Math918 Sp2012
1/23/2012 - M

Fix (R, m, k) a commutative, local Noeth. ring. Let M fg R-mod.

Koszul complexes:

① For any $r \in R$,

$$K(r;R) := 0 \to R \xrightarrow{\cdot r} R \to 0$$

$$K(r;M) := 0 \to M \xrightarrow{r} M \to 0$$

Koszul Homology: $H_i(r;M) = H_i(K(r;M))$

$H_0(r;M) = M/rM$

$H_1(r;M) = \{m \in M \mid rm = 0\} =: (0 :_M r)$

Note: $H_1(r;M) = 0 \iff r$ is a nzd on M

② Given a seq $r_1, \dots, r_n \in R$,

$$K(\underline{r};R) := (0 \to R \xrightarrow{r_1} R \to 0) \otimes_R \cdots \otimes_R (0 \to R \xrightarrow{r_n} R \to 0)$$

$$K(\underline{r};M) := K(\underline{r};R) \otimes_R M$$

Eg! $K(r,s;R) = (0 \to R \xrightarrow{r} R \to 0) \otimes_R (0 \to R \xrightarrow{s} R \to 0)$

$$= 0 \to R \xrightarrow[\partial_2]{\begin{bmatrix} -s \\ r \end{bmatrix}} R^2 \xrightarrow[\partial_1]{[r,s]} R \to 0$$

$\partial_1 \begin{bmatrix} x \\ y \end{bmatrix} = [r,s] \begin{bmatrix} x \\ y \end{bmatrix} = rx + sy$

$\partial_2 [z] = \begin{bmatrix} -sz \\ rz \end{bmatrix}$

$$K(r,s;M) = 0 \to M \xrightarrow{\begin{bmatrix} -s \\ r \end{bmatrix}} M^2 \xrightarrow{[r,s]} M \to 0$$

deg 2 1 0

$H_0(r,s;M) = M/(r,s)M$

$H_2(r,s;M) = \{m \in M \mid \begin{bmatrix} -sm \\ rm \end{bmatrix} = 0\} = \{m \in M \mid sm = 0 = rm\} = (0 :_M (r,s))$

Note: $H_2(r,s;M)=0 \iff \forall m\in M\setminus\{0\}\ \exists x\in(s,r)$ s.t. $xm\neq 0$

$\iff \exists x\in(r,s)$ that is a nzd on M.

$\Rightarrow$ is an exercise.
Needs associated primes.

Exercise: $H_1(r,s;M)=?$

$K(\underline{r};M)=K(\underline{r};R)\otimes_R M =$

$$0\to M\xrightarrow{\begin{bmatrix}-r_n\\ \vdots\\ \pm r_1\end{bmatrix}} M^n\to\cdots\to M^{\binom{n}{2}}\to M^n\xrightarrow{[r_1,\ldots,r_n]} M\to 0$$

deg $\quad n \quad n-1 \quad\quad 2 \quad 1 \quad 0$

$H_0(\underline{r};M)=M/(\underline{r})M$, $H_n(\underline{r};M)=(0:_M(\underline{r}))$

Fact: $H_i(r;M)=0 \iff (\underline{r})$ contains a nzd on M

$\iff (\underline{r})\not\subseteq \bigcup_{p\in \operatorname{Ass} M} p$

Defn: Given $I\subseteq R$ an ideal, depth$_R(I;M)$:= maximal length of an M-reg seq in I. By convention, $\operatorname{depth}_R(M):=\operatorname{depth}_R(\mathfrak{m};M)$.

Depth Sensitivity of Koszul Cxes

Let $\underline{r}=r_1,\ldots,r_n\subseteq\mathfrak{m}$ a seq of elts. Then

$$\operatorname{depth}_R((\underline{r});M)=n-\max\{i\geq 0\mid H_i(\underline{r};M)\neq 0\}$$

How much of the Koszul Cx has nonzero homology.

Rigidity of the Koszul Cx

If $H_i(\underline{r};M)=0$ for some $i\geq 0$, then $H_j(\underline{r};M)=0\ \forall j\geq i$

Fact: $(\underline{r})H_i(\underline{r};M)=0\ \forall i$

Recall: $H_0(\underline{r};M)=M/(\underline{r})M$, $H_n(\underline{r};M)=(0:_M(\underline{r}))$

Sketch of proof of depth sensitivity

Will prove $\max\{i\geq 0 \mid H_i(\underline{r};M)\neq 0\}=n-\operatorname{depth}_R((\underline{r});M)$ by induction on $\operatorname{depth}_R((\underline{r});M)$.

Base case: $\operatorname{depth}_R((\underline{r});M)=0 \iff H_n(\underline{r};M)\neq 0$

Induction step: Say $\operatorname{depth}_R((\underline{r});M)\geq 1$, and pick $r\in(\underline{r})$ nzd on M. Then $\exists$ SES of R-mods

$$0\to M\xrightarrow{r}M\longrightarrow M/rM\longrightarrow 0$$

This gives an exact seq of complexes by $-\otimes_R K(\underline{r};R)$:

$$0\longrightarrow K(\underline{r},M)\xrightarrow{r}K(\underline{r},M)\longrightarrow K(\underline{r};M/(r)M)\longrightarrow 0.$$

The homology LES reads

$$\longrightarrow H_i(\underline{r};M)\xrightarrow{r=0 \text{ by } ☆}H_i(\underline{r};M)\longrightarrow H_i(\underline{r};M/rM)\longrightarrow$$

$$H_{i-1}(\underline{r};M)\xrightarrow{r=0 \text{ by } ☆}H_{i-1}(\underline{r};M)\longrightarrow\cdots$$

so we obtain SESs

☆ $$0\longrightarrow H_i(\underline{r};M)\longrightarrow H_i(\underline{r};M/rM)\longrightarrow H_{i-1}(\underline{r};M)\longrightarrow 0$$

Since $\mathrm{depth}((\underline{r}); M/rM) = \mathrm{depth}_R(\underline{r}; M) - 1$,
the induction hypothesis implies

$$\max\{i \geq 0 \mid H_i(\underline{r}; M/\underline{r}M) \neq 0\} = n - (\mathrm{depth}_R((\underline{r}); M) - 1)$$
$$= n + 1 - \mathrm{depth}_R((\underline{r}); M)$$

So $H_j(r; M/rM) = 0 \ \forall j \geq n + 2 - \mathrm{depth}_R((\underline{r}); M)$.

$$\star \implies H_j(\underline{r}; M) = 0 \ \forall j \geq n + 1 \ \mathrm{depth}_R(\underline{r}; M).$$

Moreover,

$$H_{n+1-\mathrm{depth}_R((\underline{r}); M)}(\underline{r}; M/rM) \neq 0$$
$$\overset{\text{by } \star}{\cong}$$
$$H_{n-\mathrm{depth}_R((\underline{r}); M)}(\underline{r}; M) \qquad \square$$

<u>Cor</u> If $(r_1, \ldots, r_m) = \mathfrak{m}$, then

$$\mathrm{depth}_R M = n - \max\{i \geq 0 \mid H_i(\underline{r}; M) \neq 0\}.$$

<u>Weds</u>: Free resolutions

<u>Fri</u>: Auslander-Buchsbaum formula