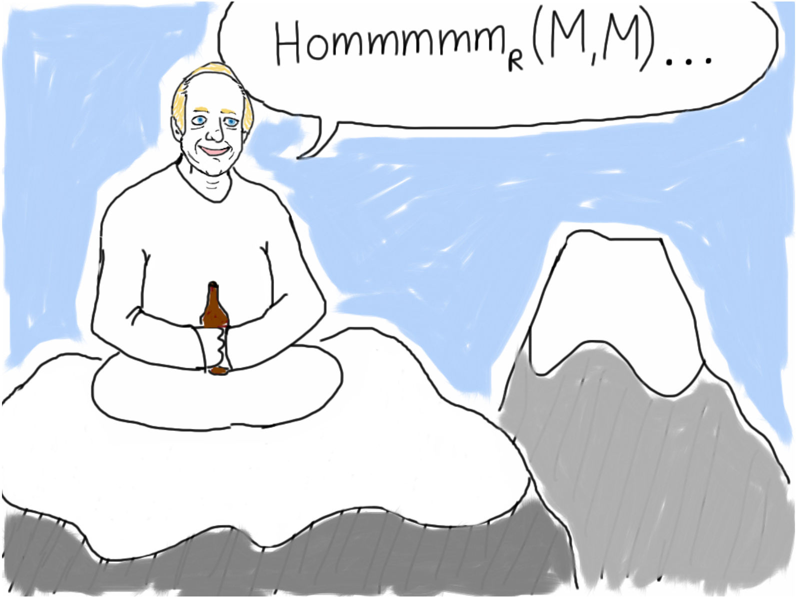
The Mountaintop Guru of Mathematics



New Directions in Boij-Söderberg Theory

The cone of Betti diagrams over a hypersurface ring of low embedding dimension

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Background

- Let R be a standard graded $\mathbb{k}$-algebra over a field $\mathbb{k}$.

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Definition

Given an R-module M, its *graded Betti numbers*, $\beta_{i,j}(M)$, are the number of degree j relations in homological degree i of a minimal free resolution of M.

$$
0 \longleftarrow
\begin{array}{c} \vdots \\ \oplus \\ R(1)^{\beta_{0,-1}(M)} \\ \oplus \\ R(0)^{\beta_{0,0}(M)} \\ \oplus \\ R(-1)^{\beta_{0,1}(M)} \\ \oplus \\ R(-2)^{\beta_{0,2}(M)} \\ \oplus \\ \vdots \end{array}
\longleftarrow
\begin{array}{c} \vdots \\ \oplus \\ R(1)^{\beta_{1,-1}(M)} \\ \oplus \\ R(0)^{\beta_{1,0}(M)} \\ \oplus \\ R(-1)^{\beta_{1,1}(M)} \\ \oplus \\ R(-2)^{\beta_{1,2}(M)} \\ \oplus \\ \vdots \end{array}
\longleftarrow
\begin{array}{c} \vdots \\ \oplus \\ R(1)^{\beta_{2,-1}(M)} \\ \oplus \\ R(0)^{\beta_{2,0}(M)} \\ \oplus \\ R(-1)^{\beta_{2,1}(M)} \\ \oplus \\ R(-2)^{\beta_{2,2}(M)} \\ \oplus \\ \vdots \end{array}
\longleftarrow \cdots
$$

Definition

The *Betti diagram of M* is a matrix with columns indexed by i and rows indexed by strands, with (i, j)th entry $\beta_{i,j+i}(M)$:

$$\beta(M) := \begin{pmatrix} \vdots & \vdots & \vdots & \\ {}^{*}\beta_{0,0}(M) & \beta_{1,1}(M) & \beta_{2,2}(M) & \cdots \\ \beta_{0,1}(M) & \beta_{1,2}(M) & \beta_{2,3}(M) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where the symbol * prepends the $(0, 0)$th entry.

Resolutions and Betti diagrams

Example

Let $R = \Bbbk[x, y]/\langle x^2 \rangle$. The module $M = R/\langle y^3 \rangle$ has a finite free resolution

$$0 \longleftarrow R \xleftarrow{(y^3)} R(-3) \longleftarrow 0.$$

In this example,

$$\beta(M) = \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^*1 & - & - & \cdots \\ - & - & - & \cdots \\ - & 1 & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Example

Let $R = \Bbbk[x,y]/\langle x^2 \rangle$. As an R-module, $\Bbbk$ has an infinite minimal free resolution, given by

$$0 \longleftarrow R \xleftarrow{(x,y)} R(-1)^2 \xleftarrow{\begin{pmatrix} -y & x \\ x & 0 \end{pmatrix}} R(-2)^2 \xleftarrow{\begin{pmatrix} x & 0 \\ y & x \end{pmatrix}} R(-3)^2 \xleftarrow{\begin{pmatrix} -y & x \\ x & 0 \end{pmatrix}} \cdots .$$

In this example,

$$\beta(\Bbbk) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ - & - & - & - & - & \cdots \\ {}^{*}1 & 2 & 2 & 2 & 2 & \cdots \\ - & - & - & - & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Example

Let $R = \mathbb{k}[x,y]/\langle x^2\rangle$ and let $N = R(-2)/\langle x\rangle \oplus \mathbb{k}$. A minimal free resolution of N is given by

$$0 \longleftarrow \begin{matrix} R \\ \oplus \\ R(-2) \end{matrix} \xleftarrow{(x,y,x)} \begin{matrix} R(-1)^2 \\ \oplus \\ R(-3) \end{matrix} \xleftarrow{\begin{pmatrix} -y & x & 0 \\ x & 0 & 0 \\ 0 & 0 & x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-4) \end{matrix} \longleftarrow \cdots .$$

We see that

$$\beta(N) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ - & - & - & - & - & \cdots \\ {}^{*}1 & 2 & 2 & 2 & 2 & \cdots \\ - & - & - & - & - & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ - & - & - & - & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The cone of Betti diagrams

Let $\mathbb{V}$ be the $\mathbb{Q}$-vector space of infinite matrices $(a_{i,j})$.

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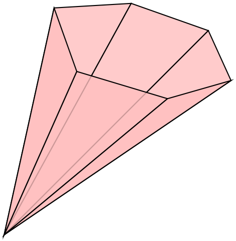
Definition

Define the *cone of Betti diagrams of finitely generated R-modules* to be

$$B_{\mathbb{Q}}(R) := \left\{ \sum_{\substack{M \text{ fg} \\ R\text{-mod}}} a_M \beta(M) \,\middle|\, \begin{array}{c} a_M \in \mathbb{Q}_{\geq 0}, \\ \text{almost all } a_M \text{ are zero} \end{array} \right\} \subseteq \mathbb{V}.$$



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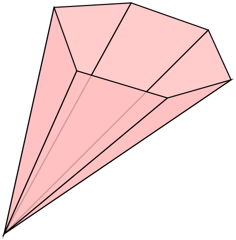

Goal

Describe $B_{\mathbb{Q}}(R)$.

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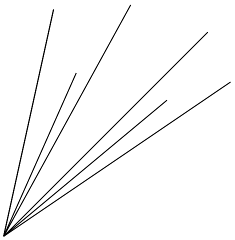

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The polynomial ring

(2008) Boij and Söderberg conjectured a description of the cone for Cohen–Macaulay modules over a graded polynomial ring.

(2009) Their conjecture was proved by Eisenbud and Schreyer. Boij and Söderberg found a description of the cone of all finitely generated modules over a polynomial ring.

What about when R has relations?

- Fix $S := \mathbb{k}[x, y]$.
- Let $q \in S$ be a homogeneous quadric polynomial.
- Set $R := S/\langle q \rangle$.

Definition

A finitely generated R-module M has a *pure resolution of type* $(d_0, d_1, d_2, \cdots)$, $d_i \in \mathbb{Z} \cup \{\infty\}$ if a minimal free R-resolution of M takes the following form

$$0 \longleftarrow R(-d_0)^{\beta_0} \longleftarrow R(-d_1)^{\beta_1} \longleftarrow R(-d_2)^{\beta_2} \longleftarrow \cdots$$

where $R(-\infty) := 0$.

$$\beta(M) = \begin{pmatrix} \vdots & \vdots & \vdots & \\ \beta_0 & & & \cdots \\ & \beta_1 & & \cdots \\ & & \beta_2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Example

Let $q = x^2$.

- The free module $R(-2)$ has a pure resolution of type $(2, \infty, \infty, \dots)$:

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- $\Bbbk$ has a pure resolution of type $(0, 1, 2, 3, \dots)$:
$$0 \leftarrow R \leftarrow R(-1)^2 \leftarrow R(-2)^2 \leftarrow R(-3)^2 \leftarrow \cdots$$

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- $N = R(-2)/\langle x \rangle \oplus \Bbbk$ does *not* have a pure resolution.

The main result

Theorem (BBEG 2011)

The cone $B_{\mathbb{Q}}(R)$ of the Betti diagrams of all finitely generated R-modules is the positive hull of Betti diagrams of free or finite length R-modules having pure resolutions of type

(i) $(d_0, \infty, \infty, \ldots)$ *with* $d_0 \in \mathbb{Z}$, *or*

(ii) $(d_0, d_1, \infty, \infty, \ldots)$ *with* $d_1 > d_0 \in \mathbb{Z}$, *or*

(iii) $(d_0, d_1, d_1 + 1, d_1 + 2, \ldots)$ *with* $d_1 > d_0 \in \mathbb{Z}$.

Definition

A free or finite length R-module is called *extremal* if its minimal free resolution is pure of type (i), (ii), or (iii).

Example

For $q = x^2$ and $N = R(-2)/\langle x \rangle \oplus \mathbb{k}$, we may write

$$\beta(N) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ - & - & - & - & - & \cdots \\ {}^*1 & 2 & 2 & 2 & 2 & \cdots \\ - & - & - & - & - & \cdots \\ 1 & 1 & 1 & 1 & 1 & \cdots \\ - & - & - & - & - & \cdots \\ - & - & - & - & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$=$$

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$$= \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^{*}1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} +$$

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$$= \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^{*}1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} + \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^{*}- & - & - & \cdots \\ - & - & - & \cdots \\ 1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} +$$

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$$= \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^*1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^*- & - & - & \cdots \\ - & - & - & \cdots \\ 1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} +$$

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For $q = x^2$ and $N = R(-2)/\langle x \rangle \oplus \mathbb{k}$, we may write

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$$= \begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^{*}1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \vdots & \vdots & \vdots & \\ - & - & - & \cdots \\ {}^{*}- & - & - & \cdots \\ - & - & - & \cdots \\ 1 & 2 & 2 & \cdots \\ - & - & - & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \vdots & \vdots & \\ - & - & \cdots \\ {}^{*}- & - & \cdots \\ - & - & \cdots \\ 1 & - & \cdots \\ - & - & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

Existence of extremal modules

Fix ℓ, a linear form that is not a scalar multiple of (WLOG) x and that does not divide q.
The modules $R(-d_0)$ and $R(-d_0)/\langle \ell^{d_1-d_0} \rangle$ are finite length modules with pure resolutions of type $(d_0, \infty, \infty, \dots)$ and $(d_0, d_1, \infty, \infty, \dots)$ respectively.

Type $(d_0, d_1, d_1 + 1, d_1 + 2, \ldots)$

It suffices to show the case when $d_0 = 0$.

Proposition

The R-module $M = R/\langle \ell^{d_1}, x\ell^{d_1 - 1} \rangle$ has a pure resolution of type $(0, d_1, d_1 + 1, d_1 + 2, \ldots)$.

Proof of the proposition

By hypothesis, as an S-module M is minimally presented by $R/\langle q, \ell^{d_1}, x\ell^{d_1-1}\rangle$. Applying the Hilbert–Burch theorem, M has a minimal free resolution

$$0 \longleftarrow S \longleftarrow \begin{matrix} S(-d_0-2) \\ \oplus \\ S(-d_1)^2 \end{matrix} \longleftarrow S(-d_1-1)^2 \longleftarrow 0.$$

Multiplication by q is 0 on M, so by the Comparison Theorem, multiplication by q homotopic to the zero map.

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & S(-2) & \longleftarrow & \begin{matrix} S(-4) \\ \oplus \\ S(-d_1-2)^2 \end{matrix} & \longleftarrow & S(-d_1-3)^2 & \longleftarrow & 0 \\
 & & \Big\downarrow {\scriptstyle \cdot q \sim 0} & & \Big\downarrow {\scriptstyle \cdot q \sim 0} & & \Big\downarrow {\scriptstyle \cdot q \sim 0} & & \\
0 & \longleftarrow & S & \longleftarrow & \begin{matrix} S(-2) \\ \oplus \\ S(-d_1)^2 \end{matrix} & \longleftarrow & S(-d_1-1)^2 & \longleftarrow & 0.
\end{array}
$$

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Fix a nullhomotopy for multiplication by q on this resolution:

$$
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 & & & \searrow & & \searrow & & & \\
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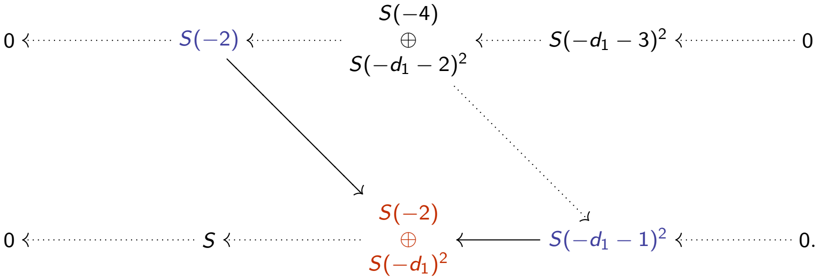

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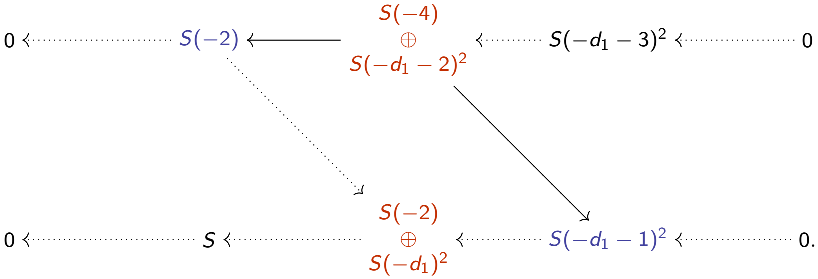

Tensoring with R, we obtain the diagram below.

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & R(-2) & \longleftarrow & \begin{matrix} R(-4) \\ \oplus \\ R(-d_1-2)^2 \end{matrix} & \longleftarrow & R(-d_1-3)^2 & \longleftarrow & 0 \\
 & & & \searrow & & \searrow & & & \\
0 & \longleftarrow & R & \longleftarrow & \begin{matrix} R(-2) \\ \oplus \\ R(-d_1)^2 \end{matrix} & \longleftarrow & R(-d_1-1)^2 & \longleftarrow & 0.
\end{array}
$$

Theorem (Shamash, 1969)

The resulting complex is a free R-resolution of M.

$$0 \leftarrow R \leftarrow \begin{matrix} R(-2) \\ \oplus \\ R(-d_1)^2 \end{matrix} \leftarrow \begin{matrix} R(-d_1-1)^2 \\ \oplus \\ R(-2) \end{matrix} \leftarrow \begin{matrix} R(-4) \\ \oplus \\ R(-d_1-2)^2 \end{matrix} \leftarrow \begin{matrix} R(-d_1-3)^2 \\ \oplus \\ R(-4) \end{matrix} \leftarrow \cdots .$$

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The maps $R(-2n) \leftarrow R(-2n)$ are the only nonminimal parts of this resolution.

Cancelling, we obtain a minimal R-free resolution for M:

$$0 \longleftarrow R \longleftarrow R(-d_1)^2 \longleftarrow R(-d_1-1)^2 \longleftarrow R(-d_1-2)^2 \longleftarrow R(-d_1-3)^2 \longleftarrow \cdots .$$

Cancelling, we obtain a minimal R-free resolution for M:

$$0 \longleftarrow R(0) \longleftarrow R(-d_1)^2 \longleftarrow R(-d_1-1)^2 \longleftarrow R(-d_1-2)^2 \longleftarrow R(-d_1-3)^2 \longleftarrow \cdots .$$

This resolution is pure of type $(0, d_1, d_1+1, d_1+2, \ldots)$. $\square$

Other directions?

Instead of generalizing to other rings, what about investigating the decomposition of Betti diagrams for classes of modules over the polynomial ring?

Example (MSRI Summer School, 2011)

Consider a complete intersection as a module over a graded polynomial ring. Do the degrees of the relations determine the rational coefficients in a Boij–Söderberg decomposition of its Betti table?

- Codimension one, two, and three: ✓
- Higher codimensions: under investigation (with J. Jeffries, S. Mayes, C. Raiciu, B. Stone, B. White)

Thanks!

"All of the difficult problems in life, philosophy, etc. can be decided by figuring out what is equal to zero, and why." –R. W.