1 Sequences

DEF: An *ordered field* is a field F and total order $<$ (for all $x, y, z \in F$):
(i) $x < y, y < x$ or $x = y$, (ii) $x < y, y < z \Rightarrow x < z$ (iii) $x < y \Rightarrow x + z < y + z$ (iv) $0 < y, x \Rightarrow 0 < xy$.[1]

DEF: The *Archimedean property* on an ordered field F is $\forall x, y \in F, x, y > 0$, there exists $N \in \mathbb{N}$ such that $n \cdot x > y$.[2]

FACT: $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{ra+sc}{rb+sd} = \frac{a}{b}$ for all $a, b, c, d, r, s \in \mathbb{Z}$ with $rb + sd \neq 0$.[3]

DEF: A real number L is the *limit* of a sequence of real numbers $(a_n)_{n=1}^{\infty}$ if for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Then, (a_n) *converges* to L.[4]

THM: "Squeeze Theorem" Suppose three sequences $(a_n), (b_n), (c_n)$ satisfy $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.[5]

PROP: If $(a_n)_{n=1}^{\infty}$ converges, then the set $\{\, a_n | n \in \mathbb{N} \,\}$ is bounded.[6]

THM: If $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$, and $\alpha \in \mathbb{R}$, then
(i) $\lim_{n\to\infty} a_n + b_n = L + M$.
(ii) $\lim_{n\to\infty} \alpha a_n = \alpha L$.
(iii) $\lim_{n\to\infty} a_n b_n = LM$.
(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.[7]

THM: For a sequence $a_n \geq 0$, we have $\lim_{n\to\infty} a_n = +\infty$ if an only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$.[8]

THM: "Least Upper Bound Principle" Every nonempty subset S of $\mathbb{R}$ that is bounded above has a supremum. Similarly, every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.[9]

THM: "Monotone Convergence Theorem" A monotone increasing sequence that is bounded above converges. A monotone decreasing sequence that is bounded below converges.[10]

THM: Let (a_n) be a sequence.
(i) If $\lim a_n$ is defined, then $\liminf a_n = \limsup a_n = \lim a_n$.
(ii) if $\liminf a_n = \limsup a_n = L$, then $\lim a_n = L$.[11]

DEF: A *subsequence* $(a_n)_{n=1}^{\infty}$ is a new sequence $(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, \dots)$ where $n_1 < n_2 < \cdots$.[12]

THM: If the sequence (a_n) converges, then every subsequence converges to the same limit.[13]

THM: Every sequence (a_n) has a monotonic subsequence.[14]

COR: Let (a_n) be a sequence. There exists a monotonic subsequence whose limit is $\limsup a_n$ and there exists a monotonic subsequence shows limit is $\liminf a_n$.[15]

DEF: Let (a_n) be a sequence in $\mathbb{R}$. A *subsequential limit* is any real number (or symbol $+\infty, -\infty$) that is the limit of some subsequence (a_{n_k}).[16]

THM: Let (a_n) be any sequence in $\mathbb{R}$, and let S denote the set of subsequential limits of (a_n).
(i) S is nonempty.
(ii) $\sup S = \limsup a_n$ and $\inf S = \liminf a_n$.
(iii) $\lim a_n$ exists if an only if S has a single element, namely $\lim a_n$.[17]

THM: Let S denote the set os subsequential limits of a sequence (a_n). Suppose (b_n) is a sequence in $S \cap \mathbb{R}$ and that $t = \lim t_n$. Then $t \in S$.[18]

LMA: "Nested Intervals Lemma" Suppose that $I_n = [a_n, b_n] = \{\, x \in \mathbb{R} \mid a_n \leq x \leq b_n \,\}$ are nonempty closed intervals such that $I_{n+1} \subseteq I_n$ for each $n \geq 1$. Then the intersection $\cap_{n\geq 1} I_n$ is nonempty.[19]

THM: "Bolzano-Weierstrass Theorem" Every bounded sequence of real numbers has a convergent subsequence.[20]

FACT: For a bounded sequence (a_n), $\limsup a_n$ ($\liminf a_n$) is the largest (smallest) possible value for a convergent subsequence.[21]

DEF: A sequence $(a_n)_{n=1}^{\infty}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$, there is an integer N such that $|a_m - a_n| < \varepsilon$ for all $m, n \geq N$.[22]

DEF: A subset S of $\mathbb{R}$ is said to be *complete* if every Cauchy sequence converges to a point in S.[23]

THM: "Completeness Theorem" A sequence of real numbers converges if and only if it is a Cauchy sequence. In particular, $\mathbb{R}$ is complete.[24]

2 Series

DEF: Given a sequence $(a_n)_{n=1}^{\infty}$, the *infinite series* $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ *converges* if the limit exists, *diverges* otherwise.[25]

FACT: Some series can be solved using a *telescoping sum*, by cancelling elements between sequence terms.[26]

THM: "nth term test" If $\lim a_n \neq 0$, then $\sum a_n$ diverges.[27]

THM: "Cauchy Criterion for Series" The following are equivalent for a series $\sum_{n=1}^{\infty} a_n$.
(i) The series converges.
(ii) For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that $\left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$ for all $n \geq N$.
(iii) For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ so that $\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$ if $n, m \geq N$.[28]

PROP: If $a_k \geq 0$ for $k \geq 1$ and $s_n = \sum_{k=1}^{n} a_k$, then either
(i) $(s_n)_{n=1}^{\infty}$ is bounded above, in which case $\sum_{n=1}^{\infty} a_n$ converges.
(ii) $(s_n)_{n=1}^{\infty}$ is unbounded, in which case $\sum_{n=1}^{\infty} a_n$ diverges.[29]

DEF: A sequence $(a_n)_{n=0}^{\infty}$ is a *geometric series* with ratio r if $a_{n+1} = ra_n$ for all $n \geq 0$, or equivalently $a_n = a_0 r^n$.

THM: "Convergence of Geometric Series" A geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if $|r| < 1$.[30]

DEF: A sequence $(a_n)_{n=0}^{\infty}$ is a *p-series* with power p if $a_n = \frac{a}{n^p}$ for some $a, p \in R$.

THM: "Convergence of p-series" A p-series $\sum_{n=1}^{\infty} \frac{a}{n^p}$ converges if and only if $p > 1$.[31]

THM: "Comparison Test" Consider two sequences $(a_n), (b_n)$ with $|a_n| \leq b_n$ for all $n \geq 1$. If (b_n) is summable, then (a_n) is summable and $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} b_n$. If (a_n) is not summable, then (b_n) is not summable.[32]

THM: "Ratio Test" A series $\sum a_n$ of nonzero terms
(i) Converges absolutely if $\limsup |a_{n+1}/a_n| < 1$,
(ii) Diverges if $\liminf |a_{n+1}/a_n| > 1$.[33]

THM: "Root Test" Suppose that $a_n \geq 0$ for all n and let $\ell = \limsup \sqrt[n]{a_n}$. If $\ell < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and if $\ell > 1$, the series diverges.[34]

THM: "Integral Test" If $f : \mathbb{R} \to \mathbb{R}$ is positive and decreasing, then $\sum_{n=A}^{\infty} f(n)$ converges if and only if $\lim_{b\to\infty} \int_A^b f(x)dx$ exists (and is finite).[35]

THM: "Limit Comparison Test" If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive numbers b_n and $\limsup \frac{|a_n|}{b_n} < \infty$ then $\sum_{n=1}^{\infty} a_n$ converges.[36]

DEF: A sequence is *alternating* if it has the form $(-1)^n a_n$ or $(-1)^{n+1} a_n$ where $a_n \geq 0$ for all $n \geq 1$.[37]

LMA: For $x > -1, m \in \mathbb{N}$, $(1 + x)^m > 1 + mx$.[38]

THM: "Leibniz Alternating Series Test" Suppose that $(a_n)_{n=1}^{\infty}$ is a monotone decreasing sequence $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and that $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.[39]

COR: Suppose that $(a_n)_{n=1}^{\infty}$ is a monotone decreasing positive sequence and that $\lim_{n\to\infty} a_n = 0$. Then the difference between the sum of the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and the Nth partial sum is at most $|a_N|$.

DEF: A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ converges. A series that converges but is not absolutely convergent is called *conditionally convergent*.[40]

DEF: A *rearrangement* of a series $\sum_{n=1}^{\infty} a_n$ is another series with the same terms in a different order. This can be described by a permutation π of the natural numbers $\mathbb{N}$ determining the series $\sum_{n=1}^{\infty} a_{\pi(n)}$.[41]

THM: For an absolutely convergent series, every rearrangement converges to the same limit.

LMA: Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Denote the positive terms as $b_1, b_2, \ldots$ and the other terms as $c_1, c_2, \ldots$.
(i) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then so are both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |c_n|$, and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |c_n|$.
(ii) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |c_n|$ both diverge.[42]

THM: "Rearrangement Theorem" If $\sum_{n=1}^{\infty}$ is a conditionally convergent series, then for every real number L, there is a rearrangement that converges to L.[43]

LMA: "Summation by Parts Lemma" Suppose (x_n) and (y_n) are sequences of real numbers. Define $X_n = \sum_{k=1}^{n} x_k$ and $Y_n = \sum_{k=1}^{n} y_n$. Then $\sum_{n=1}^{m} x_n Y_n + \sum_{n=1}^{m} X_n y_{n+1} = X_m Y_{m+1}$.[44]

THM: "Dirichlet's Test" Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers with bounded partial sums. If $(b_n)_{n=1}^{\infty}$ is a sequence of postive numbers decreasing monotonically to 0, then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.[45]

THM: "Abel's Test" Suppose that $\sum_{n=1}^{\infty} a_n$ converges and (b_n) is a monotonic convergent sequence. Then, $\sum_{n=1}^{\infty} a_n b_n$ converges.[46]

3 Topology of $\mathbb{R}^n$

DEF: The *dot product* or *inner product* of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=1}^{\infty} x_i y_i$.[47]

THM: "Schwarz Inequality" For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$. Equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are colinear.[48]

THM: "Triangle Inequality" The triangle inequality holds for the Euclidean norm on $\mathbb{R}^n$: $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Moreover, equality holds if and only if either $\mathbf{x} = 0$ or $\mathbf{y} = c\mathbf{x}$ with $c \geq 0$.[49]

DEF: A set $V = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq \mathbb{R}^n$ is *orthonormal* if $\langle v_i, v_j, = \rangle 0$ when $i \neq j$ and $\langle v_i, v_i, = \rangle 1$. If $m = 0$, then V spans $\mathbb{R}^n$ and is called an *orthonormal basis*.[50]

LMA: Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in $\mathbb{R}^n$. Then $\|\sum_{i=1}^m a_i x_i\| = \left(\sum_{i=1}^m |a_i|^2\right)^{1/2}$.[51]

DEF: A sequence of points $(\mathbf{x}_k)$ in $\mathbb{R}^n$ *converges* to a point $\mathbf{a}$ if for every $\varepsilon > 0$, there is an integer N so that $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon$ for all $k > N$. In this case, write $\lim_{n\to\infty} \mathbf{x}_k = \mathbf{a}$.[52]

LMA: Let $(\mathbf{x}_k)$ be a sequence in $\mathbb{R}^n$. Then $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$ if and only if $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{a}\| = 0$.[53]

LMA: A sequence $\mathbf{x}_k = (x_{k,1}, \ldots, x_{k,n})$ in $\mathbb{R}^n$ converges to a point $\mathbf{a} = (a_1, \ldots, a_n)$ if and only if each coordinate converges: $\lim_{k\to\infty} x_{k,i} = a_i$ for $1 \leq i \leq n$.[54]

DEF: A sequence $\mathbf{x}_k$ in $\mathbb{R}^n$ is *Cauchy* if for every $\varepsilon > 0$, there is an integer N so that $\|\mathbf{x}_k - \mathbf{x}_\ell\| < \varepsilon$ for all $k, \ell > N$. A set $S \subseteq \mathbb{R}^n$ is *complete* if every Cauchy sequence of points in S converges to a point in S.[55]

THM: "Completeness Theorem for $\mathbb{R}^n$" Every Cauchy sequence in $\mathbb{R}^n$ converges. Thus, $\mathbb{R}^n$ is complete.[56]

DEF: A point $\mathbf{x}$ is a *limit point* of a subset $A \subseteq \mathbb{R}^n$ if there is a sequence $(a_k)_{k=1}^{\infty}$ with $a_k \in A$ such that $\mathbf{x} = \lim_{k\to\infty} a_k$. A set $A \subseteq \mathbb{R}^n$ is *closed* if it contains all of its limit points.[57]

DEF: A point $\mathbf{x}$ is a *cluster point* of a subset $A \subseteq \mathbb{R}^n$ if there is a sequence $(\mathbf{a}_n)_{n=1}^{\infty}$ with $\mathbf{a}_n \in A \setminus \{\mathbf{x}\}$ such that $\mathbf{x} = \lim_{n\to\infty} \mathbf{a}_n$.[58]

PROP: If $A, B \subseteq \mathbb{R}^n$ are closed, then $A \cup B$ is closed. If $\{ A_i \mid i \in I \}$ is a family of closed subsets of $\mathbb{R}^n$, then $\cap_{i\in I} A_i$ is closed.[59]

EX: Let $A_n = \overline{B_{\frac{n-1}{n}}(0)}$. The family of sets $\{A_n\}_{n\in\mathbb{N}}$ has every set closed, but $\cup_{n\in\mathbb{N}} A_n = B_1(0)$, which is not closed.[60]

DEF: If A is a subset of $\mathbb{R}^n$, the *closure* of A is the set $\overline{A}$ consisting of all limit points of A.[61]

DEF: The *ball* about $\mathbf{a}$ in $\mathbb{R}^n$ of radius r is the set $B_r(\mathbf{a}) = \{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < r \}$. A subset $U \subseteq \mathbb{R}^n$ is *open* if for every $\mathbf{a} \in U$, there is some $r > 0$ so that the ball $B_r(\mathbf{a})$ is contained in U.[62]

PROP: Let $A \subseteq \mathbb{R}^n$. Then $\overline{A}$ is the smallest closed set containing A. In particular, $\overline{\overline{A}} = \overline{A}$.[63]

THM: "Duality of Open and Closed Sets" A set $A \subseteq \mathbb{R}^n$ is open if and only if the complement of A, $A' = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \notin A \}$ is closed.[64]

–HERE MARKS THE END OF EXAM 1 MATERIAL–

PROP: If U and V are open subsets of $\mathbb{R}^n$ then $U \cap V$ is an open subset of $\mathbb{R}^n$. If $\{ U_i \mid i \in I \}$ s a family of open subsets of $\mathbb{R}^n$, then $\cup_{i\in I} U_i$ is open.[65]

EX: Let $A_n = B_{\frac{n+1}{n}}(0)$. The family of sets $\{A_n\}_{n\in\mathbb{N}}$ has every set open, but $\cap_{n\in\mathbb{N}} A_n = \overline{B_1(0)}$, which is closed.[66]

DEF: The *interior*, $\operatorname{int} X$, of a set X is the largest open set contained in X. If $\operatorname{int} X = \emptyset$, then X has *empty interior*.[67]

DEF: A subset $A \subseteq \mathbb{R}^n$ is *compact* if every sequence $(\mathbf{a}_k)_{k=1}^{\infty}$ of points in A has a convergent subsequence $(\mathbf{a}_{k_i})_{i=1}^{\infty}$ with limit $\mathbf{a} = \lim_{i\to\infty} \mathbf{a}_{k_i}$ in A.[68]

DEF: A subset S of $\mathbb{R}^n$ is called *bounded* provided that there is a real number R such that S is contained in the ball $B_R(0)$.[69]

LMA: A compact subset of $\mathbb{R}^n$ is closed and bounded.[70]

LMA: If C is a closed subset of a compact subset of $\mathbb{R}^n$, then C is compact.[71]

LMA: The cube $[a, b]^n$ is a compact subset of $\mathbb{R}^n$.[72]

THM: "Heine-Borel Theorem" A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.[73]

THM: "Cantor's Intersection Theorem" If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a decreasing sequence of nonempty compact subsets of $\mathbb{R}^n$, then $\cap_{k\geq 1} A_k$ is not empty.[74]

DEF: A set whose closure has no interior is *nowhere dense*. A point $\mathbf{x}$ of a set A is *isolated* if there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(\mathbf{x})$ intersects A only in the singleton $\{\mathbf{x}\}$. A set A is *perfect* if each point $\mathbf{x} \in A$ is the limit of some sequence in $A \setminus \{\mathbf{x}\}$.[75]

4 Functions

DEF: Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$. If $\mathbf{a} \in S$ is a cluster point (limit point of $S \setminus \{\mathbf{a}\}$) then a point $\mathbf{v} \in \mathbb{R}^n$ is a *limit* of f at $\mathbf{a}$ if for every $\varepsilon > 0$ there is an $r > 0$ so that $\|f(\mathbf{x}) - \mathbf{v}\| < \varepsilon$ whenever $0 < \|\mathbf{x} - \mathbf{a}\| < r$ and $\mathbf{x} \in S$. Write $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{v}$.[76]

DEF: Let $S \subseteq \mathbb{R}^n$ and let $f : S \to \mathbb{R}^m$. f is *continuous at* $\mathbf{a} \in S$ if for every $\varepsilon > 0$, there is an $r > 0$ such that, for all $\mathbf{x} \in S$ with $\|\mathbf{x} - \mathbf{a}\| < r$, we have $\|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon$. Moreover, f is *continuous on* S if it is continuous at each point $\mathbf{a} \in S$. If f is not continuous at $\mathbf{a}$, f is *discontinuous at* $\mathbf{a}$.[77]

DEF: A function $f : S \to \mathbb{R}^m$ is called *Lipschitz* if there is a constant C such that $\|f(\mathbf{x}) - f(\mathbf{a})\| \leq C\|\mathbf{x} - \mathbf{a}\|$ for all $\mathbf{x}, \mathbf{a} \in S$. The *Lipschitz constant* of f is the smallest choice of C for this condition to hold.[78]

DEF: A function $f : [a, b] \to \mathbb{R}$ satisfies a *Lipschitz condition of order* $\alpha > 0$ if there is some positive constant M so that $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha$.[79]

PROP: Every Lipschitz function is continuous. [80]

COR: Every linear transformation A from $\mathbb{R}^n$ to $\mathbb{R}^m$ is Lipschitz, and therefore continuous.[81]

DEF: The *coordinate functions* are, $\pi_j : \mathbb{R}^n \to \mathbb{R}$ is given as $\pi_j(x_1, \ldots, x_n) = x_j$ for $1 \leq j \leq n$, and $\epsilon_i(t) = te_i$.[82]

DEF: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ has a *removable singularity* at $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists, but does not equal $f(\mathbf{a})$.[83]

DEF: The *limit of f as x approaches a from the right* is L if for every $\varepsilon > 0$, there is an $\delta > 0$ so that $|f(x) - L| < \varepsilon$ for all $a < x < a + \delta$, written $\lim_{x\to a^+} f(x) = L$. Similarly, define the limit from the left, $\lim_{x\to a^-} f(x) = L$.[84]

DEF: When a function f on $\mathbb{R}$ has limits from the left and right that are different, we say f has a *jump discontinuity*. A function on an interval is called *piecewise continuous* if on every finite subinterval, it has only a finite number of jump discontinuities and is continuous at all other ponts.[85]

EX: The *Heaviside function* $H(x)$ is defined to be 0 for all $x < 0$ and 1 for $x \geq 0$.[86]

EX: The *ceiling function* $\lceil x \rceil$ has countably many jump discontinuities.[87]

EX: For any subset $A \subseteq \mathbb{R}^n$, the *characteristic function* of A is $\chi_A(x)$, which equals 1 if $x \in A$, and 0 otherwise.[88]

EX: Let $f(x) = 0$ when $x \notin \mathbb{Q}$ and then $f(x) = \frac{1}{q}$ when $x = \frac{p}{q}$, in reduced form. Then, $f(x)$ is continuous only at irrational points, and discontinuous on the rationals.[89]

DEF: Say that the limit of a function $f(x)$ has x approaches a is $+\infty$ if for every positive integer N, there is an $r > 0$ so that $f(x) > N$ for all $0 < |x - a| < r$. We write $\lim_{x\to a} f(x) = +\infty$. We define the limit $\lim_{x\to a} f(x) = -\infty$ similarly.[90]

DEF: A function f is *asymptotic to the curve* g if $\lim_{x\to\infty} |f(x) - g(x)| = 0$.[91]

DEF: A subset $V \subseteq S \subseteq \mathbb{R}^n$ is *open* in S (or *relatively open*) if there is an open set U in $\mathbb{R}^n$ such that $U \cap S = V$.[92]

THM: For a function f mapping $S \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, the following are equivalent:
(i) f is continuous on S.
(ii) "Sequential characterization of continuity" For every convergent sequence $(\mathbf{x}_n)_{n=1}^\infty$ with $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{a} \in S$. $\lim_{n\to\infty} f(\mathbf{x}_n) = f(\mathbf{a})$.
(iii) "Topological characterization of continuity" For every open set U in $\mathbb{R}^m$, the set $f^{-1}(U) = \{\mathbf{x} \in S \mid f(\mathbf{x}) \in U\}$ is open in S.[93]

THM: If f, g are functions from a common domain S into $\mathbb{R}^m$, $\mathbf{a} \in S$ such that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{a}) = \mathbf{u}$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = \mathbf{v}$, then
(i) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) + g(\mathbf{x}) = \mathbf{u} + \mathbf{v}$.
(ii) $\lim_{\mathbf{x}\to\mathbf{a}} \alpha f(\mathbf{x}) = \alpha\mathbf{u}$.
When the range is contained in $\mathbb{R}$, say $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = u$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = v$, then
(iii) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})g(\mathbf{x}) = uv$, and
(iv) $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})/g(\mathbf{x}) = u/v$ provided $v \neq 0$.[94]

THM: If f, g are functions form S to $\mathbb{R}^m$ that are continuous at $\mathbf{a} \in S$ and $\alpha \in \mathbb{R}$, then
(i) $f + g$ is continuous at $\mathbf{a}$.
(ii) αf is continuous at $\mathbf{a}$.
When the range is contained in $\mathbb{R}$,
(iii) fg is continuous at $\mathbf{a}$
(iv) f/g is continuous at $\mathbf{a}$ provided that $g(\mathbf{a}) \neq 0$.[95]

DEF: A function f is a *rational function* if $f(x) = p(x)/q(x)$ where $p, q \in \mathbb{R}[x]$ and $q \neq 0$. Rational functions are continuous except where $q(x) = 0$.[96]

DEF: If a function $f : S \to T$ and $g : T \to \mathbb{R}^m$, then the *composition* of g and f, denoted $g \circ f$ is the function that sends x to $g(f(x))$.[97]

THM: Suppose $f : S \to T$ and $g : T \to \mathbb{R}^m$. If f is continuous at $\mathbf{a} \in S$ and g is continuous at $f(\mathbf{a}) \in T$, then $g \circ f$ is continuous at $\mathbf{a}$.[98]

THM: Let C be a compact subset of $\mathbb{R}^n$, and let f be a continuous function from C into $\mathbb{R}^m$. Then the image set $f(C)$ is compact.[99]

THM: "Extreme Value Theorem" Let C be a compact subset of $\mathbb{R}^n$ and let f be a continuous function from C into $\mathbb{R}$. Then there are points $\mathbf{a}$ and $\mathbf{b}$ in C attaining the minimum and maximum values of f on C. That is, $f(\mathbf{a}) \leq f(\mathbf{x}) \leq f(\mathbf{b})$ for all $\mathbf{x} \in C$.[100]

DEF: A function $f : \mathbb{R} \to \mathbb{R}^m$ is a *periodic function* if there exists a $d > 0$ such that $f(x) = f(x + d)$ for all $x \in \mathbb{R}$. The least such d is called the *period* of f. Then, f is *d-periodic*.[101]

DEF: A function $f : S \to \mathbb{R}^m$ is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ so that $\|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{a}\| < \delta$ for $\mathbf{x}, \mathbf{a} \in S$.[102]

PROP: Every Lipschitz function is uniformly continuous.[103]

COR: Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous.[104]

COR: Let f be a differentiable real-valued function on $[a, b]$ with a bounded derivative; that is, there is $M > 0$ so that $|f'(x)| \leq M$ for all $a \leq x \leq b$. Then f is uniformly continuous on $[a, b]$.[105]

THM: Suppose that $C \subseteq \mathbb{R}^n$ is conpact and $f : C \to \mathbb{R}^n$ is continuous. Then f is uniformly continuous on C.[106]

THM: "Intermediate Value Theorem" If f is a continuous real-valued function on $[a, b]$ with $f(a) < 0 < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.[107]

COR: If f is a continuous real-valued function on $[a, b]$, then $f([a, b])$ is a closed interval.[108]

DEF: A *path* in $S \subseteq \mathbb{R}^n$ from $\mathbf{a}$ to $\mathbf{b}$, both poionts in S, is the image of a continuous function γ from $[0, 1]$ into S such that $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$.[109]

COR: Suppose that $S \subseteq \mathbb{R}^n$ and f is a continuous real-valued function on S. If there is a path from $\mathbf{a}$ to $\mathbf{b}$ in S and $f(\mathbf{a}) < 0 < f(\mathbf{b})$, then there is a point $\mathbf{c}$ on the path so that $f(\mathbf{c}) = 0$.[110]

DEF: A function f is *increasing* on an interval (a, b) if $f(x) \leq f(y)$ whenever $a < x \leq y < b$. It is *strictly increasing* on (a, b) if $f(x) < f(y)$ whenever $a < x < y < b$. Similarly, define *decreasing* and *strictly decreasing* functions. All of these functions are called *monotone*.[111]

PROP: If f is an increasing function on the inerval (a, b), then the one-sided limits of f exist at each point $c \in (a, b)$ and $\lim_{x \to c^-} f(x) = L \leq f(x) \leq \lim_{x \to c^+} f(x) = M$. For decreasing functions, the inequalities are reversed.[112]

COR: The only type of discontinuity that a monotone function on an interval can have is a jump discontinuity.[113]

COR: If f is a monotone function on $[a, b]$ and the range of f intersects every nonempty open interval in $[f(a), f(b)]$ then f is continuous.[114]

THM: A monotone function on $[a, b]$ has at most countably many discontinuities.[115]

THM: Let f be a continuous strictly increasing function on $[a, b]$. Then f maps $[a, b]$ one-to-one and onto $[f(a), f(b)]$. Moreover the inverse function f^{-1} is also continuous and strictly increasing.[116]

PROP: Let $f : S \to \mathbb{R}^m$, $S \subseteq \mathbb{R}^n$ be a continuous function. If $T \subset S$ is compact, $f(T)$ is compact.[117]

EX: The *Cantor function* $c : [0, 1] \to [0, 1]$ is an onto function defined such that c is constant over the intervals not in the Cantor set, but is strictly increasing over the Cantor set. Also, $c(\mathcal{C}) = [0, 1]$.[118]

5 Intro to Calculus

DEF: A function $f:(a,b)\to\mathbb{R}$ is *differentiable at a point* $x_0\in(a,b)$ if $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists. We write $f'(x_0)$ for this limit.[119]

DEF: When f is differentiable at x_0, we define the *tangent line* to f at x_0 to be the linear function $T(x)=f(x_0)+f'(x_0)(x-x_0)$.[120]

PROP: If f is differentiable at x_0, then it is continuous at x_0. Differentiable functions are continuous.[121]

LMA: Let f be a function on $[a,b]$ that is differentiable at x_0. Let $T(x)$ be the tangent line to f at x_0. Then T is the unique linear function with the property $\lim_{x\to x_0}\frac{f(x)-T(x)}{x-x_0}=0$.[122]

COR: If $f(x)$ is a function on (a,b) and $x_0\in(a,b)$, then the following are equivalent:
(i) f is differentiable at x_0.
(ii) There is a linear function $T(x)$ and a function $\varepsilon(x)$ on (a,b) such that $\lim_{x\to x_0}\varepsilon(x)=0$ and $f(x)=T(x)+\varepsilon(x)(x-x_0)$.
(iii) There is a function $\varphi(x)$ on (a,b) such that $f(x)=f(x_0)+\varphi(x)(x-x_0)$ and $\lim_{x\to x_0}\varphi(x)$ exists.[123]

THM: "Arithmetic of Derivatives" Let f and g be differentiable functions at the point a. Each of the functions cf (c a constant), $f+g$, fg, and f/g are differentiable at a, except f/g if $g(a)=0$. The formulas are[124]
(i) $(cf)'(a)=c\cdot f'(a)$ (ii) $(f+g)'(a)=f'(a)+g'(a)$
(iii) "Product rule" $(fg)'(a)=f(a)g'(a)+f'(a)g(a)$
(iv) "Quotient rule" $(f/g)'(a)=\frac{g(a)f'(a)-f(a)g'(a)}{g^2(a)}$ if $g(a)\neq 0$.

THM: "The Chain Rule" Suppose that f is defined on $[a,b]$ and has range contained in $[c,d]$. Let g be defined on $[c,d]$. Suppose that f is differentiable at $x_0\in[a,b]$ and g is differentiable at $f(x_0)$. Then the composition $h(x)=g(f(x))$ is defined, and $h'(x_0)=g'(f(x_0))f'(x_0)$.[125]

EX: The class of functions $f(x)=x^a\sin(1/x)$ for $x>0$ and $f(0)=0$ for $a>0$ is differentiable on $[0,\infty)$, but the derivative function is not continuous at 0.[126]

DEF: A function $f(x)$ is *even* if $f(-x)=f(x)$ and *odd* when $f(-x)=-f(x)$.[127]

–HERE MARKS THE END OF EXAM 2 MATERIAL–

THM: "Fermat's Thoerem" Let f be a continuous function on an interval $[a,b]$ that takes its maximum or minimum value at a point x_0. Then, exactly one of the following holds:
(i) x_0 is an endpoint a or b, (ii) f is not differentiable at x_0, (iii) f is differentiable at x_0 and $'f(x_0)=0$.[128]

THM: "Rolle's Theorem" Suppose that f is a function that is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=f(b)=0$. THen there is a point $c\in(a,b)$ such that $f'(c)=0$.[129]

THM: "Mean Value Theorem" Suppose that f is a function that is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $c\in(a,b)$ such that $f'(C)=\frac{f(b)-f(a)}{b-a}$.[130]

COR: Let f be a differentiable function on $[a,b]$.[131]
(i) If $f'(x)$ is (strictly) positive, then f is (strictly) increasing.
(ii) If $f'(x)$ is (strictly) negative, then f is (strictly) decreasing.[132]
(iii) If $f'(x)=0$ at every $x\in(a,b)$, then $f(x)$ is constant.[133] (iv) If g is differentiable on $[a,b]$ with $g'(x)=f'(x)$, then there is a constant c such that $f(x)=g(x)+c$.[134]

DEF: If a twice differentiable function f has $f''(x)$ positive on an interval $[a,b]$ then f is *convex* or *concave up*. If $f''(x)$ is negative, then f is *concave* or *concave down*. The points where $f''(x)$ changes sign are called *inflection points*.[135]

THM: "Darboux's Theorem/Intermediate Value Theorem for Derivatives" If f is differentiable on $[a,b]$ and $f'(a)<L<f'(b)$, then there is a point x_0 in (a,b) at which $f'(x_0)=L$.[136]

THM: Let f is a one-to-one continuous function on an open interval I, and let $J=f(I)$. If f is differentiable at $x_0\in I$ and if $f'(x_0)\neq 0$, then f^{-1} is differentiable at $y_0=f(x_0)$ and $(f^{-1})'(y_0)=\frac{1}{f'(x_0)}$.[137]

6 Integration

DEF: Let $f : [a, b] \to \mathbb{R}$ be a bounded function.[138]
(i) A *partition* of $[a, b]$ is a finite set $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$. Set $\Delta_j = x_j - x_{j-1}$ and define the *mesh* of a partition P as $\text{mesh}(P) = \max_{1 \le j \le n} \Delta_j$.[139]
(ii) Let the *maximum* and *minimum* be $M_j(f, P) = \sup\{ f(x) \mid x_{j-1} \le x \le x_j \}$ and $m_j(f, P) = \inf\{ f(x) \mid x_{j-1} \le x \le x_j \}$.
(iii) Let the *upper (Darboux) sum* and *lower (Darboux) sum* be $U(f, P) = \sum_{j=1}^{n} M_j(f, P)\Delta_j$ and $L(f, P) = \sum_{j=1}^{n} m_j(f, P)\Delta_j$.
(iv) If given an *evaluation sequence* $X = \{x'_j \mid 1 \le j \le n\}$ with $x'_j \in [x_{j-1}, x_j]$ the *Riemann sum* is $I(f, P, X) = \sum_{j=1}^{n} f(x'_j)\Delta_j$.[140]
(v) A partition R is a *refinement* of a parition P provided $P \subseteq R$. If P and Q are partitions, then R is a *common refinement* if $P \cup Q \subseteq R$.

LMA: If R is a refinement of P, then $L(f, P) \le L(f, R) \le U(f, R) \le U(f, P)$.[141]

COR: If P and Q are any two partitions of $[a, b]$, $L(f, P) \le U(f, Q)$.[142]

DEF: Define the *lower Darboux integral*, $L(f) = \sup_P L(f, P)$ and the *upper darboux integral*, $U(f) = \inf_P U(f, P)$. Note that $L(f) \le U(f)$.[143] A bounded function f on a finite interval $[a, b]$ is called *Riemann integrable* if $L(f) = U(f)$. In this case, we write $L(f) = \int_a^b f(x)dx = U(f)$.[144]

THM: "Riemann's Condition" Let $f(x) : [a, b] \to \mathbb{R}$ be a bounded function. The following are equivalent:
(i) f is Riemann integrable.
(ii) For each $\varepsilon > 0$, there is a partition P so that $U(f, P) - L(f, P) < \varepsilon$.[145]

COR: Let f be a bounded real-valued function on $[a, b]$. If there is a sequence of partitions of $[a, b]$, P_n so that $\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$, then f is Riemann integrable. Moreover, if X_n is any choice of points $x'_{n,j}$ selected from each interval of P_n, then $\lim_{n\to\infty} I(f, P_n, X_n) = \int_a^b f(x)dx$.

THM: Let $f(x) : [a, b] \to \mathbb{R}$ be a bounded function. The following are equivalent:[146]
(i) f is Riemann integrable.
(ii) For each $\varepsilon > 0$, there is a partition P so that $U(f, P) - L(f, P) < \varepsilon$.
(iii) "Cauchy Criterion for Integrability" For every $\varepsilon > 0$, there is a $\delta > 0$ so that every partition Q such that $\text{mesh}(Q) < \delta$ satisfies $U(f, Q) - L(f, Q) < \varepsilon$.[147]

THM: Let $f(x) : [a, b] \to \mathbb{R}$ be a bounded function, then f is Riemann integrable with $\int_a^b f(x)dx = L$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ so that every partition Q such that $\text{mesh}(Q) < \delta$ and every choice of evaluation sequence X satisfies $|I(f, Q, X) - L| < \varepsilon$.[148]

THM: Every monotone function on $[a, b]$ is Riemann integrable.[149]

THM: Every continuous function on $[a, b]$ is integrable.[150]

DEF: Say that f is Riemann integrable on $[a, \infty)$ if the *improper integral* $\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$ exists.[151]

THM: "Arithmetic of Integrals" Let f and g be integrable function on $[a, b]$ and $c \in \mathbb{R}$. Then
(i) cf is integrable and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$. (ii) $f + g$ is integrable and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.[152]

THM: If f and g are integrable on $[a, b]$ and if $f(x) \le g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.[153]

THM: If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(x)dx| \le \int_a^b |f(x)|dx$.[154]

THM: If $f : [a, b] \to \mathbb{R}$ with $c \in (a, b)$ and f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$ with $\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)dx$.[155]

THM: "Fundamental Theorem of Calculus I" Let f be a bounded Riemann integrable function on $[a, b]$ and let $F : [a, b] \to \mathbb{R}$ be defined as $F(x) = \int_a^x f(t)dt$. Then F is continuous, and if f is continuous at a point x_0, then F is differentiable at x_0 with $F'(x_0) = f(x)$.[156]

DEF: A function f on $[a, b]$ has an *antiderivative* if there is a continuous function $F : [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$.[157]

COR: "Fundamental Theorem of Calculus II" Let f be a continuous function on $[a, b]$. Then f has an antiderivative. Moreover, if G is any antiderivative of f, then $\int_a^b f(x)dx = G(b) - G(a)$.[158]

LMA: Suppose that $f : [a, b] \to \mathbb{R}$ is an integrable function bounded by $M : [a, b] \to \mathbb{R}$. Then $|\int_a^b f(t)dt| \le M(b - a)$.[159]

THM: "Integration by Parts" GIven two differentiable functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, the integral $\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx$.[160]

THM: "Substitution/Change of Variable" Let u be a C^1 function on $[a,b]$ and let f be continuous on $[c,d]$ containing the range of u. Then $\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(t)dt$.[161]

THM: "Intermediate/Mean Value Theorem for Integrals" If f is a continuous function on $[a,b]$, then there is a point $c \in (a,b)$ such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$.[162]

6.1 Riemann-Stieltjes Integration

DEF: Consider $f, g : [a,b] \to \mathbb{R}$. Given a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and an evaluation sequence X for P, define the *Riemann-Stieltjes sum* or *R-S sum* for f with respect to g using P and X as $I_g(f,P,X) = \sum_{j=1}^n f(x_j')[g(x_j) - g(x_{j-1})]$.[163]

DEF: f is *Riemann-Stieltjes integrable* with respect to g ($f \in \mathcal{R}(g)$) if there is a number L so that for all $\varepsilon > 0$, there is a partition P_ε so that for all partitions $P \supset P_\varepsilon$ and evaluation sequences X on P, we have $|I_g(f,P,X) - L| < \varepsilon$. In this case, we say L is the *Riemann-Stieltjes integral* of f with respect to g, written $L = \int_a^b f\,dg$.[164]

THM: (i)If $f_1, f_2 \in \mathcal{R}(g)$ on $[a,b]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_2 + c_2 f_2 \in \mathcal{R}(g)$ on $[a,b]$ and $\int_a^b c_1 f_1 + c_2 f_2\,dg = c_1\int_a^b f_1\,dg + c_2\int_a^b f_2\,dg$.
(ii) If $f \in \mathcal{R}(g_1) \cap \mathcal{R}(g_2)$ on $[a,b]$, then $f \in \mathcal{R}(g_1 + g_2)$ on $[a,b]$ and $\int_a^b f\,d(g_1+g_2) = \int_a^b f\,dg_1 + \int_a^b f\,dg_2$.
(iii) If $a < b < c$ and $f \in \mathcal{R}(g)$ on both $[a,b]$ and $[b,c]$, then $f \in \mathcal{R}(g)$ on $[a,c]$ and $\int_a^c f\,dg = \int_a^b f\,dg + \int_b^c f\,dg$.[165]

THM: Suppose $f, g : [a,b] \to \mathbb{R}$ are bounded and $\varphi : [c,d]$
$to[a,b]$ is a strictly increasing, continuous function onto $[a,b]$. Let $F = f \circ \varphi$ and $G = g \circ \varphi$. If $f \in \mathcal{R}(g)$ on $[a,b]$, then $F \in \mathcal{R}(G)$ on $[c,d]$ and $\int_c^d F\,dG = \int_a^b f\,dg$.[166]

THM: "Integration by Parts" Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions. If $f \in \mathcal{R}(g)$ on $[a,b]$, then $g \in \mathcal{R}(f)$ on $[a,b]$ and $\int_a^b f\,dg + \int_a^b g\,df = g(b)f(b) - g(a)f(a)$.[167]

THM: "Substitution Rule" If $f : [a,b] \to \mathbb{R}$ has $f \in \mathcal{R}(g)$ where $g : [a,b] \to \mathbb{R}$ is C^1, then fg' is Riemann integrable on $[a,b]$ and $\int_a^b f\,dg = \int_a^b f(x)g'(x)dx$.[168]

DEF: Let f, g be bounded functions on $[a,b]$, and P a partition on $[a,b]$. Define
(i) *upper sum* with respect to g: $U_g(f,P) = \sum_{i=1}^n M_i(f_i,P)[g(x_i) - g(x_{i-1})]$.
(ii) *lower sum* with respect to g: $L_g(f,P) = \sum_{i=1}^n m_i(f_i,P)[g(x_i) - g(x_{i-1})]$.[169]

DEF: Define $L_g(f) = \sup_P L_g(f,P)$ and $U_g(f) = \inf_P U_g(f,P)$.[170]

LMA: "Refinement Lemma" Let f, g be bounded functions on $[a,b]$ and P, R partitions of $[a,b]$. Assume that g is increasing. If R is a refinement of P, then $L_g(f,P) \le L_g(f,R) \le U_g(f,R) \le U_g(f,P)$.[171]

COR: Let f, g be bounded functions on $[a,b]$ and assume that g is increasing. If P and Q are any two partitions of $[a,b]$, then $L_g(f,P) \le U_g(f,Q)$.[172]

THM: "Riemann-Stieltjes Condition" Let f, g be bounded functions on $[a,b]$ and assume that g is increasing on $[a,b]$. The following are equivalent:
(i) $f \in \mathcal{R}(g)$. (ii) $U_g(f) = L_g(f)$
(iii) For every $\varepsilon > 0$ there is a partition P so that $U_g(f,P) - L_g(f,P) < \varepsilon$.[173]

DEF: Given a function $f : [a,b] \to \mathbb{R}$ and a partition P of $[a,b]$, the *variation* of f over P is $V(f,P) = \sum_{i=1}^n |f(x_i) - f(x_{i-1})|$.

DEF: The *total variation* of f on $[a,b]$ is $V_a^b f = \sup_P V(f,P)$. f is of *bounded variation* on $[a,b]$ if $V_a^b f$ is finite.

LMA: If $a < b < c$ and $f : [a,c] \to \mathbb{R}$ is given, then $V_a^c f = V_a^b f + V_b^c f$.

THM: If $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then there are increasing functions $g, h : [a,b] \to \mathbb{R}$ so that $f = g - h$.

THM: If $f : [a,b] \to \mathbb{R}$ is bounded by M, $g : [a,b] \to \mathbb{R}$ is of bounded variation and $f \in \mathcal{R}(g)$, then $\left|\int_a^b f\,dg\right| \le M \cdot V_a^b g$.

THM: If $f : [a,b] \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is increasing, then $f \in \mathcal{R}(g)$ on $[a,b]$.

COR: If $f : [a,b] \to \mathbb{R}$ is continuous and $g : [a,b] \to \mathbb{R}$ is of bounded variation, then $f \in \mathcal{R}(g)$ on $[a,b]$.

7 Vectors and Distance

7.1 Normed Vector Spaces

DEF: Let V be a vector space over $\mathbb{R}$. A *norm* on V is a function $\|\cdot\|$ on V taking values in $[0, +\infty)$ with
(i) $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = 0$. (ii) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$. (iii) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
The pair $(V, \|\cdot\|)$ is called a *normed vector space*.

DEF: In a normed vector space V, a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ *converges* if there is $\mathbf{x} \in V$ so that $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}\| = 0$.

DEF: A sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ is *Cauchy* if $(\|\mathbf{x}_n\|)_{n=1}^{\infty}$ is Cauchy. That is, $\lim_{n\to\infty} \sup_{m\geq n} \|\mathbf{x}_n - \mathbf{x}_m\| = 0$.

DEF: V is *complete* if every Cauchy sequence in V converges to some vector $\mathbf{x} \in V$. A complete normed vector space is called a *Banach space*.

DEF: For a normed vector space V, define the *open ball* $B_r(\mathbf{x}) = \{\mathbf{v} \in V \mid \|\mathbf{v} - \mathbf{x}\| < r\}$. A subset $U \subseteq V$ is *open* if for every $\mathbf{a} \in U$ there is $r > 0$ so that $B_r(\mathbf{a}) \subseteq U$. A subset $C \subseteq V$ is *closed* if it contains all of its limit points.

PROP: A sequence $\mathbf{x}_n$ in a normed vector space V converges to a vector $\mathbf{x}$ if and only if for each open set U containing $\mathbf{x}$, there is an ingeger N so that $\mathbf{x}_n \in U$ for all $n \geq N$.

DEF: A subset K of a normed vector space V is *compact* if every sequence $(\mathbf{x}_n)$ of points in K has a convergent subsequence.

7.2 Inner Product Space

DEF: An *inner product* on a vector space V is a function $\langle \mathbf{x}, \mathbf{y} \rangle$ so that
(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = 0$.
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
(iii) For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V, a, b \in \mathbb{R}$, $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a\langle \mathbf{x}, \mathbf{z} \rangle + b\langle \mathbf{y}, \mathbf{z} \rangle$. An inner product defines a norm on V given by $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$.

EX: The space $C[0,1]$ can be given an inner product $\langle f, g \rangle = \int_a^b f(x)g(x)dx$.

THM: "Cauchy-Schwarz Inequality" For all $\mathbf{x}, \mathbf{y}$ in an inner product space V, $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$. Equality holds if and only if $\mathbf{x}, \mathbf{y}$ are colinear.

COR: For $f, g \in C[a,b]$, we have $\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b f(x)^2 dx\right)^{1/2} \left(\int_a^b g(x)^2 dx\right)^{1/2}$.

COR: An inner product space V satisfies the triangle inequality. Moreover, if equality occurs, then $\mathbf{x}$ and $\mathbf{y}$ are colinear.

COR: Let V be an inner product space with induced norm $\|\cdot\|$. Then the inner product is continuous (i.e. if $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$, then $\langle \mathbf{x}_n, \mathbf{y}_n \rangle \to \langle \mathbf{x}, \mathbf{y} \rangle$).

DEF: A normed vector space is *strictly convex* if $\|\mathbf{u}\| = \|\mathbf{v}\| = \|(\mathbf{u}+\mathbf{v})/2\| = 1$ for vectors $\mathbf{u}, \mathbf{v} \in V$ implies that $\mathbf{u} = \mathbf{v}$.

PROP: All inner product spaces are strictly convex.

7.3 Orthonormality

DEF: Two vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. A collection of vectors $\{\mathbf{v}_i \mid i \in I\}$ in V are *orthonormal* if $\|\mathbf{v}_i\| = 1$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$. This set is called an *orthonormal basis* if it is a maximal orthonormal set.

PROP: An orthonormal set is linearly independent. An orthonormal basis in a finite-dimensional inner product space is a basis.

LMA: The functions $\{1, \sqrt{2}\sin n\theta, \sqrt{2}\cos n\theta \mid n \geq 1\}$ form an orthonormal set in $C[-\pi, \pi]$ with the inner product $\langle f, g \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(\theta)\,d\theta$.

DEF: A *trigonometric polynomial* is a finite sum $f(\theta) = A_0 + \sum_{k=1}^{N} A_k \cos k\theta + B_k \sin k\theta$.

DEF: Denote the *Fourier series* of $f \in C[-\pi, \pi]$ by $f \sim A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta$, where $A_0 = \langle f, 1 \rangle, A_n = \langle f, \sqrt{2}\cos n\theta \rangle, B_n = \langle f, \sqrt{2}\sin n\theta \rangle$. The sequences A_n and B_n are the *Fourier coefficients* of f.

LMA: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal set in an inner product space V. If M is the subspace spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then every vector $\mathbf{x} \in M$ can be written uniquely as $\sum_{i=1}^{n} a_i \mathbf{v}_i$, where $a_i = \langle \mathbf{x}, \mathbf{v}_i \rangle$. Moreover, for $\mathbf{x}, \mathbf{y} \in M$ with $a_i = \langle \mathbf{x}, \mathbf{v}_i \rangle, b_i = \langle \mathbf{y}, \mathbf{v}_i \rangle$, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} a_i b_i$. In particular, $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} a_i^2$.

COR: If V is an inner product space of finite dimension n, then it has an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and the inner product and norm are defined by the lemma.

7.4 Projections

DEF: A *projection* is a linear map P so that $P^2 = P$. In addition, P is an *orthogonal projection* if $\ker P \perp \operatorname{im} P$.

THM: "Projection Theorem" Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal set in an inner product space V and let $M = \operatorname{span} B$. Define $P : V \to M$ by $P\mathbf{y} = \sum_{i=1}^{n} \langle \mathbf{y}, \mathbf{v}_i \rangle \mathbf{v}_i$. Then P is the *orthogonal projection* onto M and $\|\mathbf{y}\|^2 \geq \sum_{i=1}^{n} \langle \mathbf{y}, \mathbf{v}_i \rangle^2$.

Moreover, for all $\mathbf{v} \in M$, $\|\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y} - P\mathbf{y}\|^2 + \|P\mathbf{y} - \mathbf{v}\|^2$. In particular, $P\mathbf{y}$ is the closest vector in M to $\mathbf{y}$.

THM: "Bessel's Inequality" Let $\{\mathbf{v}_i \mid i \in I\}$ be an orthonormal set in an inner product space V. For each vector $\mathbf{x} \in V$, $\sum_{i \in I} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2 \leq \|\mathbf{x}\|^2$.

DEF: A complete inner product space is called a *Hilbert space*.

EX: The space ℓ^2 consists of all sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$ such that $\|\mathbf{x}\|_2 := \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}$ is finite. The inner product on ℓ^2 is given by $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n=1}^{\infty} x_n y_n$.

THM: The space ℓ^2 is complete.

DEF: In a Hilbert space, the *closed span* of a set of vectors S, denoted $\overline{\operatorname{span} S}$ is the closure of the linear subspace spanned by S.

THM: "Parseval's Theorem" Let $I \subseteq \mathbb{N}$ and $E = \{\mathbf{v}_i \mid i \in I\}$ be an orthonormal set in a Hilbert space H. Then the subspace $M = \overline{\operatorname{span} E}$ consists of all vectors $\mathbf{x} = \sum_{i \in I} a_i \mathbf{v}_i$ where the coefficient sequence $(a_i)_{i=1}^{\infty}$ belongs to ℓ^2. Further, if $\mathbf{x} \in H$, then $\mathbf{x} \in M$ if and only if $\sum_{i \in I} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2 = \|\mathbf{x}\|^2$.

COR: Let $E = \{\mathbf{v}_i \mid i \in I\}$ be an orthonormal set in a Hilbert space H. Then there is a continuous linear orthogonal projection P_E of H onto $M = \overline{\operatorname{span} E}$ given by $P_E \mathbf{x} = \sum_{i \in I} \langle \mathbf{x}, \mathbf{v}_i \rangle \mathbf{v}_i$.

COR: If $E = \{\mathbf{v}_i \mid i \in \mathbb{N}\}$ is an orthonormal basis for a Hilbert space H, every vector $\mathbf{x} \in H$ may be uniquely expressed as $\mathbf{x} = \sum_{i=1}^{\infty} a_i \mathbf{v}_i$ where $a_i = \langle \mathbf{x}, \mathbf{v}_i \rangle$.

DEF: If M is a closed subspace of a Hilbert space H, define the *orthogonal complement* of M to be $M^{\perp} = \{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{v} \rangle = 0 \forall \mathbf{v} \in M\}$.

PROP: Every vector in H can be written uniquely as $\mathbf{x} = \mathbf{v} + \mathbf{y}$ where $\mathbf{v} \in M$ and $\mathbf{y} \in M^{\perp}$. Moreover, $\|\mathbf{x}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{y}\|^2$.

PROP: $(M^{\perp})^{\perp} = M$.

7.5 Finite Dimensions

LMA: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set in a normed vector space V, then there exist positive constants $0 < c < C$ so that for all $\mathbf{a} \in \mathbb{R}^n$ we have $c\|\mathbf{a}\|_2 \leq \|\sum_{i=1}^{n} a_i \mathbf{v}_i\| \leq C\|\mathbf{a}\|_2$.

COR: "Hiene-Borel for Finite-Dimensional Normed Vector Spaces" A subset of a finite-dimensional normed vector space is compact if and only if it is closed and bounded.

COR: A finite-dimensional subspace of a normed vector space is complete, and in particular it is closed.

THM: Let V be a normed vector space, and let W be a finite dimensional subspace of V. Then for any $\mathbf{v} \in V$ there is at least one closest point $\mathbf{w}^* \in W$ so that $\|\mathbf{v} - \mathbf{w}^*\| = \inf\{\|\mathbf{v} - \mathbf{w}\| \mid \mathbf{w} \in W\}$.

7.6 Limits Of Functions

DEF: Let (f_k) be a sequence of functions from $S \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$. This sequence *converges pointwise* to a function f if $\lim_{n\to infty} f_n(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in S$.

DEF: Let (f_k) be a sequence of functions from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. This sequence *converges uniformly* if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for all $n \geq N$ and all $\mathbf{x} \in S$, $\|f_n(\mathbf{x}) - f(\mathbf{x})\| < \varepsilon$.

THM: For a sequence of functions (f_n) in $C_b(S, \mathbb{R}^m)$, (f_n) converges uniformly to f if and only if $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

THM: "Dini's Theorem" Suppose that f and f_n are continuous functions on $[a, b]$ so that $f_n \leq f_{n+1}$ for all $n \geq 1$ and (f_n) converges to f pointwise. Then (f_n) converges to f uniformly.

THM: Let (f_k) be a sequence of continuous functions mapping a subset $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ that converges uniformly to a function f. Then f is continuous.

THM: "Completeness Theorem for $C(K)$" If K is a compact set, the space $C(K)$ of all continuous functions on K with the ∞ norm is complete.

THM: "Integral Convergence Theorem" Let (f_k) be a sequence of continuous functions on $[a, b]$ converging uniformly to $f(x)$ and fix $c \in [a, b]$. Then the functions $F_k(x) = \int_c^x f_k(t)dt$, $k \geq 1$ converge uniformly on $[a, b]$ to the function $F(x) = \int_c^x f(t)dt$.

COR: Suppose that (f_n) is a sequence of continuously differentiable functions on $[a, b]$ such that (f_n') converges uniformly to a function g and there is a point $c \in [a, b]$ so that $\lim_{n\to\infty} f_n(c) = \gamma$ exists. Then (f_n) converges uniformly to a differentiable function f with $f(c) = \gamma$ and $f' = g$.

PROP: Let $f(x, t)$ be a continuous function on $[a, b] \times [c, d]$. Define $F(x) = \int_c^d f(x, t)dt$. Then F is continuous on $[a, b]$.

THM: "Liebniz's Rule" Suppose that $f(x, t)$ and $\frac{\partial}{\partial x} f(x, t)$ are continuous functions on $[a, b] \times [c, d]$. Then $F(x) = \int_c^d f(x, t)dt$ is differentiable and $F'(x) = \int_c^d \frac{\partial}{\partial x} f(x, t)dt$.

THM: Let (f_k) be a sequence of functions from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. If $\sum_{k=1}^\infty f_k(x)$ converges uniformly, then it is continuous.

DEF: Let $S \subseteq \mathbb{R}^n$. We say that a series of functions f_k from S to $\mathbb{R}^m$ is *uniformly Cauchy* on S if for every $\varepsilon > 0$, there is an N so that $\left\|\sum_{k=m+1}^{\ell} f_k(\mathbf{x})\right\| < \varepsilon$ whenever $\mathbf{x} \in S$ and $\ell > m \geq N$.

THM: A series of functions converges uniformly if and only if it is uniformly Cauchy.

THM: "Weierstrass M-Test" Suppose that $a_n(\mathbf{x})$ is a sequence of functions on $S \subseteq \mathbb{R}^k$ to $\mathbb{R}^m$ and (M_n) is a sequence of real numbers so that $\|a_n\|_\infty \leq M_n$. If $\sum_{n=1}^\infty M_n$ converges, then $\sum_{n=1}^\infty a_n(\mathbf{x})$ converges uniformly on S.

THM: "Hadamard's Theorem" Given a power series $\sum_{n=0}^\infty a_n x^n$ there is R in $[0, +\infty) \cup \{+\infty\}$ so that the series converges for all x with $|x| < R$ and diverges for all x with $|x| > R$. Moreover, the series converges uniformly on each interval $[a, b]$ contained in $(-R, R)$. Finally, if $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$, then

$$R = \begin{cases} +\infty & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha = +\infty \\ \frac{1}{\alpha} & \text{if } \alpha \in (0, +\infty). \end{cases}$$

We call R the *radius of convergence* of the power series.

THM: "Term-By-Term Differentiation" If $f(x) = \sum_{n=0}^\infty a_n x^n$ has radius of convergence $R > 0$, then $\sum_{n=1}^\infty n a_n x^{n-1}$ has radius of convergence R, f is differentiable on $(-R, R)$, and for $x \in (-R, R)$, $f'(x) = \displaystyle\sum_{n=1}^\infty n a_n x^{n-1}$. Further, $\sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1}$ has radius of convergence R and, for $x \in (-R, R)$, $\int_0^x f(t)dt = \sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1}$.

7.7 Compactness of $S \subseteq C(K, \mathbb{R}^m)$

DEF: A family of functions $\mathcal{F}$ mapping $S \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ is *equicontinuous* at a point $\mathbf{a} \in S$ if for every $\varepsilon > 0$ there is an $r > 0$ such that $\|f(x) - f(a)\| < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{a}\| < r$ and $f \in \mathcal{F}$. The family $\mathcal{F}$ is *equicontinuous on S* if it is equicontinuous at every $\mathbf{a} \in S$. The family $\mathcal{F}$ is *uniformly equicontinuous* on S if for each $\varepsilon > 0$, there is an $r > 0$ so that $\|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{y}\| < r, \mathbf{x}, \mathbf{y} \in S$ and $f \in \mathcal{F}$.

LMA: Let K be a compact subset of $\mathbb{R}^n$. A compact subset $\mathcal{F}$ of $C(K, \mathbb{R}^m)$ is equicontinuous.

PROP: If $\mathcal{F}$ is an equicontinuous family of functions on a compact set, then it is uniformly equicontinuous.

DEF: A subset S of K is called an ε-net of K if $K \subseteq \cup_{\mathbf{a}\in S} B_\varepsilon(\mathbf{a})$. A set K is *totally bounded* if it has a finite ε-net for every $\varepsilon > 0$.

LMA: Let K be a compact subset of $\mathbb{R}^m$. Then K is totally bounded.

COR: Let K be a compact subset of $\mathbb{R}^m$. Then K contains a sequence $\{\mathbf{x}_i \mid i \geq 1\}$ that is dense in K. Moreover, for any $\varepsilon > 0$, there is an integer N so that $\{x_1, \ldots, x_N\}$ forms an ε-net for K.

THM: "Arzela-Ascoli Theorem" Let K be a compact subset of $\mathbb{R}^n$. A subset $\mathcal{F}$ of $C(K, \mathbb{R}^m)$ is compact if and only if it is closed, bounded, and equicontinuous.

Notes

[1]Donsig, 825 Notes, 08/29.
[2]Dongig, 825 Notes, 08/29.
[3]Donsig, 825 Notes, 08/31.
[4]D&D Definition 2.3.1. p. 38.
[5]D&D. Theorem 2.3.6. p. 39.
[6]D&D. Proposition 2.4.2. p. 42; Ross, Theorem 9.1. p. 43.
[7]D&D. Theorem 2.4.3. p. 42; Ross, Thm 9.2-6. pp.43-46.
[8]Ross. Theorem 9.10. p. 51.
[9]D&D. Theorem 2.5.3. p. 46.
[10]D&D. Theorem 2.5.4. p. 47; Ross. Thm 10.2. p. 55.
[11]Ross. Theorem 10.7. p. 58.
[12]Ross. Definition 11.1. p. 63.
[13]Ross. Theorem 11.2. p. 67.
[14]Ross. Theorem 11.3. p. 67.
[15]Ross. Corollary 11.4. p. 68.
[16]Ross. Definition 11.6. p. 70.
[17]Ross. Theorem 11.7. p. 71.
[18]Ross. Theorem 11.8. p. 72.
[19]D&D. Lemma 2.6.3. p. 51.
[20]D&D. Theorem 2.6.4. p. 52.
[21]Donsig, 825 Notes, 09/17.
[22]D&D. Definition 2.7.2. p. 56; Ross. Definition 10.8. p. 60.
[23]D&D. Definition 2.7.3. p. 56.
[24]D&D. Theorem 2.7.4. p. 56; Ross. Theorem 10.11. p. 61.
[25]D&D, Def 3.1.1. p. 66.
[26]D&D, Example 3.1.3. p. 68.
[27]Donsig, 825 Notes, 09/19.
[28]D&D Theorem 3.1.5. p. 69; Ross, Definition 14.3, Theorem 14.4. pp. 92-93.
[29]D&D, Prop 3.2.1., p. 71.
[30]D&D, Theorem 3.2.2., p. 71.
[31]H,W&T, Example 3, p. 525.
[32]D&D, Theorem 3.2.3. p. 71; Ross, Theorem 14.6. p. 93.
[33]Ross, Theorem 14.8. p. 94.
[34]D&D, Theorem 3.2.4., p.72.; Ross, Theorem 14.9. p. 94.
[35]Donsig, 825 Notes, 09/26.
[36]Donsig, 825 Notes, 09/26.
[37]D&D, Definition 3.2.5. p. 72.
[38]Donsig, 825 Notes, 09/26. D&D. p. 77.
[39]D&D. Theorem 3.2.6. p. 72.
[40]D&D. Definition 3.4.2. p. 81.
[41]D&D. Definition 3.4.4. p. 81.
[42]D&D. Lemma 3.4.7. p. 82.
[43]D&D, Theorem 3.4.8. p. 83.
[44]D&D. Lemma 3.4.9. p. 85.
[45]D&D. Theorem 3.4.10. p. 85.
[46]D&D. Problem 3.4.G. Solution in 825 Problem Set 5.
[47]D&D. p.89.
[48]D&D. Theorem 4.1.1. p.89.
[49]D&D. Theorem 4.1.2. p.89.
[50]D&D. p.91.
[51]D&D. Lemma 4.1.3. p.91.
[52]D&D. Definition 4.2.1. p.92.
[53]D&D. Lemma 4.2.2. p. 93.
[54]D&D. Lemma 4.2.3. p.93.
[55]D&D. Definition 4.2.4. p.93.
[56]D&D. Theorem 4.2.5. p.94.
[57]D&D. Definition 4.3.1. p. 96.
[58]D&D. 4.4.N. p. 101.
[59]D&D. Proposition 4.3.3. p. 97.
[60]D.S. 10/15.
[61]D&D. Definition 4.3.4. p.97.
[62]D&D. Definition 4.3.6. p.98.
[63]D&D. Proposition 4.3.5. p.97.
[64]D&D. Theorem 4.3.8. p.98.
[65]D&D. Proposition 4.3.9. p. 99.
[66]D.S. 10/15.
[67]D&D. p.99.
[68]D&D. Definition 4.4.1. p. 101.
[69]D&D. p.102.
[70]D&D. Lemma 4.4.3. p. 102.
[71]D&D. Lemma 4.4.4. p. 102.
[72]D&D. Lemma 4.4.5. p. 102.
[73]D&D. Theorem 4.4.6. p. 103.
[74]D&D. Theorem 4.4.7. p. 103.
[75]D&D. p. 104
[76]D&D. Def. 5.1.1. p. 108.
[77]D&D. Def. 5.1.2. p. 109.
[78]D&D. Def. 5.1.3. p. 111.
[79]D&D. 5.5.K. p.133.
[80]D&D. Prop. 5.1.4. p. 111.
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[86]D&D. Ex. 5.2.2. p.114.
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[88]D&D. Ex. 5.2.9. p.118.
[89]D&D. Ex. 5.2.10. p.118.
[90]D&D. Def. 5.2.5. p.116.
[91]D&D. p.117.
[92]D&D. p.120.
[93]D&D. Thm 5.3.1. p.120.
[94]D&D. Thm 5.3.2. p.121.
[95]D&D. Thm. 5.3.3. p.122.
[96]D&D. p.122.
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[98]D&D. p.122.
[99]D&D. Thm 5.4.3. p.125.
[100]D&D. Thm 5.4.4. p.125.
[101]D&D. 5.4.H. p.126.
[102]D&D. Def. 5.5.1. p.127.
[103]D&D. Prop. 5.5.4. p.129.
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[106]D&D. Thm. 5.5.9. p.131.
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[108]D&D. Cor. 5.6.2. p.134.
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[110]D&D. Cor. 5.6.3. p.134.
[111]D&D. Def. 5.7.1. p.135.
[112]D&D. Prop. 5.7.2. p.135.
[113]D&D. Cor. 5.7.3. p.136.
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[115]D&D. Thm. 5.7.5. p.136.
[116]D&D. Thm. 5.7.6. p.137.
[117]Not in the book or notes.
[118]D&D. Example 5.7.8. pp. 137-140.
[119]D&D Def 6.1.1. p.141; Ross Def 28.1 pp.205-6
[120]D&D p.141.
[121]D&D. Prop 6.1.2 p.142; Ross Thm 28.2 p.207
[122]D&D Lemma 6.1.3 p.142
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[124]Ross Thm 28.3 p.208
[125]D&D Thm 6.1.6 p.145; Ross Thm 28.4 p.209
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[128]D&D Thm 6.2.1. p.148; Ross Thm 29.1 p.214
[129]D&D Thm 6.2.2. p.148; Ross Thm 29.2 p. 214
[130]D&D Thm 6.2.3. p.149; Ross Thm 29.3 p.215
[131]D&D Cor 6.2.5. p.150
[132]Ross Cor 29.7 p217
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[136]D&D 6.2.I p.152; Ross Thm 29.8 p.217
[137]Ross Thm 29.9 pp.218-9
[138]D&D Def 6.3.1 p.154; Ross Def 32.1 pp.243-4
[139]Ross Def 32.6 p.249
[140]Ross Def 32.8 p.250
[141]D&D Lma 6.3.2 p.155; Ross Lma 32.2 p.246
[142]D&D Cor 6.3.3 p.155; Ross Lma 32.3 p.247
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[144]D&D Def 6.3.4 p.156; Ross Def 32.1 p.244
[145]D&D Thm 6.3.5 p.156; Ross Thm 32.5 p.248
[146]D&D Thm 6.3.8 p.158
[147]Ross Thm 32.7 p.249
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[152]Ross Thm 33.3 p.254
[153]Ross Thm 33.4 p.257
[154]Ross Thm 33.5 p.257
[155]Ross Thm 33.6 p.258
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[167]D&D Thm 6.7.5
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8 Problems

QUESTION: Define a sequence (x_n) by $x_1 = 2$ and, for $n \geq 2$, $x_n = 1 + \frac{1}{x_1 \cdots x_{n-1}}$. Show that there is an integer m so that $x_1 \cdots x_m > 100$.

QUESTION: Fix an integer $N \geq 2$. Consider the remainders $q(n)$ obtained by dividing the Fibonacci number $F(n)$ by N, so that $0 \leq q(n) < N$. Prove that this sequence is periodic with period $d \leq N^2$ as follows:
(i) Show that there are integers $0 \leq i < j \leq N^2$ such that $q(i) = q(j)$ and $q(i+1) = q(j+1)$.
(ii) Show that if $q(i+d) = q(i)$ and $q(i+1+d) = q(i+1)$, then $q(n+d) = q(n)$ for all $n \geq i$. (iii) Show that if $q(i+d) = q(i)$ and $q(i+1+d) = q(i+1)$, then $q(n+d) = q(n)$ for all $n \geq 0$.

QUESTION: If m and n are integers, show that $\left|\sqrt{3} - \frac{m}{n}\right| \geq \frac{1}{5n^2}$.

QUESTION: Compute the limit, and for $\varepsilon = 10^{-6}$, find an integer N that satisfies the limit equation: $\lim_{n\to\infty} \frac{n^2+2n+1}{2n^2-n+2}$. *Optional:* What is the smallest value of N that satisfies the limit definition for $\varepsilon = 10^{-6}$?

QUESTION: (i) Prove that if $a_n \leq b_n$ for $n \geq 1, L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$, then $L \leq M$.
(ii) Find convergent sequences (a_n) and (b_n) so that
(a) $a_n \leq b_n$ for all n, (b) there is no N so that for all $n \geq N$, $a_n \leq \lim\limits_{n\to\infty} b_n$, and
(c) there is no N so that for all $n \geq N$, $b_n \geq \lim\limits_{n\to\infty} a_n$.

QUESTION: Consider $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$. Show that the new sequence $(x_1, y_2, x_2, y_2, \ldots)$ converges to a number L if and only if the two original sequences both converge to L.

QUESTION: Define a sequence $(a_n)_{n=1}^{\infty}$ so that $\lim_{n\to\infty} a_{n^2}$ exists but $\lim_{n\to\infty}$ does not exist.

QUESTION: (i) Let $x_n = \sqrt[n]{n} - 1$. Use the fact that $(1+x_n)^n = n$ to show that $x_n^2 \leq 2/n$.
(ii) Hence compute $\lim_{n\to\infty} n^{1/n}$.

QUESTION: Show that the set $S = \{n + m\sqrt{2} | m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

QUESTION: Suppose that $\lim_{n\to\infty} a_n = L$. Show that $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = L$.

QUESTION: (i) Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence. Define a sequence $b_n = \sup\{a_k : k > n\}$ for $n \geq 1$. Prove that (b_n) converges. This limit is called the *limit superior* of (a_n), almost always abbreviated to $\limsup a_n$.
(ii) Without redoing the proof, do the same for the *limit inferior* of (a_n), which is defined as $\liminf a_n := \lim_{n\to\infty}(\inf_{k\geq n} a_k)$.

QUESTION: Given two sequences of real numbers (x_n) and (y_n), prove that $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$. Give an example where all three lim sup's are finite and the inequality is strict.

QUESTION: Define $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$ for $n \geq 1$.
(i) Find a formula for $x_{n+1}^2 - 5$ in terms of $x_n^2 - 5$.
(ii) Hence evaluate $\lim_{n\to\infty} x_n$.
(iii) Compute the first ten terms on a calculator.
(iv) Show that the tenth term approximates the limit to over 600 decimal places.

QUESTION: Construct a sequence $(x_n)_{n=1}^{\infty}$ so that for every real number L, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} = L$.

QUESTION: Suppose that (a_n) is a sequence such that $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$ f or all $n \geq 0$. Show that this sequence is Cauchy if and only if $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$.

QUESTION: Suppose that, for a sequence (a_n), there is $\lambda \in (0,1)$ so that $|a_{n+2} - a_{n+1}| \leq \lambda |a_{n+1} - a_n|$. Show that (a_n) is Cauchy.

QUESTION: (i) Show that a sequence $(a_n)_{n=1}^{\infty}$ converges if and only if $\limsup a_n = \liminf a_n$.
(ii) Suppose a sequence (a_n) has the property that for any sequence (b_n), we have $\limsup a_n + b_n = \limsup a_n + \limsup b_n$. Show that (a_n) converges.

QUESTION: Find the sum of $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$.

QUESTION: Construct a convergent series of positive terms with $\limsup \frac{a_{n+1}}{a_n} = \infty$. [174]

QUESTION: If $a_n \geq 0$ for all n, prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.[175]

QUESTION: Suppose that (a_n) is a strictly decreasing positive sequence, i.e., $0 < a_{n+1} < a_n$ for all n.
(i) Suppose that (g_k) is a strictly increasing sequence of integers and there is a constant C so that for $k = 2, 3, \ldots$, we have $g_{k+1} - g_k \leq C(g_k - g_{k-1})$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=1}^{\infty}(g_{k+1} - g_k)a_{g_k}$ converges.
(ii) By a suitable choice of (g_k), prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{n^2}$ converges.
(iii) Similarly, prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} n a_n^2$ converges.[176]

QUESTION: Suppose (a_n) is a decreasing positive sequence, i.e. $0 < a_{n+1} \leq a_n$.
(i)Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.
(ii) Give a sequence (a_n) as above so that $\lim_{n\to\infty} n a_n = 0$ but $\sum_{n=1}^{\infty} a_n$ diverges.[177]

QUESTION: Use summation by parts to prove **Abel's Test**: Suppose that $\sum_{n=1}^{\infty} a_n$ converges and (b_n) is a monotonic convergent sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.[178]

QUESTION: Suppose that $\mathbf{x}$ and $\mathbf{y}$ are unit vectors in $\mathbb{R}^n$. Show that if $\left\| \frac{\mathbf{x}+\mathbf{y}}{2} \right\| = 1$, then $\mathbf{x} = \mathbf{y}$.

QUESTION: (i) Show that if $(\mathbf{x}_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^n$ such that $\sum_{n \geq 1} ||\mathbf{x}_n - \mathbf{x}_{n+1}|| < \infty$, then $(\mathbf{x}_n)$ is Cauchy.
(ii) Give an example of a Cauchy sequence for which this condition fails.
(iii) However, show that every Cauchy sequence $(\mathbf{x}_k)_{k=1}^{\infty}$ has a subsequence $(\mathbf{x}_{k_i})_{i=1}^{\infty}$ such that $\sum_{i \geq 1} ||\mathbf{x}_{k_i} - \mathbf{x}_{k_{i+1}}|| < \infty$.

QUESTION: Suppose that $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ are sequences so that $\sum_{n=1}^{\infty} x_n^2$ and $\sum_{n=1}^{\infty} y_n^2$ converge. Show that $|\sum_{n=1}^{\infty} x_n y_n| \leq \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} y_n^2\right)^{1/2}$. In particular, you are showing that the series on the left-hand side converges.

QUESTION: (i) Show that the sum of a closed subset and a compact subset of $\mathbb{R}^n$ is closed. Recall that $A + B = \{ \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B \}$.
(ii) Is this true for the sum of two compact sets and a closed set?
(iii) Is this true for the sum of two closed sets?

QUESTION: Let $(\mathbf{x}_n)_{n=1}^{\infty}$ be a sequence in a compact set K that is *not* convergent. Show that there are two subsequences of this sequence that are convergent to *different* limit points.

QUESTION: Let A and B be *disjoint* closed subsets of $\mathbb{R}^n$. Define $d(A, B) = \inf\{\|\mathbf{a} - \mathbf{b}\| \mid \mathbf{a} \in A, \mathbf{b} \in B\}$.
(i) If $A = \{\mathbf{a}\}$ is a singleton, show that $d(A, B) > 0$.
(ii) If A is compact, show that $d(A, B) > 0$.
(iii) Find an example of two disjoint closed sets in $\mathbb{R}^2$ with $d(A, B) = 0$.

QUESTION: Define a function on the set $S = \{0\} \cup \{ \frac{1}{n} \mid n \geq 1 \}$ by $f(\frac{1}{n}) = a_n$ and $f(0) = L$. Prove that f is continuous on S if and only if $\lim_{n \to \infty} a_n = L$.

QUESTION: Find a bounded continuous function on $\mathbb{R}$ that is not Lipschitz.

QUESTION: Suppose that $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are open. Show that the set

$$A \times B = \{ (x_1, \dots, x_{n+m}) \mid (x_1, \dots, x_n) \in A, (x_{n+1}, \dots, x_{n+m}) \in B \}$$

is open in $\mathbb{R}^{m+n}$.

QUESTION: Define f on $\mathbb{R}$ by $f(x) = x\chi_{\mathbb{Q}}(x)$, where $\chi_{\mathbb{Q}}(x)$ is the characteristic function on $\mathbb{Q}$. Show that f is continuous at 0 and that this is the *only* point where it is continuous.

QUESTION: Let f and g be continuous mappings of $S \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$. Show that the inner product $h(\mathbf{x}) = \langle f(\mathbf{x}), g(\mathbf{x}) \rangle$ is continuous.

QUESTION: Suppose that f is a continuous function on $[a, b]$ and g is a continuous function on $[b, c]$ such that $f(b) = g(b)$. Show that

$$h(x) = \begin{cases} f(x) & \text{if } a \leq x \leq b \\ g(x) & \text{if } b \leq x \leq c \end{cases}$$

is continuous on $[a, c]$.

QUESTION: Give an example of a continuous function f and an open set U such that $f(U)$ is not open.

QUESTION: (January 2007 Qual) Let f be a positive continuous function defined on $\mathbb{R}$ such that $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$. Show that f attains its maximum value, that is there is $b \in \mathbb{R}$ so that $f(b) = \sup f(\mathbb{R})$.

QUESTION: Let f be a continuous function on $(0, 1]$. Show that f is uniformly continuous if and only if $\lim\limits_{x \to 0^+} f(x)$ exists.

QUESTION: Let f be a continuous function from $B = \{ \mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1 \}$, the closed ball in $\mathbb{R}^2$, into $\mathbb{R}$. Show that f cannot be one-to-one.

QUESTION: or $x \in [0, 1]$, express it as a decimal $x = x_0.x_1x_2x_3\dots$. Use a finite decimal expansion without repeating 9s when there is a choice. Then define a function f by $f(x) = x_0.0x_10x_20x_3\dots$.
(i) Show that f is strictly increasing.
(ii) Compute $\lim\limits_{x \to 1^-} f(x)$.
(iii) Show that $\lim\limits_{x \to a^+} f(x) = f(a)$ for $0 \leq a < 1$.
(iv) Find all discontinuities of f.

QUESTION: Let f be a real uniformly continuous function on the bounded set $E \subseteq \mathbb{R}$. Prove that f is bounded on E. Show that the conclusion may be false if boundedness of E is not assumed.

QUESTION: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q^3} & \text{if } x = \frac{p}{q} \text{ in lowest terms, and } q > 0. \end{cases}$$

Show that $f'(\sqrt{3})$ exists and is zero. (HINT: Exercise 2.2.H could be useful).

QUESTION: Suppose that $f : [0, +\infty) \to \mathbb{R}$ is twice differentiable on $[0, +\infty)$ and satisfies $f(0) = 0, f'(0) = 1$, and $f''(x) \leq 0$ for all x.
(i) Prove that $f(x) \leq x$ for all $x > 0$.
(ii) Prove that $f(x)/x$ is decreasing on $(0, +\infty)$.

QUESTION: Suppose that f is continuous on an interval $[a, b]$ and is differentiable at all points of (a, b) except possibly at a single point $x_0 \in (a, b)$. If $\lim_{x \to x_0} f'(x)$ exists, show that $f'(x_0)$ exists and $f'(x_0) = \lim_{x \to x_0} f'(x)$. HINT: Consider the intervals $[x_0 - h, x_0]$ and $[x_0, x_0 + h]$.

QUESTION: Suppose that f is differentiable on $[a, b]$ and $f'(a) < 0 < f'(b)$.
(i) Show that there are points $a < c < d < b$ such that $f(c) < f(a)$ and $f(d) < f(b)$.
(ii) Show that the minimum on $[a, b]$ occurs at an interior point.
(iii) Hence show that there is a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

QUESTION: Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and $g : [a, b] \to \mathbb{R}$ has $g(x) = f(x)$ for all x in $[a, b]$ except at $c_0, \ldots, c_n$. Prove that g is integrable and $\int_a^b g(x)dx = \int_a^b f(x)dx$.

QUESTION: Suppose that f is Lipschitz with constant L on $[0, 1]$. Prove that

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \right| \leq \frac{L}{n}.$$

QUESTION: If f and g are both Riemann integrable on $[a, b]$, show that fg is also integrable. HINT: Use the identity $f(x)g(x) - f(t)g(t) = f(x)(g(x) - g(t)) + (f(x) - f(t))(g(t))$ to show that $M_i(fg, P) - m_i(fg, P)$ is bounded by

$$\|f\| \left(M_i(g, P) - m_i(g, P)\right) + \|g\| \left(M_i(f, P) - m_i(f, P)\right).$$

QUESTION: Let f be a continuous function on $\mathbb{R}$, and fix $\varepsilon > 0$. Define a function G by

$$G(x) = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(t)dt.$$

Show that G is C^1 and compute G'.

Notes

[174]D&D, 3.2.K.
[175]D&D, 3.2.N.
[176]Donsig, 825 Problem Set 5, # 3.
[177]Donsig, 825 Problem Set 5, # 4.
[178]D&D, 3.4.G.