(\mathbb{N}\) determining the series \(\sum_{n=1}^{\infty} a_{\pi(n)}\).\(^{41}\)

THM: For an absolutely convergent series, every rearrangement converges to the same limit.

THM: “Summation by Parts Lemma” Suppose \((x_n)\) and \((y_n)\) are sequences of real numbers. Define \(X_n = \sum_{k=1}^{n} x_k\) and \(Y_n = \sum_{k=1}^{n} y_k\). Then \(\sum_{n=1}^{\infty} x_k y_{n+1} = \sum_{n=1}^{\infty} X_n y_{n+1}\).\(^{44}\)

THM: “Dirichlet’s Test” Suppose that \((a_n)_{n=1}^{\infty}\) is a sequence of real numbers with bounded partial sums. If \((b_n)_{n=1}^{\infty}\) is a sequence of positive numbers decreasing monotonically to 0, then the series \(\sum_{n=1}^{\infty} a_n b_n\) converges.\(^{45}\)

THM: “Abel’s Test” Suppose that \(\sum_{n=1}^{\infty} a_n\) converges and \((b_n)\) is a monotonic convergent sequence. Then, \(\sum_{n=1}^{\infty} a_n b_n\)

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3 Topology of $\mathbb{R}^n$

**Def:** The dot product or inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{\infty} x_i y_i$.  

**Thm:** “Schwarz Inequality” For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$. Equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are colinear.

**Thm:** “Triangle Inequality” The triangle inequality holds for the Euclidean norm on $\mathbb{R}^n$: $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Moreover, equality holds if and only if either $\mathbf{x} = 0$ or $\mathbf{y} = c\mathbf{x}$ with $c \geq 0$.

**Def:** A set $V = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^n$ is orthonormal if $\langle v_i, v_j \rangle = 0$ when $i \neq j$ and $\langle v_i, v_i \rangle = 1$. If $m = 0$, then $V$ spans $\mathbb{R}^n$ and is called an orthonormal basis.

**LMA:** Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in $\mathbb{R}^n$. Then $\|\sum_{i=1}^{n} a_i x_i \| = (\sum_{i=1}^{n} |a_i|^2)^{1/2}$.

**Def:** A sequence of points $(x_k)$ in $\mathbb{R}^n$ converges to a point $a$ if for every $\varepsilon > 0$, there is an integer $N$ so that $||x_k - a|| < \varepsilon$ for all $k > N$. In this case, write $\lim_{k \to \infty} x_k = a$.

**LMA:** Let $(x_k)$ be a sequence in $\mathbb{R}^n$. Then $\lim_{k \to \infty} x_k = a$ if and only if $\lim_{k \to \infty} ||x_k - a|| = 0$.

**LMA:** A sequence $x_k = (x_{k,1}, \ldots, x_{k,n})$ in $\mathbb{R}^n$ converges to a point $a = (a_1, \ldots, a_n)$ if and only if each coordinate converges: $\lim_{k \to \infty} x_{k,i} = a_i$ for $1 \leq i \leq n$.

**Def:** A sequence $x_k$ in $\mathbb{R}^n$ is Cauchy if for every $\varepsilon > 0$, there is an integer $N$ so that $||x_k - x_{\ell}|| < \varepsilon$ for all $k, \ell > N$. A set $S \subseteq \mathbb{R}^n$ is complete if every Cauchy sequence of points in $S$ converges to a point in $S$.

**Thm:** “Completeness Theorem for $\mathbb{R}^n$” Every Cauchy sequence in $\mathbb{R}^n$ converges. Thus, $\mathbb{R}^n$ is complete.

**Def:** A point $x$ is a limit point of a subset $A \subseteq \mathbb{R}^n$ if there is a sequence $(a_k)_{k=1}^{\infty}$ with $a_k \in A$ such that $x = \lim_{k \to \infty} a_k$.

A set $A \subseteq \mathbb{R}^n$ is closed if it contains all of its limit points.

**Def:** A point $x$ is a cluster point of a subset $A \subseteq \mathbb{R}^n$ if there is a sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in A \setminus \{x\}$ such that $x = \lim_{n \to \infty} a_n$.

**Prop:** If $A, B \subseteq \mathbb{R}^n$ are closed, then $A \cup B$ is closed. If $\{A_i \mid i \in I\}$ is a family of closed subsets of $\mathbb{R}^n$, then $\bigcap_{i \in I} A_i$ is closed.

**Ex:** Let $A_0 = B_{\frac{1}{n^2}}(0)$. The family of sets $\{A_n\}_{n \in \mathbb{N}}$ has every set closed, but $\bigcup_{n \in \mathbb{N}} A_n = B_1(0)$, which is not closed.

**Def:** If $A$ is a subset of $\mathbb{R}^n$, the closure of $A$ is the set $\overline{A}$ consisting of all limit points of $A$.

**Def:** The ball about $a$ in $\mathbb{R}^n$ of radius $r$ is the set $B_r(a) = \{x \in \mathbb{R}^n \mid ||x - a|| < r\}$. A subset $U \subseteq \mathbb{R}^n$ is open if for every $a \in U$, there is some $r > 0$ so that the ball $B_r(a)$ is contained in $U$.

**Prop:** Let $A \subseteq \mathbb{R}^n$. Then $\overline{A}$ is the smallest closed set containing $A$. In particular, $\overline{\mathbb{R}^n} = \mathbb{R}^n$.

**Thm:** “Duality of Open and Closed Sets” A set $A \subseteq \mathbb{R}^n$ is open if and only if the complement of $A$, $A' = \{x \in \mathbb{R}^n \mid x \notin A\}$, is closed.

**Prop:** If $U$ and $V$ are open subsets of $\mathbb{R}^n$ then $U \cap V$ is an open subset of $\mathbb{R}^n$. If $\{U_i \mid i \in I\}$ is a family of open subsets of $\mathbb{R}^n$, then $\bigcup_{i \in I} U_i$ is open.

**Ex:** Let $A_0 = B_{\frac{1}{n^2}}(0)$. The family of sets $\{A_n\}_{n \in \mathbb{N}}$ has every set open, but $\bigcap_{n \in \mathbb{N}} A_n = \overline{B}_1(0)$, which is closed.

**Def:** The interior, $\text{int} \, X$, of a set $X$ is the largest open set contained in $X$. If $\text{int} \, X = \emptyset$, then $X$ has empty interior.

**Def:** A subset $A \subseteq \mathbb{R}^n$ is compact if every sequence $(a_k)_{k=1}^{\infty}$ of points in $A$ has a convergent subsequence $(a_{k_j})_{j=1}^{\infty}$ with limit $a = \lim_{j \to \infty} a_{k_j}$.

**Def:** A subset $S$ of $\mathbb{R}^n$ is called bounded provided that there is a real number $R$ such that $S$ is contained in the ball $B_R(0)$.

**LMA:** A compact subset of $\mathbb{R}^n$ is closed and bounded.

**LMA:** If $C$ is a closed subset of a compact subset of $\mathbb{R}^n$, then $C$ is compact.

**LMA:** The cube $[a, b]^n$ is a compact subset of $\mathbb{R}^n$.

**Thm:** “Heine-Borel Theorem” A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

**Thm:** “Cantor’s Intersection Theorem” If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a decreasing sequence of nonempty compact subsets of $\mathbb{R}^n$, then $\bigcap_{k \geq 1} A_k$ is not empty.

**Def:** A set whose closure has no interior is nowhere dense. A point $x$ of a set $A$ is isolated if there is an $\varepsilon > 0$ such that the ball $B_{\varepsilon}(x)$ intersects $A$ only in the singleton $\{x\}$. A set $A$ is perfect if each point $x \in A$ is the limit of some sequence in $A \setminus \{x\}$.
4 Functions

**Def:** Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$. If $a \in S$ is a cluster point (limit point of $S \setminus \{a\}$) then a point $v \in \mathbb{R}^n$ is a limit of $f$ at $a$ if for every $\varepsilon > 0$ there is an $r > 0$ so that $\|f(x) - v\| < \varepsilon$ whenever $0 < \|x - a\| < r$ and $x \in S$. Write $\lim_{x \to a} f(x) = v$.  

**Def:** Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$. $f$ is **continuous at** $a \in S$ if for every $\varepsilon > 0$, there is an $r > 0$ such that, for all $x \in S$ with $\|x - a\| < r$, we have $\|f(x) - f(a)\| < \varepsilon$. Moreover, $f$ is **continuous on** $S$ if it is continuous at each point $a \in S$. If $f$ is not continuous at $a$, $f$ is **discontinuous at** $a$.  

**Def:** A function $f : S \to \mathbb{R}^m$ is **piecewise continuous** if for any subset $A \subseteq S$, $f$ is continuous on $A$. The **limit** of $f$ at $a$, $\lim_{x \to a} f(x) = L$, if for every sequence $(x_n)$ in $S$ such that $x_n \to a$, $f(x_n) \to L$. Moreover, $f$ is **continuous at** $a$ if and only if $\lim_{x \to a} f(x) = f(a)$.  

**Thm:** If $f, g$ are functions from a common domain $S$ into $\mathbb{R}^m$, $a \in S$ such that $\lim_{x \to a} f(x) = u$ and $\lim_{x \to a} g(x) = v$, then
(i) $\lim_{x \to a} (f(x) + g(x)) = u + v$.
(ii) $\lim_{x \to a} c f(x) = c u$.
(iii) $\lim_{x \to a} f(x) g(x) = u v$, and
(iv) $\lim_{x \to a} f(x)/g(x) = u/v$ provided $v \neq 0$.

**Thm:** If $f, g$ are functions from $S$ to $\mathbb{R}^m$ that are continuous at $a \in S$ and $\alpha \in \mathbb{R}$, then
(i) $f + g$ is continuous at $a$.
(ii) $\alpha f$ is continuous at $a$.
When the range is contained in $\mathbb{R}$,
(iii) $fg$ is continuous at $a$
(iv) $f/g$ is continuous at $a$ provided that $g(a) \neq 0$.

**Def:** A function $f$ is a **rational function** if $f(x) = p(x)/q(x)$ where $p, q \in \mathbb{R}[x]$ and $q \neq 0$. Rational functions are continuous except where $q(x) = 0$.

**Def:** If a function $f : S \to T$ and $g : T \to \mathbb{R}^m$, then the composition of $g$ and $f$, denoted $g \circ f$ is the function that sends $x$ to $g(f(x))$. 

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THM: Suppose $f : S \to T$ and $g : T \to \mathbb{R}^m$. If $f$ is continuous at $a \in S$ and $g$ is continuous at $f(a) \in T$, then $g \circ f$ is continuous at $a$.

THM: Let $C$ be a compact subset of $\mathbb{R}^n$, and let $f$ be a continuous function from $C$ into $\mathbb{R}^m$. Then the image set $f(C)$ is compact.

THM: “Extreme Value Theorem” Let $C$ be a compact subset of $\mathbb{R}^n$ and let $f$ be a continuous function from $C$ into $\mathbb{R}$. Then there are points $a$ and $b$ in $C$ attaining the minimum and maximum values of $f$ on $C$. That is, $f(a) \leq f(x) \leq f(b)$ for all $x \in C$.

THM: Suppose that $C \subseteq \mathbb{R}^n$ is compact and $f : C \to \mathbb{R}^n$ is continuous. Then $f$ is uniformly continuous on $C$.

THM: “Intermediate Value Theorem” If $f$ is a continuous real-valued function on $[a, b]$ with $f(a) < 0 < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

COR: Every Lipschitz function is uniformly continuous.

COR: Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous.

THM: Suppose that $C \subseteq \mathbb{R}^n$ is compact and $f : C \to \mathbb{R}^n$ is continuous. Then $f$ is uniformly continuous on $C$.

THM: “Intermediate Value Theorem” If $f$ is a continuous real-valued function on $[a, b]$ with $f(a) < 0 < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

COR: If $f$ is a continuous real-valued function on $[a, b]$, then $f([a, b])$ is a closed interval.

DEF: A path in $S \subseteq \mathbb{R}^n$ from $a$ to $b$, both points in $S$, is the image of a continuous function $\gamma$ from $[0, 1]$ into $S$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

COR: Suppose that $S \subseteq \mathbb{R}^n$ and $f$ is a continuous real-valued function on $S$. If there is a path from $a$ to $b$ in $S$ and $f(a) < 0 < f(b)$, then there is a point $c$ on the path so that $f(c) = 0$.

DEF: A function $f$ is increasing on an interval $(a, b)$ if $f(x) \leq f(y)$ whenever $a < x \leq y < b$. It is strictly increasing on $(a, b)$ if $f(x) < f(y)$ whenever $a < x < y < b$. Similarly, define decreasing and strictly decreasing functions. All of these functions are called monotone.

PROP: If $f$ is an increasing function on the interval $(a, b)$, then the one-sided limits of $f$ exist at each point $c \in (a, b)$ and $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = M$. For decreasing functions, the inequalities are reversed.

COR: The only type of discontinuity that a monotone function on an interval can have is a jump discontinuity.

COR: If $f$ is a monotone function on $[a, b]$ and the range of $f$ intersects every nonempty open interval in $[f(a), f(b)]$ then $f$ is continuous.

THM: A monotone function on $[a, b]$ has at most countably many discontinuities.

THM: Let $f$ be a continuous strictly increasing function on $[a, b]$. Then $f$ maps $[a, b]$ one-to-one and onto $[f(a), f(b)]$. Moreover the inverse function $f^{-1}$ is also continuous and strictly increasing.

PROP: Let $f : S \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^n$ be a continuous function. If $T \subseteq S$ is compact, $f(T)$ is compact.

EX: The Cantor function $c : [0, 1] \to [0, 1]$ is an onto function defined such that $c$ is constant over the intervals not in the Cantor set, but is strictly increasing over the Cantor set. Also, $c([0, 1]) = [0, 1]$. 

Derrick Stolee
5 Intro to Calculus

DEF: A function \( f: (a, b) \to \mathbb{R} \) is differentiable at a point \( x_0 \in (a, b) \) if \( \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} \) exists. We write \( f'(x_0) \) for this limit. \( ^{119} \)

DEF: When \( f \) is differentiable at \( x_0 \), we define the tangent line to \( f \) at \( x_0 \) to be the linear function \( T(x) = f(x_0) + f'(x_0)(x-x_0) \). \( ^{120} \)

PROP: If \( f \) is differentiable at \( x_0 \), then it is continuous at \( x_0 \). Differentiable functions are continuous. \( ^{121} \)

LMA: Let \( f \) be a function on \([a, b]\) that is differentiable at \( x_0 \). Let \( T(x) \) be the tangent line to \( f \) at \( x_0 \). Then \( T \) is the unique linear function with the property \( \lim_{x \to x_0} \frac{f(x)-T(x)}{x-x_0} = 0 \). \( ^{122} \)

COR: If \( f(x) \) is a function on \((a, b)\) and \( x_0 \in (a, b) \), then the following are equivalent:

(i) \( f \) is differentiable at \( x_0 \).
(ii) There is a linear function \( T(x) \) and a function \( \varepsilon(x) \) on \((a, b)\) such that \( f(x) = f(x_0) + \varepsilon(x)(x-x_0) \) and \( \lim_{x \to x_0} \varepsilon(x) = 0 \) and \( f(x) = T(x) + \varepsilon(x)(x-x_0) \).

(iii) There is a function \( \varphi(x) \) on \((a, b)\) such that \( f(x) = f(x_0) + \varphi(x)(x-x_0) \) and \( \lim_{x \to x_0} \varphi(x) = 0 \).

(iv) If \( f \) is differentiable at \( x_0 \), then there is a point \( c \) such that \( f(x) = f(x_0) + \varphi(x)(x-x_0) \) and \( \lim_{x \to x_0} \varphi(x) = 0 \).

LMA: Let \( f \) and \( g \) be differentiable functions at the point \( a \). Each of the functions \( e \) \( f \) is a constant, \( f + g \), \( fg \), and \( f/g \) are differentiable at \( a \), except \( f/g \) if \( g(a) = 0 \). The formulas are \( ^{124} \)

(i) \( (c)f'(a) = c\cdot f'(a) \)
(ii) \( (f+g)'(a) = f'(a) + g'(a) \)
(iii) \( (fg)'(a) = f(a)g'(a) + f'(a)g(a) \)
(iv) \( (f/g)'(a) = (g(a)f'(a) - f(a)g'(a))/g^2(a) \) if \( g(a) \neq 0 \).

THM: “Arithmetic of Derivatives” Let \( f \) and \( g \) be differentiable functions at the point \( a \). Each of the functions \( f \) \( g \) is an end point \( (a, b) \) \( g \) is differentiable at \( f(x_0) \). Then the composition \( h(x) = g(f(x)) \) is defined, and \( h'(x_0) = g'(f(x_0))f'(x_0) \). \( ^{125} \)

EX: The class of functions \( f(x) = x^n \sin(1/x) \) for \( x > 0 \) and \( f(0) = 0 \) for \( a > 0 \) is differentiable on \([0, \infty)\), but the derivative function is not continuous at \( 0 \). \( ^{126} \)

DEF: A function \( f(x) \) is even if \( f(-x) = f(x) \) and odd if \( f(-x) = -f(x) \). \( ^{127} \)

HERE MARKS THE END OF EXAM 2 MATERIAL-

THM: “Fermat’s Theorem” Let \( f \) be a continuous function on an interval \([a, b]\) that takes its maximum or minimum value at a point \( x_0 \). Then, exactly one of the following holds:

(i) \( x_0 \) is an endpoint \( a \) or \( b \), (ii) \( f \) is not differentiable at \( x_0 \), (iii) \( f \) is differentiable at \( x_0 \) and \( f'(x_0) = 0 \). \( ^{128} \)

THM: “Rolle’s Theorem” Suppose that \( f \) is a function that is continuous on \([a, b]\) and differentiable on \((a, b)\) such that \( f(a) = f(b) = 0 \). Then there is a point \( c \in (a, b) \) such that \( f'(c) = 0 \). \( ^{129} \)

THM: “Mean Value Theorem” Suppose that \( f \) is a function that is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there is a point \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b-a} \). \( ^{130} \)

COR: Let \( f \) be a differentiable function on \([a, b]\). \( ^{131} \)

(i) If \( f'(x) \) is (strictly) positive, then \( f \) is (strictly) increasing.
(ii) If \( f'(x) \) is (strictly) negative, then \( f \) is (strictly) decreasing. \( ^{132} \)
(iii) If \( f'(x) = 0 \) at every \( x \in (a, b) \), then \( f(x) \) is constant. \( ^{133} \)
(iv) If \( g \) is differentiable on \([a, b]\) with \( g'(x) = f'(x) \), then there is a constant \( c \) such that \( f(x) = g(x) + c \). \( ^{134} \)

DEF: If a twice differentiable function \( f \) has \( f''(x) \) positive on an interval \([a, b]\) then \( f \) is convex or concave up. If \( f''(x) \) is negative, then \( f \) is concave or concave down. The points where \( f''(x) \) changes sign are called inflection points. \( ^{135} \)

THM: “Darboux’s Theorem/Intermediate Value Theorem for Derivatives” If \( f \) is differentiable on \([a, b]\) and \( f'(a) < L < f'(b) \), then there is a point \( x_0 \) in \((a, b)\) at which \( f'(x_0) = L \). \( ^{136} \)

THM: Let \( f \) is a one-to-one continuous function on an open interval \( I \), and let \( J = f(I) \). If \( f \) is differentiable at \( x_0 \in I \) and if \( f'(x_0) \neq 0 \), then \( f^{-1} \) is differentiable at \( y_0 = f(x_0) \) and \( (f^{-1})'(y_0) = \frac{1}{f'(x_0)} \). \( ^{137} \)

Derrick Stolee
6 Integration

**Def:** Let \( f : [a, b] \to \mathbb{R} \) be a bounded function.\(^{138}\)
(i) A partition of \([a, b]\) is a finite set \( P = \{ x_0 < x_1 < \cdots < x_{n-1} < x_n = b \}. \) Set \( \Delta_j = x_j - x_{j-1} \) and define the mesh of a partition \( P \) as \( \text{mesh}(P) = \max_{1 \leq j \leq n} \Delta_j. \)
(ii) Let the maximum and minimum be \( M_j(f, P) = \sup \{ f(x) \mid x_{j-1} \leq x \leq x_j \} \) and \( m_j(f, P) = \inf \{ f(x) \mid x_{j-1} \leq x \leq x_j \}. \)
(iii) Let the upper (Darboux) sum and lower (Darboux) sum be \( U(f, P) = \sum_{j=1}^{n} M_j(f, P) \Delta_j \) and \( L(f, P) = \sum_{j=1}^{n} m_j(f, P) \Delta_j. \)
(iv) If given an evaluation sequence \( X = \{ x'_j \mid 1 \leq j \leq n \} \) with \( x'_j \in [x_{j-1}, x_j] \) the Riemann sum is \( I(f, P, X) = \sum_{j=1}^{n} f(x'_j) \Delta_j. \)
(v) A partition \( R \) is a refinement of a partition \( P \) provided \( P \subseteq R \). If \( P \) and \( Q \) are partitions, then \( R \) is a common refinement if \( P \cup Q \subseteq R \).

**Thm:** “Riemann’s Condition” Let \( f(x) : [a, b] \to \mathbb{R} \) be a bounded function. The following are equivalent: \(^{141}\)
(i) \( f \) is Riemann integrable.
(ii) For each \( \varepsilon > 0 \), there is a partition \( P \) so that \( U(f, P) - L(f, P) < \varepsilon. \)

**Cor:** If \( P \) and \( Q \) are any two partitions of \([a, b]\), \( L(f, P) \leq U(f, Q) \).\(^{142}\)

**Def:** Define the lower Darboux integral, \( L(f) = \sup_P L(f, P) \) and the upper Darboux integral, \( U(f) = \inf_P U(f, P). \) Note that \( L(f) \leq U(f). \) A bounded function \( f \) on a finite interval \([a, b]\) is called Riemann integrable if \( L(f) = U(f) \).

In this case, we write \( L(f) = \int_a^b f(x)dx = U(f). \)

**Thm:** Let \( f(x) : [a, b] \to \mathbb{R} \) be a bounded function. The following are equivalent: \(^{146}\)
(i) \( f \) is Riemann integrable.
(ii) For each \( \varepsilon > 0 \), there is a partition \( P \) so that \( U(f, P) - L(f, P) < \varepsilon. \)

**Thm:** Let \( f \) be a bounded real-valued function on \([a, b]\). If there is a sequence of partitions \( \{ P_n \} \) so that \( \lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0 \), then \( f \) is Riemann integrable. Moreover, if \( X_n \) is any choice of points \( x'_{n,j} \) selected from each interval of \( P_n \), then \( \lim_{n \to \infty} I(f, P_n, X_n) = \int_a^b f(x)dx. \)

**Thm:** Let \( f(x) : [a, b] \to \mathbb{R} \) be a bounded function, then \( f \) is Riemann integrable with \( \int_a^b f(x)dx = L \) if and only if for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) so that every partition \( Q \) such that \( \text{mesh}(Q) < \delta \) satisfies \( U(f, Q) - L(f, Q) < \varepsilon. \)

**Thm:** Every monotone function on \([a, b]\) is Riemann integrable.\(^{149}\)

**Thm:** Every continuous function on \([a, b]\) is integrable.\(^{150}\)

**Def:** Say that \( f \) is Riemann integrable on \([a, \infty)\) if the improper integral \( \int_a^\infty f(x)dx := \lim_{b \to \infty} \int_a^b f(x)dx \) exists.\(^{151}\)

**Thm:** “Arithmetic of Integrals” Let \( f \) and \( g \) be integrable function on \([a, b]\) and \( c \in \mathbb{R} \).
(i) \( cf \) is integrable and \( \int_a^b cf(x)dx = c \int_a^b f(x)dx. \)
(ii) \( f + g \) is integrable and \( \int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \)

**Thm:** If \( f \) and \( g \) are integrable on \([a, b]\) and if \( f(x) \leq g(x) \) for all \( x \in [a, b] \), then \( \int_a^b f(x)dx \leq \int_a^b g(x)dx. \)

**Thm:** If \( f \) is integrable on \([a, b]\), then \( |f| \) is integrable on \([a, b]\) and \( |\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx. \)

**Thm:** If \( f : [a, b] \to \mathbb{R} \) with \( c \in (a, b) \) and \( f \) is integrable on \([a, c]\) and \([c, b]\) then \( f \) is integrable on \([a, b]\) with \( \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \)

**Thm:** “Fundamental Theorem of Calculus I” Let \( f \) be a bounded Riemann integrable function on \([a, b]\) and let \( F : [a, b] \to \mathbb{R} \) be defined as \( F(x) = \int_a^x f(t)dt \). Then \( F \) is continuous, and if \( f \) is continuous at a point \( x_0 \), then \( F \) is differentiable at \( x_0 \) with \( F'(x_0) = f(x_0). \)

**Def:** A function \( f : [a, b] \to \mathbb{R} \) has an antiderivative if there is a continuous function \( F : [a, b] \to \mathbb{R} \) such that \( F'(x) = f(x) \) for all \( x \in [a, b]. \)

**Cor:** “Fundamental Theorem of Calculus II” Let \( f \) be a continuous function on \([a, b]\). Then \( f \) has an antiderivative.

Moreover, if \( G \) is any antiderivative of \( f \), then \( \int_a^b f(x)dx = G(b) - G(a). \)

**Lma:** Suppose that \( f : [a, b] \to \mathbb{R} \) is an integrable function bounded by \( M : [a, b] \to \mathbb{R} \). Then \( |\int_a^b f(x)dx| \leq M(b - a). \)

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*Derrick Stolee*
6 INTEGRATION

6.1 Riemann-Stieltjes Integration

**Def:** Consider $f, g : [a, b] \to \mathbb{R}$. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and an evaluation sequence $X$ for $P$, define the **Riemann-Stieltjes sum** or **R-S sum** for $f$ with respect to $g$ using $P$ and $X$ as $I_g(f, P, X) = \sum_{i=1}^{n} f(x_i)(g(x_i) - g(x_{i-1}))$.  

**Def:** $f$ is **Riemann-Stieltjes integrable** with respect to $g$ ($f \in \mathcal{R}(g)$) if there is a number $L$ so that for all $\varepsilon > 0$, there is a partition $P_\varepsilon$ so that for all partitions $P \supseteq P_\varepsilon$ and evaluation sequences $X$ on $P$, we have $|I_g(f, P, X) - L| < \varepsilon$. In this case, we say $L$ is the **Riemann-Stieltjes integral** of $f$ with respect to $g$, written $L = \int_a^b f \, dg$.

**Thm:** If $f_1, f_2 \in \mathcal{R}(g)$ on $[a, b]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2 \in \mathcal{R}(g)$ on $[a, b]$ and $\int_a^b c_1 f_1 + c_2 f_2 \, dg = c_1 \int_a^b f_1 \, dg + c_2 \int_a^b f_2 \, dg$.

**Def:** Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions. If $f \in \mathcal{R}(g)$ on $[a, b]$, then $g \in \mathcal{R}(f)$ on $[a, b]$ and $\int_a^b f \, dg + \int_a^b g \, df = \int_a^b f \, df - \int_a^b g \, dg$.

**Thm:** **Integration by Parts** Let $f, g : [a, b] \to \mathbb{R}$ have $f \in \mathcal{R}(g)$ where $g : [a, b] \to \mathbb{R}$ is $C^1$, then $f g'$ is Riemann integrable on $[a, b]$ and $\int_a^b f g' \, dx = \int_a^b f(x)g'(x) \, dx$.

**Def:** Let $f, g$ be bounded functions on $[a, b]$, and $P$ a partition on $[a, b]$. Define

(i) upper sum with respect to $g$: $U_g(f, P) = \sum_{i=1}^{n} M_i(f_i, P)[g(x_i) - g(x_{i-1})]$.

(ii) lower sum with respect to $g$: $L_g(f, P) = \sum_{i=1}^{n} m_i(f_i, P)[g(x_i) - g(x_{i-1})]$.

**Def:** Define $L_g(f) = \sup_P L_g(f, P)$ and $U_g(f) = \inf_P U_g(f, P)$.

**LMa:** **Refinement Lemma** Let $f, g$ be bounded functions on $[a, b]$ and $P, Q$ partitions of $[a, b]$. Assume that $g$ is increasing. If $P$ is a refinement of $Q$, then $L_g(f, P) \leq L_g(f, Q) \leq U_g(f, Q) \leq U_g(f, P)$.

**Cor:** Let $f, g$ be bounded functions on $[a, b]$, and assume that $g$ is increasing. If $P$ and $Q$ are any two partitions of $[a, b]$, then $L_g(f, P) \leq U_g(f, Q)$.

**Thm:** **Riemann-Stieltjes Condition** Let $f, g$ be bounded functions on $[a, b]$ and assume that $g$ is increasing on $[a, b]$. The following are equivalent:

(i) $f \in \mathcal{R}(g)$. (ii) $U_g(f) = L_g(f)$.

(iii) For every $\varepsilon > 0$ there is a partition $P$ so that $U_g(f, P) - L_g(f, P) < \varepsilon$.

**Def:** Given a function $f : [a, b] \to \mathbb{R}$ and a partition $P$ of $[a, b]$, the variation of $f$ over $P$ is $V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$.

**Def:** The total variation of $f$ on $[a, b]$ is $V_a^b f = \sup_P V(f, P)$. $f$ is of bounded variation on $[a, b]$ if $V_a^b f$ is finite.

**LMa:** If $a < b < c$ and $f : [a, c] \to \mathbb{R}$ is given, then $V_a^b f = V_a^c f + V_c^b f$.

**Thm:** If $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then there are increasing functions $g, h : [a, b] \to \mathbb{R}$ so that $f = g - h$.

**Thm:** If $f : [a, b] \to \mathbb{R}$ is bounded by $M$, $g : [a, b] \to \mathbb{R}$ is of bounded variation and $f \in \mathcal{R}(g)$, then $\left| \int_a^b f \, dg \right| \leq M \cdot V_a^b g$.

**Thm:** If $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$ is increasing, then $f \in \mathcal{R}(g)$ on $[a, b]$.

**Cor:** If $f : [a, b] \to \mathbb{R}$ is continuous and $g : [a, b] \to \mathbb{R}$ is of bounded variation, then $f \in \mathcal{R}(g)$ on $[a, b]$.

Derrick Stolee
7 VECTORS AND DISTANCE

7.1 Normed Vector Spaces

DEF: Let \( V \) be a vector space over \( \mathbb{R} \). A norm on \( V \) is a function \( ||\cdot|| \) on \( V \) taking values in \([0, +\infty)\) with

(i) \( ||x|| = 0 \iff x = 0 \).
(ii) \( ||\alpha x|| = |\alpha||x|| \).
(iii) \( ||x + y|| \leq ||x|| + ||y|| \).

The pair \((V, ||\cdot||)\) is called a normed vector space.

DEF: In a normed vector space \( V \), a sequence \((x_n)_{n=1}^{\infty}\) converges if there is \( x \in V \) so that \( \lim_{n \to \infty} ||x_n - x|| = 0 \).

DEF: A sequence \((x_n)_{n=1}^{\infty}\) is Cauchy if \((||x_n||)_{n=1}^{\infty}\) is Cauchy. That is, \( \lim_{n \to \infty} \sup_{m \geq n} ||x_n - x_m|| = 0 \).

DEF: \( V \) is complete if every Cauchy sequence in \( V \) converges to some vector \( x \in V \). A complete normed vector space is called a Banach space.

DEF: For a normed vector space \( V \), define the open ball \( B_r(x) = \{v \in V \mid ||v - x|| < r\} \). A subset \( U \subseteq V \) is open if for every \( a \in U \) there is \( r > 0 \) so that \( B_r(a) \subseteq U \). A subset \( C \subseteq V \) is closed if it contains all of its limit points.

PROP: A sequence \( x_n \) in a normed vector space \( V \) converges to a vector \( x \) if and only if for each open set \( U \) containing \( x \), there is an integer \( N \) so that \( x_n \in U \) for all \( n \geq N \).

DEF: A subset \( K \) of a normed vector space \( V \) is compact if every sequence \((x_n)\) of points in \( K \) has a convergent subsequence.

7.2 Inner Product Spaces

DEF: An inner product on a vector space \( V \) is a function \( \langle \cdot, \cdot \rangle \) so that

(i) \( \langle x, x \rangle \geq 0 \), and \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \).
(ii) \( \langle x, y \rangle = \langle y, x \rangle \).
(iii) For all \( x, y, z \in V, a, b \in \mathbb{R} \), \( \langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle \). An inner product defines a norm on \( V \) given by \( ||x|| = \langle x, x \rangle^{1/2} \).

EX: The space \( C[0,1] \) can be given an inner product \( \langle f, g \rangle = \int_a^b f(x)g(x)dx \).

THM: “Cauchy-Schwarz Inequality” For all \( x, y \) in an inner product space \( V \), \( ||\langle x, y \rangle|| \leq ||x|| ||y|| \). Equality holds if and only if \( x, y \) are colinear.

COR: For \( f, g \in C[a, b] \), we have \( \left| \int_a^b f(x)g(x)dx \right| \leq \left( \int_a^b f(x)^2dx \right)^{1/2} \left( \int_a^b g(x)^2dx \right)^{1/2} \).

COR: An inner product space \( V \) satisfies the triangle inequality. Moreover, if equality occurs, then \( x \) and \( y \) are colinear.

COR: Let \( V \) be an inner product space with induced norm \( ||\cdot|| \). Then the inner product is continuous (i.e. \( x_n \to x \) and \( y_n \to y \), then \( \langle x_n, y_n \rangle \to \langle x, y \rangle \)).

DEF: A normed vector space is strictly convex if \( ||u|| = ||v|| = \frac{1}{2} ||(u + v)|| = 1 \) for vectors \( u, v \in V \) implies that \( u = v \).

PROP: All inner product spaces are strictly convex.

7.3 Orthonormality

DEF: Two vectors \( x \) and \( y \) are orthogonal if \( \langle x, y \rangle = 0 \). A collection of vectors \( \{v_i \mid i \in I\} \) in \( V \) are orthonormal if \( ||v_i|| = 1 \) and \( \langle v_i, v_j \rangle = \delta_{ij} \). This set is called an orthonormal basis if it is a maximal orthonormal set.

PROP: An orthonormal set is linearly independent. An orthonormal basis in a finite-dimensional inner product space is a basis.

LMA: The functions \( \{1, \sqrt{2} \sin n\theta, \sqrt{2} \cos n\theta \mid n \geq 1\} \) form an orthonormal set in \( C[-\pi, \pi] \) with the inner product \( \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)g(\theta) \, d\theta \).

DEF: A trigonometric polynomial is a finite sum \( f(\theta) = A_0 + \sum_{k=1}^{N} A_k \cos k\theta + B_k \sin k\theta \).

DEF: Denote the Fourier series of \( f \in C[-\pi, \pi] \) by \( f \sim A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta \), where \( A_0 = \langle f, 1 \rangle \), \( A_n = \langle f, \sqrt{2} \cos n\theta \rangle \), \( B_n = \langle f, \sqrt{2} \sin n\theta \rangle \). The sequences \( A_n \) and \( B_n \) are the Fourier coefficients of \( f \).

LMA: Let \( \{v_1, \ldots, v_n\} \) be an orthonormal set in an inner product space \( V \). If \( M \) is the subspace spanned by \( \{v_1, \ldots, v_n\} \), then every vector \( x \in M \) can be written uniquely as \( \sum_{i=1}^{n} a_i v_i \), where \( a_i = \langle x, v_i \rangle \). Moreover, for \( x, y \in M \) with \( a_i = \langle x, v_i \rangle, b_i = \langle y, v_i \rangle, \langle x, y \rangle = \sum_{i=1}^{n} a_i b_i \). In particular, \( ||x||^2 = \sum_{i=1}^{n} a_i^2 \).

COR: If \( V \) is an inner product space of finite dimension \( n \), then it has an orthonormal basis \( \{v_1, \ldots, v_n\} \) and the inner product and norm are defined by the lemma.

Derrick Stolee
7.4 Projections

DEF: A projection is a linear map \( P \) so that \( P^2 = P \). In addition, \( P \) is an orthogonal projection if \( \ker P \perp \im P \).

THM: "Projection Theorem" Let \( B = \{v_1, \ldots, v_n\} \) be an orthonormal set in an inner product space \( V \) and let \( M = \text{span} B \). Define \( P : V \to M \) by \( Py = \sum_{i=1}^{n} \langle y, v_i \rangle v_i \). Then \( P \) is the orthogonal projection onto \( M \) and \( \|y\|^2 = \sum_{i=1}^{n} \langle y, v_i \rangle^2 \).

Moreover, for all \( v \in M \), \( \|y-v\|^2 = \|y-Py\|^2 + \|Py-v\|^2 \). In particular, \( Py \) is the closest vector in \( M \) to \( y \).

THM: "Bessel's Inequality" Let \( \{v_i | i \in I\} \) be an orthonormal set in an inner product space \( V \). For each vector \( x \in V \), \( \sum_{i \in I} |\langle x, v_i \rangle|^2 \leq \|x\|^2 \).

DEF: A complete inner product space is called a Hilbert space.

EX: The space \( l^2 \) consists of all sequences \( x = (x_n)_{n=1}^{\infty} \) such that \( \|x\|_2 = (\sum_{n=1}^{\infty} x_n^2)^{1/2} \) is finite. The inner product on \( l^2 \) is given by \( \langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n \).

THM: The space \( l^2 \) is complete.

DEF: In a Hilbert space, the closed span of a set of vectors \( S \), denoted \( \overline{\text{span}} S \) is the closure of the linear subspace spanned by \( S \).

THM: "Parseval's Theorem" Let \( I \subseteq \mathbb{N} \) and \( E = \{v_i | i \in I\} \) be an orthonormal set in a Hilbert space \( H \). Then the subspace \( M = \overline{\text{span}} E \) consists of all vectors \( x = \sum_{i \in I} a_i v_i \) where the coefficient sequence \( (a_i)_{i=1}^{\infty} \) belongs to \( l^2 \).

Further, if \( x \in H \), then \( x \in M \) if and only if \( \sum_{i \in I} |\langle x, v_i \rangle|^2 = \|x\|^2 \).

COR: Let \( E = \{v_i | i \in I\} \) be an orthonormal set in a Hilbert space \( H \). Then there is a continuous linear orthogonal projection \( P_E \) of \( H \) onto \( M = \overline{\text{span}} E \) given by \( P_E x = \sum_{i \in I} \langle x, v_i \rangle v_i \).

COR: If \( E = \{v_i | i \in \mathbb{N}\} \) is an orthonormal basis for a Hilbert space \( H \), every vector \( x \in H \) may be uniquely expressed as \( x = \sum_{i=1}^{\infty} a_i v_i \) where \( a_i = \langle x, v_i \rangle \).

DEF: If \( M \) is a closed subspace of a Hilbert space \( H \), define the orthogonal complement of \( M \) to be \( M^\perp = \{x | \langle x, v \rangle = 0 \forall v \in M\} \).

PRO: Every vector in \( H \) can be written uniquely as \( x = v + y \) where \( v \in M \) and \( y \in M^\perp \). Moreover, \( \|x\|^2 = \|v\|^2 + \|y\|^2 \).

PRO: \( (M^\perp)^\perp = M \).

7.5 Finite Dimensions

LMA: If \( \{v_1, \ldots, v_n\} \) is a linearly independent set in a normed vector space \( V \), then there exist positive constants \( 0 < c < C \) so that for all \( a \in \mathbb{R}^n \) we have \( c\|a\| \leq \|\sum_{i=1}^{n} a_i v_i\| \leq C\|a\| \).

COR: "Hilbert-Borel for Finite-Dimensional Normed Vector Spaces" A subset of a finite-dimensional normed vector space is compact if and only if it is closed and bounded.

COR: A finite-dimensional subspace of a normed vector space is complete, and in particular it is closed.

THM: Let \( V \) be a normed vector space, and let \( W \) be a finite dimensional subspace of \( V \). Then for any \( v \in V \) there is at least one closest point \( w^* \in W \) so that \( \|v - w^*\| = \inf \{\|v - w\| | w \in W\} \).
7.6 Limits Of Functions

DEF: Let \((f_k)\) be a sequence of functions from \(S \subseteq \mathbb{R}^n\) into \(\mathbb{R}^m\). This sequence converges pointwise to a function \(f\) if \(\lim_{n \to \infty} f_n(x) = f(x)\) for all \(x \in S\).

DEF: Let \((f_k)\) be a sequence of functions from \(S \subseteq \mathbb{R}^n\) to \(\mathbb{R}^m\). This sequence converges uniformly to \(f\) if for every \(\varepsilon > 0\) there is \(N \in \mathbb{N}\) so that for all \(n \geq N\) and all \(x \in S\), \(\|f_n(x) - f(x)\| < \varepsilon\).

THM: For a sequence of functions \((f_n)\) in \(C_b(S, \mathbb{R}^m)\), \((f_n)\) converges uniformly to \(f\) if and only if \(\lim_{n \to \infty} \|f_n - f\|_\infty = 0\).

THM: “Dini’s Theorem” Suppose that \(f\) and \(f_n\) are continuous functions on \([a, b]\) so that \(f_n \leq f_{n+1}\) for all \(n \geq 1\) and \((f_n)\) converges to \(f\) pointwise. Then \((f_n)\) converges to \(f\) uniformly.

THM: Let \((f_k)\) be a sequence of continuous functions mapping a subset \(S \subseteq \mathbb{R}^n\) to \(\mathbb{R}^m\) that converges uniformly to a function \(f\). Then \(f\) is continuous.

THM: “Completeness Theorem for \(C(K)\)” If \(K\) is a compact set, the space \(C(K)\) of all continuous functions on \(K\) with the \(\infty\) norm is complete.

THM: “Integral Convergence Theorem” Let \((f_k)\) be a sequence of continuous functions on \([a, b]\) converging uniformly to \(f(x)\) and fix \(c \in [a, b]\). Then the functions \(F_k(x) = \int_a^c f_k(t)dt\), \(k \geq 1\) converge uniformly on \([a, b]\) to the function \(F(x) = \int_a^c f(t)dt\).

COR: Suppose that \((f_n)\) is a sequence of continuously differentiable functions on \([a, b]\) such that \((f_n')\) converges uniformly to a function \(g\) and there is a point \(c \in [a, b]\) so that \(\lim_{n \to \infty} f_n(c) = \gamma\) exists. Then \((f_n)\) converges uniformly to a differentiable function \(f\) with \(f'(x) = \gamma\) and \(f'(c) = g\).

PROP: Let \(f(x, t)\) be a continuous function on \([a, b] \times [c, d]\). Define \(F(x) = \int_c^d f(x, t)dt\). Then \(F\) is continuous on \([a, b]\).

THM: “Liebniz’s Rule” Suppose that \(f(x, t)\) and \(\frac{\partial f}{\partial x}(x, t)\) are continuous functions on \([a, b] \times [c, d]\). Then \(F(x) = \int_c^d f(x, t)dt\) is differentiable and \(F'(x) = \int_c^d \frac{\partial f}{\partial x}(x, t)dt\).

THM: Let \((f_k)\) be a sequence of functions from \(S \subseteq \mathbb{R}^n\) to \(\mathbb{R}^m\). If \(\sum_{k=1}^{\infty} f_k(x)\) converges uniformly, then it is continuous.

DEF: Let \(S \subseteq \mathbb{R}^n\). We say that a sequence of functions \(f_k\) from \(S\) to \(\mathbb{R}^m\) is uniformly Cauchy on \(S\) if for every \(\varepsilon > 0\), there is an \(N\) so that \(\sum_{k=m+1}^{\infty} f_k(x)\) is uniformly Cauchy on \(S\).

THM: “Weierstrass M-Test” Suppose that \(a_n(x)\) is a sequence of functions on \(S \subseteq \mathbb{R}^k\) to \(\mathbb{R}^m\) and \((M_n)\) is a sequence of real numbers so that \(\|a_n\|_\infty \leq M_n\). If \(\sum_{n=1}^{\infty} M_n\) converges, then \(\sum_{n=1}^{\infty} a_n(x)\) converges uniformly on \(S\).

THM: “Hadamard’s Theorem” Given a power series \(\sum_{n=0}^{\infty} a_n x^m\) there is \(R\) in \([0, +\infty)\) so that the series converges for all \(x\) with \(|x| < R\) and diverges for all \(x\) with \(|x| > R\). Moreover, the series converges uniformly on each interval \([a, b]\) contained in \((-R, R)\). Finally, if \(\alpha = \limsup_{n \to \infty} |a_n|^{1/m}\), then

\[
R = \begin{cases} \infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \\ \frac{1}{\alpha} & \text{if } \alpha \in (0, +\infty) \\
\end{cases}
\]

We call \(R\) the radius of convergence of the power series.

THM: “Term-By-Term Differentiation” If \(f(x) = \sum_{m=0}^{\infty} a_n x^n\) has radius of convergence \(R > 0\), then \(\sum_{m=1}^{\infty} na_n x^{n-1}\) has radius of convergence \(R\), \(f\) is differentiable on \((-R, R)\), and for \(x \in (-R, R)\), \(f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}\). Further,

\[
\sum_{m=0}^{\infty} \frac{a_m}{m+1} x^{m+1}
\]

has radius of convergence \(R\) and, for \(x \in (-R, R)\), \(\int_0^x f(t)dt = \sum_{m=0}^{\infty} \frac{a_m}{m+1} x^{m+1}\).

7.7 Compactness of \(S \subseteq C(K, \mathbb{R}^m)\)

DEF: A family of functions \(\mathcal{F}\) mapping \(S \subseteq \mathbb{R}^n\) into \(\mathbb{R}^m\) is equicontinuous at a point \(a \in S\) if for every \(\varepsilon > 0\) there is an \(r > 0\) such that \(||f(x) - f(a)|| < \varepsilon\) whenever \(||x - a|| < r\) and \(f \in \mathcal{F}\). The family \(\mathcal{F}\) is equicontinuous on \(S\) if it is equicontinuous at every \(a \in S\). The family \(\mathcal{F}\) is uniformly equicontinuous on \(S\) if for each \(\varepsilon > 0\), there is an \(r > 0\) so that \(||f(x) - f(y)|| < \varepsilon\) whenever \(||x - y|| < r\) and \(f \in \mathcal{F}\).

LMA: Let \(K\) be a compact subset of \(\mathbb{R}^m\). A compact subset \(\mathcal{F}\) of \(\mathcal{C}(K, \mathbb{R}^m)\) is equicontinuous.

PROP: If \(\mathcal{F}\) is an equicontinuous family of functions on a compact set, then it is uniformly equicontinuous.

DEF: A subset \(S\) of \(K\) is called an \(\varepsilon\)-net of \(K\) if \(K \subseteq \bigcup_{a \in S} B_\varepsilon(a)\). A set \(K\) is totally bounded if it has a finite \(\varepsilon\)-net for every \(\varepsilon > 0\).

LMA: Let \(K\) be a compact subset of \(\mathbb{R}^m\). Then \(K\) is totally bounded.

COR: Let \(K\) be a compact subset of \(\mathbb{R}^m\). Then \(K\) contains a sequence \(\{x_i\}_{i \geq 1}\) that is dense in \(K\). Moreover, for any \(\varepsilon > 0\), there is an integer \(N\) so that \(\{x_1, \ldots, x_N\}\) forms an \(\varepsilon\)-net for \(K\).

Derrick Stolee
**Thm:** “Arzela-Ascoli Theorem” Let $K$ be a compact subset of $\mathbb{R}^n$. A subset $F$ of $C(K, \mathbb{R}^m)$ is compact if and only if it is closed, bounded, and equicontinuous.
NOTES

Analysis Study Guide

Derrick Stolee
8 Problems

**Question:** Define a sequence \((x_n)\) by \(x_1 = 2\) and, for \(n \geq 2\), \(x_n = 1 + \frac{1}{x_{n-1}}\). Show that there is an integer \(m\) so that \(x_m \cdots x_{m+1} > 100\).

**Question:** Fix an integer \(N \geq 2\). Consider the remainders \(q(n)\) obtained by dividing the Fibonacci number \(F(n)\) by \(N\), so that \(0 \leq q(n) < N\). Prove that this sequence is periodic with period \(d \leq N^2\) as follows:

(i) Show that there are integers \(0 \leq i < j \leq N^2\) such that \(q(i) = q(j)\) and \(q(i + 1) = q(j + 1)\).

(ii) Show that if \(q(i + d) = q(i)\) and \(q(i + 1 + d) = q(i + 1)\), then \(q(n + d) = q(n)\) for all \(n \geq i\). (iii) Show that if \(q(i + d) = q(i)\) and \(q(i + 1 + d) = q(i + 1)\), then \(q(n + d) = q(n)\) for all \(n \geq 0\).

**Question:** If \(m\) and \(n\) are integers, show that \(\left|\frac{\sqrt{3} - \frac{n}{m}}{\sqrt{3}}\right| \geq \frac{1}{m^2}\).

**Question:** Compute the limit, and for \(\varepsilon = 10^{-6}\), find an integer \(N\) that satisfies the limit equation: \(\lim_{n \to \infty} \frac{n^2 + 2n + 1}{2m^2 - n^2 + 2} = 2\).

**Optional:** What is the smallest value of \(N\) that satisfies the limit definition for \(\varepsilon = 10^{-6}\)?

**Question:** (i) Prove that if \(a_n \leq b_n\) for \(n \geq 1\), \(L = \lim_{n \to \infty} a_n\) and \(M = \lim_{n \to \infty} b_n\), then \(L \leq M\).

(ii) Find convergent sequences \((a_n)\) and \((b_n)\) so that

(a) \(a_n \leq b_n\) for all \(n\), (b) there is no \(N\) so that for all \(n \geq N\), \(a_n \leq \lim b_n\), and

(c) there is no \(N\) so that for all \(n \geq N\), \(b_n \geq \lim_{n \to \infty} a_n\).

**Question:** Consider \((x_1, x_2, \ldots)\) and \((y_1, y_2, \ldots)\). Show that the new sequence \((x_1, y_2, x_2, y_3, \ldots)\) converges to a number \(L\) if and only if the two original sequences both converge to \(L\).

**Question:** Define a sequence \((a_n)_{n=1}^\infty\) so that \(\lim_{n \to \infty} a_n = a\) exists but \(\lim_{n \to \infty} a_n = b\) does not exist.

**Question:** (i) Let \(x_n = \sqrt[n]{n} - 1\). Use the fact that \((1 + x_n)^n = n\) to show that \(x_n^2 \leq 2/n\).

(ii) Hence compute \(\lim_{n \to \infty} x_n\).

**Question:** Show that the set \(S = \{ n + m\sqrt{2} | m, n \in \mathbb{Z} \}\) is dense in \(\mathbb{R}\).

**Question:** Suppose that \(\lim_{n \to \infty} a_n = L\). Show that \(\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L\).

**Question:** (i) Let \((a_n)_{n=1}^\infty\) be a bounded sequence. Define a sequence \(b_n = \sup\{a_k : k > n\}\) for \(n \geq 1\). Prove that \((b_n)\) converges. This limit is called the **limit superior** of \((a_n)\), almost always abbreviated to \(\limsup a_n\).

(ii) Without redoing the proof, do the same for the **limit inferior** of \((a_n)\), which is defined as \(\liminf a_n := \lim_{n \to \infty} (\inf_{k \geq n} a_k)\).

**Question:** Given two sequences of real numbers \((x_n)\) and \((y_n)\), prove that \(\lim sup(x_n + y_n) \leq \lim sup x_n + \lim sup y_n\).

Give an example where all three \(\lim sup's\) are finite and the inequality is strict.

**Question:** Define \(x_1 = 2\) and \(x_{n+1} = \frac{1}{2}(x_n + 5/x_n)\) for \(n \geq 1\).

(i) Find a formula for \(x_{n+1}^2 - 5\) in terms of \(x_n^2 - 5\).

(ii) Hence evaluate \(\lim_{n \to \infty} x_n\).

(iii) Compute the first ten terms on a calculator.

(iv) Show that the tenth term approximates the limit to over 600 decimal places.

**Question:** Construct a sequence \((x_n)_{n=1}^\infty\) so that for every real number \(L\), there is a subsequence \((x_{n_k})_{k=1}^\infty\) with \(\lim_{k \to \infty} x_{n_k} = L\).

**Question:** Suppose that \((a_n)\) is a sequence such that \(a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}\) for all \(n \geq 0\). Show that this sequence is Cauchy if and only if \(\lim_{n \to \infty} |a_n - a_{n+1}| = 0\).

**Question:** Suppose that, for a sequence \((a_n)\), there is \(\lambda \in (0, 1)\) so that \(|a_{n+2} - a_{n+1}| \leq \lambda |a_{n+1} - a_n|\). Show that \((a_n)\) is Cauchy.

**Question:** (i) Show that a sequence \((a_n)_{n=1}^\infty\) converges if and only if \(\lim sup a_n = \lim inf a_n\).

(ii) Suppose a sequence \((a_n)\) has the property that for any sequence \((b_n)\), we have \(\lim sup a_n + b_n = \lim sup a_n + \lim sup b_n\). Show that \((a_n)\) converges.

**Question:** Find the sum of \(\sum_{n=1}^\infty \frac{1}{2(n+1)^2}\).

**Question:** Construct a convergent series of positive terms with \(\lim sup \frac{a_{n+1}}{a_n} = \infty\).

**Question:** If \(a_n \geq 0\) for all \(n\), prove that \(\sum_{n=1}^\infty a_n\) converges if and only if \(\sum_{n=1}^\infty \frac{a_n}{1+a_n}\) converges.

**Question:** Suppose that \((a_n)\) is a strictly decreasing positive sequence, i.e., \(0 < a_{n+1} < a_n\) for all \(n\).

(i) Suppose that \((g_k)\) is a strictly increasing sequence of integers and there is a constant \(C\) so that for \(k = 2, 3, \ldots\), we have \(g_{k+1} - g_k \leq C(g_k - g_{k-1})\). Prove that \(\sum_{k=1}^\infty a_k\) converges if and only if \(\sum_{k=1}^\infty (g_{k+1} - g_k) a_{g_k}\) converges.

(ii) By a suitable choice of \((g_k)\), prove that \(\sum_{k=1}^\infty a_k\) converges if and only if \(\sum_{k=1}^\infty 2^k a_{2^n}\) converges.

(iii) Similarly, prove that \(\sum_{k=1}^\infty a_k\) converges if and only if \(\sum_{k=1}^\infty k a_{k}\) converges.

**Question:** Suppose \((a_n)\) is a decreasing positive sequence, i.e. \(0 < a_{n+1} \leq a_n\).

(i) Prove that if \(\sum_{n=1}^\infty a_n\) converges, then \(\lim_{n \to \infty} n a_n = 0\).

(ii) Give a sequence \((a_n)\) as above so that \(\lim_{n \to \infty} n a_n = 0\) but \(\sum_{n=1}^\infty a_n\) diverges.

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Derrick Stolee
**Question:** Use summation by parts to prove Abell’s Test: Suppose that \( \sum_{n=1}^{\infty} a_n \) converges and \((b_n)\) is a monotonic convergent sequence. Show that \( \sum_{n=1}^{\infty} a_n b_n \) converges.\(^{178}\)

**Question:** Suppose that \( x \) and \( y \) are unit vectors in \( \mathbb{R}^n \). Show that if \( \|x+y\| = 1 \), then \( x = y \).

**Question:** (i) Show that if \((x_n)_{n=1}^{\infty}\) is a sequence in \( \mathbb{R}^n \) such that \( \sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty \), then \((x_n)\) is Cauchy.

(ii) Give an example of a Cauchy sequence for which this condition fails.

(iii) However, show that every Cauchy sequence \((x_n)_{n=1}^{\infty}\) has a subsequence \((x_{n_k})_{k=1}^{\infty}\) such that \( \sum_{k=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| < \infty \).

**Question:** Suppose that \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) are sequences so that \( \sum_{n=1}^{\infty} x_n^2 \) and \( \sum_{n=1}^{\infty} y_n^2 \) converge.

Show that \( \sum_{n=1}^{\infty} x_n y_n \leq \left( \sum_{n=1}^{\infty} x_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} y_n^2 \right)^{1/2} \). In particular, you are showing that the series on the left-hand side converges.

**Question:** (i) Show that the sum of a closed subset and a compact subset of \( \mathbb{R}^n \) is closed. Recall that \( A + B = \{ a + b \mid a \in A, b \in B \} \).

(ii) Is this true for the sum of two compact sets and a closed set?

(iii) Is this true for the sum of two closed sets?

**Question:** Let \((x_n)_{n=1}^{\infty}\) be a sequence in a compact set \( K \) that is not convergent. Show that there are two subsequences of this sequence that are convergent to different limit points.

**Question:** Let \( A \) and \( B \) be disjoint closed subsets of \( \mathbb{R}^n \). Define \( d(A, B) = \inf \{ \|a - b\| \mid a \in A, b \in B \} \).

(i) If \( A = \{ a \} \) is a singleton, show that \( d(A, B) > 0 \).

(ii) If \( A \) is compact, show that \( d(A, B) > 0 \).

(iii) Find an example of two disjoint closed sets in \( \mathbb{R}^2 \) with \( d(A, B) = 0 \).

**Question:** Define a function on the set \( S = \{ 0 \} \cup \{ \frac{1}{n} \mid n \geq 1 \} \) by \( f\left(\frac{1}{n}\right) = a_n \) and \( f(0) = L \). Prove that \( f \) is continuous on \( S \) if and only if \( \lim_{n \to \infty} a_n = L \).

**Question:** Find a bounded continuous function on \( \mathbb{R} \) that is not Lipschitz.

**Question:** Suppose that \( A \subseteq \mathbb{R}^n \) and \( B \subseteq \mathbb{R}^m \) are open. Show that the set

\[
A \times B = \{ (x_1, \ldots, x_{n+m}) \mid (x_1, \ldots, x_n) \in A, (x_{n+1}, \ldots, x_{n+m}) \in B \}
\]

is open in \( \mathbb{R}^{n+m} \).

**Question:** Define \( f \) on \( \mathbb{R} \) by \( f(x) = x \chi_\mathbb{Q}(x) \), where \( \chi_\mathbb{Q}(x) \) is the characteristic function on \( \mathbb{Q} \). Show that \( f \) is continuous at 0 and that this is the only point where it is continuous.

**Question:** Let \( f \) and \( g \) be continuous mappings of \( S \subseteq \mathbb{R}^n \) into \( \mathbb{R}^m \). Show that the inner product \( h(x) = \langle f(x), g(x) \rangle \) is continuous.

**Question:** Suppose that \( f \) is a continuous function on \([a, b]\) and \( g \) is a continuous function on \([b, c]\) such that \( f(b) = g(b) \).

Show that

\[
h(x) = \begin{cases} f(x) & \text{if } a \leq x \leq b \\ g(x) & \text{if } b \leq x \leq c \end{cases}
\]

is continuous on \([a, c]\).

**Question:** Give an example of a continuous function \( f \) and an open set \( U \) such that \( f(U) \) is not open.

**Question:** (January 2007 Qual) Let \( f \) be a positive continuous function defined on \( \mathbb{R} \) such that \( \lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0 \). Show that \( f \) attains its maximum value, that is there is \( b \in \mathbb{R} \) so that \( f(b) = \sup f(\mathbb{R}) \).

**Question:** Let \( f \) be a continuous function on \((0, 1] \). Show that \( f \) is uniformly continuous if and only if \( \lim_{x \to 0^+} f(x) \) exists.

**Question:** Let \( f \) be a continuous function from \( B = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \} \), the closed ball in \( \mathbb{R}^2 \), into \( \mathbb{R} \). Show that \( f \) cannot be one-to-one.

**Question:** or \( x \in [0, 1] \), express it as a decimal \( x = x_0.x_1x_2x_3 \ldots \). Use a finite decimal expansion without repeating 9s when there is a choice. Then define a function \( f \) by \( f(x) = x_0.0x_10x_20x_3 \ldots \).

(i) Show that \( f \) is strictly increasing.

(ii) Compute \( \lim_{x \to 0^+} f(x) \).

(iii) Show that \( \lim_{x \to a^+} f(x) = f(a) \) for \( 0 \leq a < 1 \).

(iv) Find all discontinuities of \( f \).

**Question:** Let \( f \) be a real uniformly continuous function on the bounded set \( E \subseteq \mathbb{R} \). Prove that \( f \) is bounded on \( E \).

Show that the conclusion may be false if boundedness of \( E \) is not assumed.

---

*Derrick Stolee*
QUESTION: Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0 & \text{if } x \notin \mathbb{Q} \\
\frac{1}{x^2} & \text{if } x = \frac{p}{q} \text{ in lowest terms, and } q > 0.
\end{cases}
\]

Show that \( f'(\sqrt{3}) \) exists and is zero. (Hint: Exercise 2.2.H could be useful).

QUESTION: Suppose that \( f : [0, +\infty) \rightarrow \mathbb{R} \) is twice differentiable on \([0, +\infty)\) and satisfies \( f(0) = 0, f'(0) = 1, \) and \( f''(x) \leq 0 \) for all \( x \).

(i) Prove that \( f(x) \leq x \) for all \( x > 0 \).

(ii) Prove that \( f(x)/x \) is decreasing on \((0, +\infty)\).

QUESTION: Suppose that \( f \) is continuous on an interval \([a, b]\) and is differentiable at all points of \((a, b)\) except possibly at a single point \(x_0 \in (a, b)\). If \( \lim_{x \to x_0} f'(x) \) exists, show that \( f'(x_0) \) exists and \( f'(x_0) = \lim_{x \to x_0} f'(x) \). (Hint: Consider the intervals \([x_0 - h, x_0]\) and \([x_0, x_0 + h]\).)

QUESTION: Suppose that \( f \) is differentiable on \([a, b]\) and \( f'(a) < 0 < f'(b) \).

(i) Show that there are points \( a < c < d < b \) such that \( f(c) < f(a) \) and \( f(d) < f(b) \).

(ii) Show that the minimum on \([a, b]\) occurs at an interior point.

(iii) Hence show that there is a point \( x_0 \in (a, b) \) such that \( f'(x_0) = 0 \).

QUESTION: Suppose that \( f : [a, b] \rightarrow \mathbb{R} \) is integrable and \( g : [a, b] \rightarrow \mathbb{R} \) has \( g(x) = f(x) \) for all \( x \) in \([a, b]\) except at \( c_0, \ldots, c_n \). Prove that \( g \) is integrable and \( \int_a^b g(x)dx = \int_a^b f(x)dx \).

QUESTION: Suppose that \( f \) is Lipschitz with constant \( L \) on \([0, 1]\). Prove that

\[
\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{j=1}^n f \left( \frac{j}{n} \right) \right| \leq \frac{L}{n}.
\]

QUESTION: If \( f \) and \( g \) are both Riemann integrable on \([a, b]\), show that \( fg \) is also integrable. (Hint: Use the identity \( f(x)g(x) - f(t)g(t) = f(x)(g(x) - g(t)) + (f(x) - f(t))(g(t)) \) to show that \( M_i(f, g, P) - m_i(f, g, P) \) is bounded by \( \|f\| \cdot (M_i(g, P) - m_i(g, P)) + \|g\| \cdot (M_i(f, P) - m_i(f, P)) \)).

QUESTION: Let \( f \) be a continuous function on \( \mathbb{R} \), and fix \( \varepsilon > 0 \). Define a function \( G \) by

\[
G(x) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(t)dt.
\]

Show that \( G \) is \( C^1 \) and compute \( G' \).

Notes

\footnotesize{\(^{174}\)D&D, 3.2.K. \(^{175}\)D&D, 3.2.N. \(^{176}\)Donsig, 825 Problem Set 5, # 3. \(^{177}\)Donsig, 825 Problem Set 5, # 4. \(^{178}\)D&D, 3.4.G.

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