

Outline:

- Background: a taste of homological algebra
- Another construction of matrix factorizations for MCMs
- From a matrix factorization to a free R -resolution
- More general construction (Shamash/Eisenbud)

§ Background [Can skim over definition of complex for this audience]

Let Q be any local ring.

For this talk, all complexes C will have the property that $C_i = 0$ when $i < 0$.

Consider a complex of Q -mods with differential ∂ .

If $\text{im } \partial_{j+1} = \text{ker } \partial_j \forall j \geq 1$, the complex is called an exact sequence.

If $\text{im } \partial_2 \neq \text{ker } \partial_1$, but $\text{im } \partial_{j+1} = \text{ker } \partial_j$ otherwise, the complex is called acyclic.

Morphisms of complexes Let C, D be cxes of Q -mods.

A morphism of cxes \mathcal{P} is a collection of Q -mod homomorphism

$\varphi_i: C_i \rightarrow D_i$ satisfying $\varphi_{i-1} \partial_i^C = \partial_i^D \varphi_i \forall i$.

(each square commutes:)

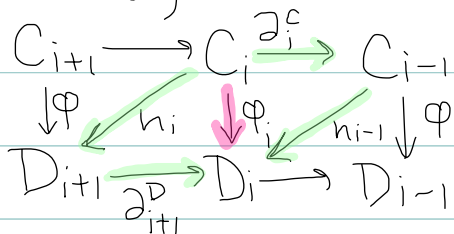
$$\begin{array}{ccc} C_i & \xrightarrow{\partial_i^C} & C_{i-1} \\ \downarrow \varphi_i & \circlearrowleft & \downarrow \varphi_{i-1} \\ D_i & \xrightarrow{\partial_i^D} & D_{i-1} \end{array}$$

[In ptc, a morphism is a degree 0 homomorphism of complexes.]

A morphism φ is said to be **nullhomotopic** if \exists a degree 1 homomorphism of cxes h (where $h_i: C_i \rightarrow D_{i+1}$) s.t.

$$\varphi = \partial^D h + h \partial^C$$

(schematic:)



Two morphisms are **homotopic** if their difference is nullhomotopic.

With the vocab of cxes above, for a \mathbb{Q} -mod M , a complex F is a free resolution of M if it is acyclic and $H_0(F) = M$, i.e., the complex

$$F \xrightarrow{\epsilon} M \rightarrow 0$$

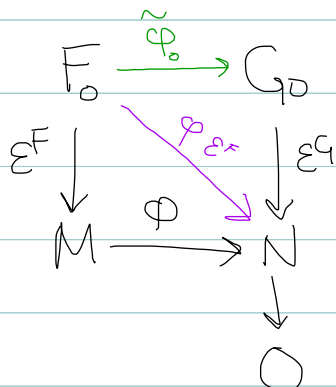
is exact.

Thm (Comparison theorem)

Let M and N be \mathbb{Q} -modules and $F \xrightarrow{\sim} M, G \xrightarrow{\sim} N$ free resolutions of M and N respectively. Given a \mathbb{Q} -module homomorphism $\varphi: M \rightarrow N$, there is a morphism of complexes $\tilde{\varphi}: F \rightarrow G$ that lifts φ and is unique up to homotopy.

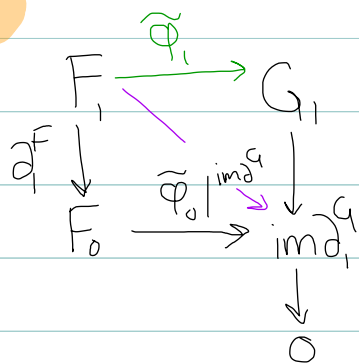
Pf: (Existence)

Step 1:



$\tilde{\varphi}_0$ exists because F_0 is projective.

finish (verify step 2)



Step 2 $\tilde{\varphi}_1$ lifts $\tilde{\varphi}_0|_{\text{im} \partial_1^G}$

RIS $\tilde{\varphi}_0 \partial_1^F(F_1) \subseteq \text{im} \partial_1^G = \text{ker} \varepsilon^G$

Well, let $x \in F_1$

$$\begin{array}{ccc} x & & \\ \downarrow & & \\ \partial_1^F(x) & \longmapsto & \tilde{\varphi}_0 \partial_1^F(x) \\ \downarrow & & \downarrow \varepsilon^G \\ \varepsilon^F \partial_1^F(x) = 0 & \longmapsto & 0 \end{array}$$

$$\dots \tilde{\varphi}_0 \partial_1^F \subseteq \text{ker} \varepsilon^G$$

* So $\tilde{\varphi}_1$ lifts $\tilde{\varphi}_0 \partial_1^F$ itself

Step n ... répetez!

uniqueness is more subtle (ignoring it for this talk!)

Where the talk starts getting interesting

Another construction of matrix factorizations

Let (S, \mathfrak{m}, k) be a regular local ring. Let $f \in \mathfrak{m}$, and set $R = S/(f)$.

Let $\text{MCM}(R)$ denote the category of MCM R -mods and let $\underline{\text{MCM}}(R)$ denote the category of MCM R -mods having no free direct summands.

basic fact about RLR:
 $0 \leq \text{pd}_S M \leq \text{edim } S$

Let $M \in \text{MCM}(R)$. Recall from Jason's talk that $\text{pd}_S M < \infty$. Therefore by the Auslander-Buchsbaum formula,

$$\text{pd}_S M + \text{depth}_S M = \text{depth}_S S = \text{dim } S$$

Claim: $\text{pd}_S M = 1$.

Facts: ① $\text{depth}_S N = \text{depth}_R N$ for any R -mod N
[BH]

$$\text{② } \text{depth}_S R = \text{depth}_S S - 1$$

Sketch: ② $f \in \mathfrak{m} \Rightarrow f$ is S -regular.

[we'll see this in Sn's class]

① f is a zero divisor on N

→ not rigorous (use Ext to show?)

So, substituting in (\star) ,

we have LHS:

$$\begin{aligned} \text{pd}_S M + \text{depth}_S M &= \text{pd}_S M + \text{depth}_R M \\ &= \text{pd}_S M + \text{depth}_R R \\ &= \text{pd}_S M + \text{depth}_S - 1 = \text{RHS} \\ &= \text{depth}_S \end{aligned}$$

This proves the claim.

Now let $F: 0 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ be a free res of M .

Consider the map $\cdot f: M \rightarrow M$. This is identical to the 0 map.

By the comparison theorem, there exists a morphism

$\tilde{\cdot f}: F \rightarrow F$ that lifts $\cdot f$ and is null homotopic:

$$\begin{array}{ccccccc} 0 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & 0 \\ & & \tilde{\cdot f}_1 \downarrow & \nearrow h_0 & \tilde{\cdot f}_0 \downarrow & & \\ 0 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & 0 \end{array}$$

where $\tilde{\cdot f}_0 = \partial_1 h_0 + 0$

and $\tilde{\cdot f}_1 = 0 + h_0 \partial_1$

Aha! Choosing a basis, (∂_1, h_0) is a matrix factorization of f .

Check: reduced $\Leftrightarrow M \in \underline{\text{MCM}}(R)$.

§ To a minimal free R -res

Let $\bar{\quad}$ denote $\square \otimes_S R$.

Thm. Let $M \in \text{MCM}(R)$. The complex

$$\widehat{F}: \quad \dots \xrightarrow{\bar{h}_0} \bar{F}_1 \xrightarrow{\bar{a}_1} \bar{F}_0 \xrightarrow{\bar{h}_0} \bar{F}_1 \xrightarrow{\bar{a}_1} \bar{F}_0 \rightarrow 0$$

is a min'l free R -res of M .

Proof We need to show a bunch of stuff:

① \widehat{F} is a complex

② $M = H_0(\widehat{F})$

③ $\ker \bar{a}_1 = \text{im } \bar{h}_0$

④ $\ker \bar{h}_0 = \text{im } \bar{a}_1$

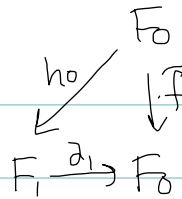
⑤ since $\partial_1 h_0 = f = h_0 \partial_1$, $\bar{\partial}_1 h_0 = \frac{0}{f} = \bar{h}_0 \bar{a}_1$.

For ②,

$$\begin{aligned} H_0(\widehat{F}) &= \text{coker } \bar{a}_1 = \overline{\text{coker } a_1} \\ &= M \otimes_S R = M. \end{aligned}$$

This means that M is actually coker of the differential of \widehat{F} at every other spot.

For (2), we use the triangle

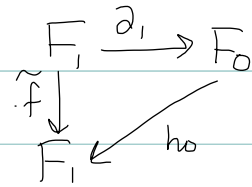


Let $x \in F_1$ st $\bar{x} \in \ker \bar{d}_1$.

$$\begin{aligned} \text{Then } \bar{d}_1(x) &= f \cdot y \text{ for some } y \in F_0, \\ &= \bar{d}_1(h_0(y)) \end{aligned}$$

$$\bar{d}_1 \text{ inj} \Rightarrow x = h_0(y) \quad \therefore \bar{x} \in \text{im } \bar{d}_1.$$

For (3) we use the other triangle



Let $x \in F_0$ st $\bar{x} \in \ker h_0$

$$\begin{aligned} \text{So } h_0(x) &= f \cdot y \text{ for some } y \in F_1 \\ &= h_0 \bar{d}_1(y) \end{aligned}$$

Now apply \bar{d}_1 :

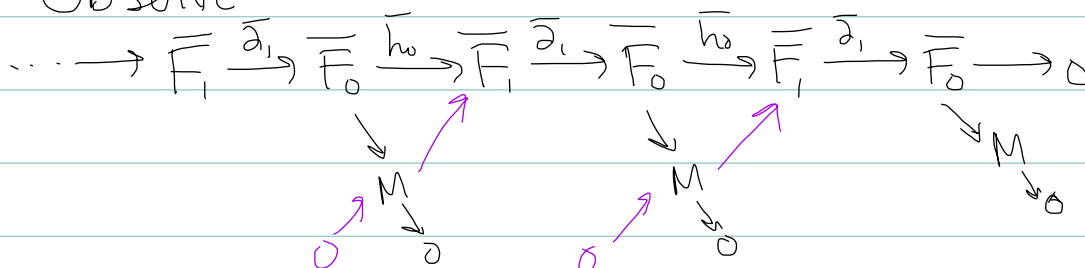
$$f x = \bar{d}_1 h_0(x) = \bar{d}_1 h_0 \bar{d}_1(y) = f \bar{d}_1(y)$$

and f nzd on $S \Rightarrow$

$$x = \bar{d}_1(y).$$

so $\bar{x} \in \text{im } \bar{d}_1$. □

Observe:



§ Generalization. [Shamash '69, Eisenbud '80]
 Let S, R as before. Let M be any R -mod and F a minimal free S -res of M :

$$F: 0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$$

By the comparison theorem, there is a morphism $\tilde{f}: F \rightarrow F$ that lifts $f: M \rightarrow M$ and is null homotopic:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & F_n & \rightarrow & F_{n-1} & \rightarrow & \dots & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & 0 \\ & & \swarrow 0 & \downarrow & \swarrow h_{n-1} & \downarrow & \swarrow h_{n-1} & \downarrow & \swarrow h_1 & \downarrow & \swarrow h_0 & \downarrow & \swarrow 0 \\ 0 & \rightarrow & F_n & \rightarrow & F_{n-1} & \rightarrow & \dots & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & 0 \end{array}$$

Thm: The complex

$$\begin{array}{ccccccc} \dots \rightarrow & \bigoplus_{\text{odd}} \bar{F}_1 & \rightarrow & \bigoplus_{\text{even}} \bar{F}_1 & \rightarrow & \dots \rightarrow & \bar{F}_0 \rightarrow \bar{F}_1 \rightarrow \bar{F}_2 \rightarrow \bar{F}_3 \rightarrow \bar{F}_2 \rightarrow \bar{F}_1 \rightarrow \bar{F}_0 \rightarrow 0 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \\ & & & & & & \nearrow \bar{h}_0 \quad \searrow \bar{h}_1 \end{array}$$

is a (not nec min'l) R -res of M . [proof - another CARs talk. 😊]

Cor Every R -res obtained in this way is eventually 2-periodic [PF: $\text{Syz}^n M \in \text{MCM}(R)$]