1. Does the sequence $\{a_n\}_{n=1}^{\infty}$ converge or diverge? If it converges, find what the sequence converges to.

$$a_n = \frac{(n-1)!(n+1)}{n!}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n-1)!\,(n+1)}{n!} = \lim_{n\to\infty} \frac{(n-1)!\,(n+1)}{n\,(n-1)!}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1.$$

$\boxed{\text{The sequence converges to } 1.}$

2. Does the sequence $\sum_{k=3}^{\infty} \frac{1}{\pi^k}$ converge or diverge? If it converges, find its sum <u>exactly</u>.

$$\sum_{k=3}^{\infty} \frac{1}{\pi^k} = \frac{1}{\pi^3} + \frac{1}{\pi^4} + \frac{1}{\pi^5} + \frac{1}{\pi^6} + \cdots$$

This is a geometric series, with first term $a = \frac{1}{\pi^3}$ and multiplier $r = \frac{1}{\pi}$, so it can be written in the form

$$\sum_{k=0}^{\infty} ar^k, \quad \text{i.e.,} \quad \sum_{k=0}^{\infty} \left(\frac{1}{\pi^3}\right)\left(\frac{1}{\pi}\right)^k.$$

This series $\boxed{\text{converges}}$ because $|r| = \left|\frac{1}{\pi}\right| < 1$. The sum of the series is

$$\frac{a}{1-r} = \boxed{\frac{1/\pi^3}{1-1/\pi}}.$$

3. (a) Write the Taylor series for $f(x) = \ln(x)$ about $a = 1$.

$$
\begin{aligned}
f(x) &= \ln(x) & f(1) &= \ln(1) = 0 \\
f'(x) &= \frac{1}{x} = x^{-1} & f'(1) &= 1^{-1} = 1 \\
f''(x) &= -x^{-2} & f''(1) &= -1^{-2} = -1 \\
f^{(3)}(x) &= 2x^{-3} & f^{(3)}(1) &= 2 \cdot 1^{-3} = 2 \\
f^{(4)}(x) &= -6x^{-4} & f^{(4)}(1) &= -6 \cdot 1^{-4} = -6 \\
f^{(5)}(x) &= 24x^{-5} & f^{(5)}(1) &= 24 \cdot 1^{-5} = 24 \\
&\vdots & &\vdots
\end{aligned}
$$

Taylor series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \frac{0}{0!}(x-1)^0 + \frac{1}{1!}(x-1)^1 + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5 + \cdots$$

$$= \underbrace{0(x-1)^0}_{=0} + 1(x-1)^1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n}\right)(x-1)^n}.$$

3.(b) Find the number of terms in the Taylor series for $f(x) = \ln(x)$ about $a = 1$ that will guarantee an accuracy of at least 10^{-10} in the approximation of $\ln(1.1)$, and use this to approximate $\ln(1.1)$.

The Taylor series is an alternating series, so the error in using the nth partial sum as an approximation of the true sum is less than the absolute value of the $(n+1)$ st term; that is,

$$|R_n| < |a_{n+1}| = \left|(-1)^{n+2}\left(\tfrac{1}{n+1}\right)(x-1)^{n+1}\right| = \left(\tfrac{1}{n+1}\right)|x-1|^{n+1}.$$

We want the error for $x = 1.1$ to be less than 10^{-10}, so we want

$$\left(\tfrac{1}{n+1}\right)|1.1-1|^{n+1} \leq 10^{-10},$$

i.e.,

$$\left(\tfrac{1}{n+1}\right)(0.1)^{n+1} \leq 10^{-10}.$$

Note that $\left(\tfrac{1}{9+1}\right)(0.1)^{9+1} < 10^{-10}$, so $\boxed{n=9}$ works.

So

$$\ln(1.1) \approx \sum_{n=1}^{9} (-1)^{n+1}\left(\tfrac{1}{n}\right)(1.1-1)^n \approx \boxed{0.0953101798.}$$

4.(a) Determine whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+3}}$ converges absolutely, converges conditionally, or diverges. Justify your answer.

By the alternating series test:
- Alternating series ✓
- $\lim_{k\to\infty} \frac{(-1)^{k+1}}{\sqrt{k+3}} = 0$ ✓
- $\left|\frac{(-1)^{k+1}}{\sqrt{k+3}}\right| \geq \left|\frac{(-1)^{(k+1)+1}}{\sqrt{(k+1)+3}}\right|$ ✓

So the series converges.

Now consider the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+3}}$ (where all the terms are positive).

Integral test:

$$\int_1^{\infty} \frac{1}{\sqrt{x+3}}\,dx = \int_4^{\infty} \frac{1}{\sqrt{u}}\,du = \lim_{b\to\infty} \left(2\sqrt{u}\Big|_4^b\right)$$

$u = x+3$
$du = dx$

$$= \lim_{b\to\infty} \left(2\sqrt{b} - 4\right) = \infty$$

Since the integral diverges, the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+3}}$ diverges.

So $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k+3}}$ $\boxed{\text{converges conditionally.}}$

4.(b) Determine an $n \geq 1$ such that the difference between the sum S of the series given in part (a) and its nth partial sum $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{\sqrt{k+3}}$ is less than 0.001.

This is an alternating series, so, as in 3(b), we have

$$|R_n| < |a_{n+1}| = \left| \frac{(-1)^{(n+1)+1}}{\sqrt{(n+1)+3}} \right| = \frac{1}{\sqrt{n+4}}.$$

We want $|R_n| < 0.001$, so we want

$$\frac{1}{\sqrt{n+4}} < 0.001 = \frac{1}{1000},$$

so $\sqrt{n+4} > 1000$, i.e., $n+4 > 1\,000\,000$.

So $\boxed{n = 999\,997}$ works.

5. Determine the radius and the interval of convergence of the series given below.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+5)2^k}(x+1)^k$$

Using the ratio test with the absolute values of the terms:

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left|\frac{(x+1)^{k+1}}{(k+6)2^{k+1}}\right| \cdot \left|\frac{(k+5)2^k}{(x+1)^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{(x+1)^{k+1}}{(x+1)^k}\right| \cdot \left(\frac{2^k}{2^{k+1}}\right) \cdot \left(\frac{k+5}{k+6}\right)$$

$$= \lim_{k\to\infty} |x+1| \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{k+5}{k+6}\right)$$

$$= \frac{1}{2}|x+1| \lim_{k\to\infty} \left(\frac{k+5}{k+6}\right) \quad \sim \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \frac{1}{2}|x+1| \lim_{k\to\infty} \left(\frac{1}{1}\right) = \frac{1}{2}|x+1|.$$

In order to converge, we need:

$$\frac{1}{2}|x+1| < 1$$
$$|x+1| < 2$$
$$-2 < x+1 < 2$$
$$-3 < x < 1.$$

So the radius of convergence is $\boxed{\frac{3}{2}.}$

5. — cont'd.

Test endpoints: $\underline{x=-3}$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+5)2^k}(-2)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k(2)^k}{(k+5)\,2^k}$$

$$= \sum_{k=1}^{\infty} \frac{-1}{k+5}$$

Limit comparison test with $\sum \frac{1}{k}$:

$$\lim_{k\to\infty} \frac{1/k}{-1/(k+5)} = \lim_{k\to\infty}\left(-\frac{k+5}{k}\right) \overset{L'H}{=} -1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does $\sum_{k=1}^{\infty} \frac{-1}{k+5}$.

$\underline{x=1}$: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+5)2^k}(2)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+5}$

Alternating series test:
- Alternating series ✓
- $\lim_{k\to\infty} \frac{(-1)^{k+1}}{k+5} = 0$ ✓
- $\left|\frac{(-1)^{k+1}}{k+5}\right| \geq \left|\frac{(-1)^{(k+1)+1}}{(k+1)+5}\right|$ ✓

So $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+5}$ converges.

Therefore the interval of convergence is

$$\boxed{-3 < x \leq 1}.$$

6. Use the limit comparison test to determine whether the following series converges or not.

$$\sum_{k=1}^{\infty} \frac{3k^2+5k}{3^k(k^2+1)}$$

Limit comparison test with $\sum \frac{1}{3^k}$:

$$\lim_{k\to\infty} \frac{\frac{3k^2+5k}{3^k(k^2+1)}}{\frac{1}{3^k}} = \lim_{k\to\infty} \frac{3k^2+5k}{3^k(k^2+1)} \cdot 3^k$$

$$= \lim_{k\to\infty} \frac{3k^2+5k}{k^2+1}$$

$$= \lim_{k\to\infty} \frac{3+\frac{5}{k}}{1+\frac{1}{k^2}} \quad \left[\text{multiply by } \frac{1/k^2}{1/k^2}\right]$$

$$= 3.$$

Now $\sum_{k=1}^{\infty} \frac{1}{3^k}$ is a geometric series with $r = \frac{1}{3}$, and $\left|\frac{1}{3}\right| < 1$, so it converges. The limit comparison test shows that $\sum_{k=1}^{\infty} \frac{3k^2+5k}{3^k(k^2+1)}$ and $\sum_{k=1}^{\infty} \frac{1}{3^k}$ have the same behavior (since $0 < 3 < \infty$), so $\sum_{k=1}^{\infty} \frac{3k^2+5k}{3^k(k^2+1)}$ also $\boxed{\text{converges.}}$

7. Use a Taylor polynomial of order 2 for $f(x) = \cos x$ to compute $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

We know $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$,
so

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{1}{x^2}\right)\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \lim_{x\to 0} \left(-\frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots\right)$$

$$= \boxed{-\frac{1}{2}.}$$

(Note: Here we just used the whole Taylor series. If you wanted to use the Taylor polynomial of order 2, that would be just $1 - \frac{x^2}{2!}$.)

8. Determine whether the following series converge or diverge. Justify your answers.

(a) $\sum_{k=1}^{\infty} ke^{-k^2}$

Integral test:

$$\int_1^{\infty} xe^{-x^2}\,dx \underset{\substack{u=x^2 \\ du=2x\,dx \\ \frac{1}{2}du = x\,dx}}{=} \frac{1}{2}\int_1^{\infty} e^{-u}\,du$$

$$= \frac{1}{2}\lim_{b\to\infty}\left(-e^{-u}\Big|_1^b\right)$$

$$= \frac{1}{2}\lim_{b\to\infty}\left(-e^{-b} + e^{-1}\right)$$

$$= \frac{1}{2}\left(0 + e^{-1}\right) = \frac{1}{2e}.$$

Since the integral converges, so does the series.

8. (b) $\sum_{k=1}^{\infty} \frac{2+\cos k}{k^2}$

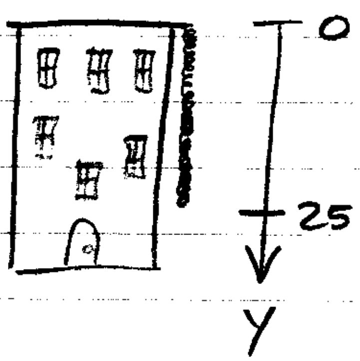
Direct comparison test with $\sum \frac{3}{k^2}$:

Since $-1 \le \cos k \le 1$ for all k,
we have $1 \le 2+\cos k \le 3$ for all k,
so
$$0 < \frac{2+\cos k}{k^2} \le \frac{3}{k^2} \quad \text{for all } k.$$

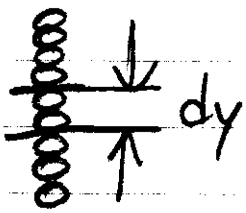
Furthermore, $\sum_{k=1}^{\infty} \frac{3}{k^2} = 3\sum_{k=1}^{\infty} \frac{1}{k^2}$, and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by the p-test (here $p = 2 > 1$), so $\sum_{k=1}^{\infty} \frac{3}{k^2}$ also converges.

Therefore $\sum_{k=1}^{\infty} \frac{2+\cos k}{k^2}$ $\boxed{\text{converges.}}$

9. A 25-foot chain weighs 4 pounds per foot and is hanging over the side of a building. How much work is done lifting the chain onto the roof?



We will divide up the chain into short pieces. Each piece has a constant weight and moves a constant distance (nearly).



For each short piece of the chain:

length = dy

so force = weight = $4\,dy$

distance = y

(since the piece at $y = 20$ has to be moved 20 feet, for example)

so work = force · distance = $4y\,dy$

Now, adding all the pieces together with an integral:

$$W = \int_0^{25} 4y\,dy = 2y^2\Big|_0^{25} = \boxed{1250 \text{ ft-lb}}$$

Don't forget appropriate units!