CARS talk
1/24/12 "Matrix Factorizations Revisited"

Outline:

- Background: a taste of homological algebra
- Another construction of matrix factorizations for MCMs
- From a matrix factorization to a free R-resolution
- More general construction~~s~~ (Shamash/Eisenbud)

§ Background

[Can skim over definition of complex for this audience]

Let Q be any local ring.

For this talk, all complexes C will have the property that $C_i = 0$ when $i<0$.

Consider a complex of Q-mods with differential ∂.

If $\operatorname{im}\partial_{j+1} = \ker\partial_j \ \forall j \geq 1$, the complex is called an **exact sequence**.

If $\operatorname{im}\partial_2 \neq \ker\partial_1$ but $\operatorname{im}\partial_{j+1} = \ker\partial_j$ otherwise, the complex is called **acyclic**.

Morphisms of complexes Let C, D be cxes of Q-mods.

A **morphism** of cxes φ is a collection of Q-mod homomorphism $\varphi_i : C_i \longrightarrow D_i$ satisfying $\varphi_{i-1}\partial_i^C = \partial_i^D \varphi_i \quad \forall i$.

(each square commutes:)

$$\begin{tikzcd} C_i \arrow[r, "\partial_i^C"] \arrow[d, "\varphi_i"] \arrow[dr, phantom, "\circlearrowleft"] & C_{i-1} \arrow[d, "\varphi_{i-1}"] \\ D_i \arrow[r, "\partial_i^D"] & D_{i-1} \end{tikzcd}$$

[In ptc, a morphism is a degree 0 homomorphism of complexes.]

A morphism φ is said to be nullhomotopic if $\exists$ a degree 1 homomorphism of cxes h [where $h_i : C_i \to D_{i+1}$ is a module hom'm $\forall i$ & parallelograms commute] s.t.

$$\varphi = \partial^D h + h \partial^C$$

(schematic:)

$$\begin{tikzcd}
C_{i+1} \arrow[r] \arrow[d, "\varphi"'] & C_i \arrow[r, "\partial_i^C"] \arrow[d, "\varphi_i"] \arrow[dl, "h_i"'] & C_{i-1} \arrow[d, "\varphi"] \arrow[dl, "h_{i-1}"'] \\
D_{i+1} \arrow[r, "\partial_{i+1}^D"'] & D_i \arrow[r] & D_{i-1}
\end{tikzcd}$$

Two morphisms are homotopic if their difference is nullhomotopic.

With the vocab of cxes above, for a Q-mod M, a complex F is a free resolution of M if it is acyclic and $H_0(F) = M$; ie, the complex

$$F \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact.

Thm (Comparison theorem)

Let M and N be Q-modules and $F \xrightarrow{\sim} M$, $G \xrightarrow{\sim} N$ free resolutions of M and N respectively. Given a Q-module homomorphism $\varphi : M \longrightarrow N$, there is a morphism of complexes $\widetilde{\varphi} : F \longrightarrow G$ that lifts φ and is unique up to homotopy.

Pf: (Existence)

Step 1:

$$\begin{tikzcd} F_0 \arrow[r, "\tilde{\varphi}_0"] \arrow[d, "\varepsilon^F"'] \arrow[dr, "\varphi\varepsilon^F"] & G_0 \arrow[d, "\varepsilon^G"] \\ M \arrow[r, "\varphi"] & N \arrow[d] \\ & 0 \end{tikzcd}$$

$\tilde{\varphi}_0$ exists because F_0 is projective.

finish (verify step 2)

$$\begin{tikzcd} F_1 \arrow[r, "\tilde{\varphi}_1"] \arrow[d, "\partial_1^F"'] \arrow[dr] & G_1 \arrow[d] \\ F_0 \arrow[r, "\tilde{\varphi}_0|^{\mathrm{im}\,\partial_1^G}"] & \mathrm{im}\,\partial_1^G \arrow[d] \\ & 0 \end{tikzcd}$$

Step 2 $\tilde{\varphi}_1$ lifts $\tilde{\varphi}_0|^{\mathrm{im}\,\partial_1^G}$;

RTS $\tilde{\varphi}_0\partial_1^F(F_1) \subseteq \mathrm{im}\,\partial_1^G = \ker \varepsilon^G$.

Well, let $x \in F_1$

$$\begin{array}{ccc} & x & \\ & \downarrow & \\ & \partial_1^F(x) & \longmapsto \tilde{\varphi}_0\partial_1^F(x) \\ & \downarrow & \downarrow \varepsilon^G \\ & \varepsilon^F\partial_1^F(x) = 0 & \longmapsto 0 \end{array}$$

$\therefore \tilde{\varphi}_0\partial_1^F \subseteq \ker \varepsilon^G$

* So $\tilde{\varphi}_1$ lifts $\tilde{\varphi}_0\partial_1^F$ itself.

Step n ... répètez!

uniqueness is more subtle (ignoring it for this talk!)

Where the talk starts getting interesting

§ Another construction of matrix factorizations

Let (S, n, k) be a regular local ring. Let $f \in n$, and set $R = S/(f)$.
Let $MCM(R)$ denote the category of MCM R-mods and let $\underline{MCM}(R)$ denote the category of MCM R-mods having no free direct summands.

basic fact about RLR: $0 \le pd_S M \le edim S$

Let $M \in MCM(R)$. Recall from Jason's talk that $pd_S M < \infty$. Therefore by the Auslander–Buchsbaum formula,

$$pd_S M + depth_S M = depth_S S = dim S$$

Claim: $pd_S M = \underline{1}$.

Facts: ① $depth_S N = depth_R N$ for any R-mod N
[BH]

② $depth_S R = depth_S S - 1$

Sketch: ② $f \in n \Rightarrow f$ is S-regular.

[we'll see this in Sri's class]

① f is a zero divisor on N

not rigorous (use Ext to show?)

So, substituting in (☆),

we have LHS:

$$\begin{aligned} \mathrm{pd}_S M + \mathrm{depth}_S M &= \mathrm{pd}_S M + \mathrm{depth}_R M \\ &= \mathrm{pd}_S M + \mathrm{depth}_R R \\ &= \mathrm{pd}_S M + \mathrm{depth}_S S - 1 = \mathrm{RHS} \\ &\qquad\qquad\qquad\qquad\qquad\quad = \mathrm{depth}_S S \end{aligned}$$

This proves the claim.

Now let $F: \; 0 \to F_1 \to F_0 \to 0$ be a free res of M.

Consider the map $\cdot f : M \to M$. This is identical to the 0 map.
By the comparison theorem, there exists a morphism $\widetilde{\cdot f} : F \to F$ that lifts $\cdot f$ and is null homotopic:

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r] \arrow[d, "\widetilde{\cdot f_1}"'] & F_0 \arrow[r] \arrow[d, "\widetilde{\cdot f_0}"'] \arrow[dl, "h_0"'] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "\partial_1"'] & F_0 \arrow[r] & 0 \end{tikzcd}$$

where $\widetilde{\cdot f_0} = \partial_1 h_0 + 0$

and $\widetilde{\cdot f_1} = 0 + h_0 \partial_1$

Aha! Choosing a basis, (∂_1, h_0) is a matrix factorization of f.

Check: reduced $\iff$ $M \in \underline{\mathrm{MCM}}(R)$.

§ To a minimal free R-res

Let $\overline{}$ denote $\square \otimes_S R$.

Thm. Let $M \in \underline{MCM}(R)$. The complex

$$\widehat{F}: \quad \cdots \xrightarrow{\overline{h_0}} \overline{F_1} \xrightarrow{\overline{\partial_1}} \overline{F_0} \xrightarrow{\overline{h_0}} \overline{F_1} \xrightarrow{\overline{\partial_1}} \overline{F_0} \longrightarrow 0$$

is a min'l free R-res of M.

Proof We need to show a bunch of stuff:

0. $\widehat{F}$ is a complex
1. $M = H_0(\widehat{F})$
2. $\ker \overline{\partial_1} = \operatorname{im} \overline{h_0}$
3. $\ker \overline{h_0} = \operatorname{im} \overline{\partial_1}$

0 since $\partial_1 h_0 = f = h_0 \partial_1$, $\overline{\partial_1 h_0} = \overset{0}{\cancel{f}} = \overline{h_0 \partial_1}$.

For 1,

$$H_0(\widetilde{F}) = \operatorname{coker} \overline{\partial}_1 = \overline{\operatorname{coker} \partial_1}$$
$$= M \otimes_S R = M.$$

This means that M is actually coker of the differential of $\widehat{F}$ at every other spot.

For ②, we use the triangle
$$\begin{tikzcd} & F_0 \arrow[dl, "h_0"'] \arrow[d, "f"] \\ F_1 \arrow[r, "\partial_1"] & F_0 \end{tikzcd}$$

Let $x \in F_1$ st $\bar{x} \in \ker \overline{\partial_1}$.

Then $\partial_1(x) = f \cdot y$ for some $y \in F_0$,
$$= \partial_1 h_0(y)$$
∂_1 inj $\Rightarrow$ $x = h_0(y)$. $\quad \therefore \bar{x} \in \operatorname{im} \overline{\partial_1}$.

For ③ we use the other triangle
$$\begin{tikzcd} F_1 \arrow[r, "\partial_1"] \arrow[d, "\tilde{f}"'] & F_0 \arrow[dl, "h_0"] \\ F_1 & \end{tikzcd}$$

Let $x \in F_0$ st $\bar{x} \in \ker \overline{h_0}$

So $\quad h_0(x) = f \cdot y \quad$ for some $y \in F_1$
$$= h_0 \partial_1(y)$$

Now apply ∂_1:
$$f x = \partial_1 h_0(x) = \partial_1 h_0 \partial_1(y) = f \partial_1(y)$$
and f nzd on $S \Rightarrow$
$$x = \partial_1(y).$$
so $\bar{x} \in \operatorname{im} \overline{\partial_1}$. $\square$

Observe:
$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \overline{F_1} \arrow[rr, "\overline{\partial_1}"] & & \overline{F_0} \arrow[rr, "\overline{h_0}"] \arrow[dr] & & \overline{F_1} \arrow[rr, "\overline{\partial_1}"] & & \overline{F_0} \arrow[rr, "\overline{h_0}"] \arrow[dr] & & \overline{F_1} \arrow[rr, "\overline{\partial_1}"] & & \overline{F_0} \arrow[rr] \arrow[dr] & & 0 \\ & & & & M \arrow[ur] \arrow[dr] & & & & M \arrow[ur] \arrow[dr] & & & & M \arrow[dr] & \\ & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & & & & 0 \end{tikzcd}$$

§ Generalization. [Shamash '69, Eisenbud '80]

Let S, R as before. Let M be any R-mod and F a minimal free S-res of M:

$$F: 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0$$

By the comparison theorem, there is a morphism $\hat{f}: F \to F$ that lifts $f: M \to M$ and is null homotopic:

$$\begin{tikzcd}
0 \arrow[r] & F_n \arrow[r] \arrow[d] \arrow[dl, "0"'] & F_{n-1} \arrow[r] \arrow[d] \arrow[dl, "h_{n-1}"'] & \cdots \arrow[r] \arrow[dl, "h_{n-1}"'] & F_1 \arrow[r] \arrow[d] \arrow[dl, "h_1"'] & F_0 \arrow[r] \arrow[d] \arrow[dl, "h_0"'] & 0 \arrow[dl, "0"'] \\
0 \arrow[r] & F_n \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[r] & 0
\end{tikzcd}$$

Thm: The complex

$$\cdots \to \bigoplus_{\text{odd}} \bar{F}_i \to \bigoplus_{\text{even}} \bar{F}_i \to \cdots \to \begin{matrix} \bar{F}_0 \\ \oplus \\ \bar{F}_2 \\ \oplus \\ \bar{F}_4 \end{matrix} \to \begin{matrix} \bar{F}_1 \\ \oplus \\ \bar{F}_3 \end{matrix} \xrightarrow{\begin{pmatrix} \bar\partial_1 & \\ \bar h_1 & \bar\partial_3 \end{pmatrix}} \begin{matrix} \bar{F}_0 \\ \oplus \\ \bar{F}_2 \end{matrix} \xrightarrow{\begin{pmatrix} \bar h_0 & \bar\partial_2 \end{pmatrix}} \bar{F}_1 \xrightarrow{\bar\partial_1} \bar{F}_0 \to 0$$

is a (not nec min'l) R-res of M. [Proof — another CARs talk. ☺]

Cor Every R-res obtained in this way is eventually 2-periodic [Pf: $\mathrm{Syz}^{n+1} M \in \underline{\mathrm{MCM}}(R)$]