Practice Exam 1 Solutions

1a) $\int_0^\infty \frac{x}{e^x}dx = \lim_{t\to\infty}\int_0^t \frac{x}{e^x}dx = \lim_{t\to\infty} \frac{-x}{e^x}\Big|_0^t - \int_0^t \frac{-1}{e^x}dx$

$u = x \quad dv = \frac{dx}{e^x} = e^{-x}dx$

$du = dx \quad v = -e^{-x} = \frac{-1}{e^x}$

$= \lim_{t\to\infty} \frac{-t}{e^t} - \left(\frac{-0}{e^0}\right) - \frac{1}{e^x}\Big|_0^t = \lim_{t\to\infty} \frac{-t}{e^t} - \frac{1}{e^t} + \frac{1}{e^0}$

$= 0 - 0 + 1 = \boxed{1 \quad \text{converges}}$

1b) $\int_{-2}^3 \frac{1}{\sqrt[3]{x^4}}dx = \lim_{t\to 0^-}\int_{-2}^t \frac{1}{\sqrt[3]{x^4}}dx + \lim_{s\to 0^+}\int_s^3 \frac{1}{\sqrt[3]{x^4}}dx$

$\sqrt[3]{x^4} = x^{4/3}$

$= \lim_{t\to 0^-} \frac{-3}{\sqrt[3]{x}}\Big|_{-2}^t + \lim_{s\to 0^+} \frac{-3}{\sqrt[3]{x}}\Big|_s^3$

$= \lim_{t\to 0^-} -\left(\frac{3}{\sqrt[3]{t}}\right) + \frac{3}{\sqrt[3]{-2}} + \lim_{s\to 0^+} \frac{-3}{\sqrt[3]{3}} + \frac{3}{\sqrt[3]{s}}$

$= -(-\infty) + \frac{3}{\sqrt[3]{-2}} - \frac{3}{\sqrt[3]{3}} + \infty = \boxed{+\infty}$

$\boxed{\text{Diverges}}$

2) $\frac{dy}{dt} = K(y - T_0)$

$y(t) = T_0 + Ce^{kt}$

$T_0 = 50°F$

$y(0) = 150° = 50 + C(e^0)$

$150 = 50 + C$

$100 = C$

$y(10) = 120 = 50 + 100e^{10k}$

$70 = 100e^{10k}$

$.7 = e^{10k}$

$\ln(.7) = 10K$

$K = \frac{1}{10}\ln(.7)$

$y(t) = 50 + 100e^{(\frac{1}{10}\ln .7)t}$

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2 cont) When does $y(t) = 99°F$?

$$99 = 50 + 100e^{(\frac{1}{10}\ln .7)t} \qquad \text{solve for } t$$

$$49 = 100e^{(\frac{1}{10}\ln .7)t}$$

$$.49 = e^{(\frac{1}{10}\ln .7)t}$$

$$\ln .49 = \left(\frac{1}{10}\ln .7\right)t$$

$$\boxed{t = \frac{10\ln(.49)}{\ln(.7)}}$$

(notice this is positive)

3a) $\frac{dy}{dx} = x^2\sqrt{y}$, $y > 0$

$$\frac{dy}{\sqrt{y}} = x^2 dx$$

$$\int \frac{dy}{\sqrt{y}} = \int x^2 dx$$

$$\frac{2}{1} y^{\frac{1}{2}} = \frac{x^3}{3} + C \quad (C \text{ is constant})$$

$$\sqrt{y} = \frac{x^3}{6} + \frac{C}{2}$$

$$y = \left(\frac{x^3}{6} + \frac{C}{2}\right)^2$$

$$y = \left(\frac{x^3}{6} + D\right)^2 \quad (D \text{ is constant})$$

3b) $\sec x \frac{dy}{dx} = e^{y + \sin x}$

$$\frac{dy}{dx} = \frac{e^y e^{\sin x}}{\sec x}$$

$$\frac{dy}{e^y} = e^{\sin x} \cos x \, dx$$

$$\int \frac{dy}{e^y} = \int e^{\sin x} \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$\frac{-1}{e^y} = e^{\sin x} + C$$

$$e^y = \frac{-1}{e^{\sin x} + C} = \frac{1}{-e^{\sin x} - C}$$

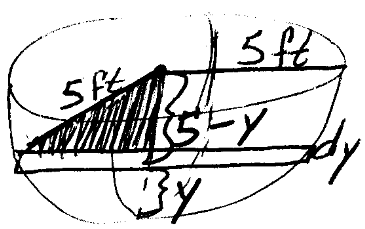
$$y = \ln\left|\left(\frac{+1}{-e^{\sin x} \mp C}\right)\right|$$

$$y = \ln|+1| - \ln|e^{\sin x} \mp C|$$

$$y = 0 - \ln|e^{\sin x} \mp C|$$

$$y = -\ln|e^{\sin x} \mp C|$$

4)



ⓐ Tank is full of water

Cross section is a disk
For a cross section a distance y from the bottom it's radius is one leg of the right triangle shaded.

$$\text{Radius} = r(y) = \sqrt{25-(5-y)^2}$$
$$= \sqrt{25-(25-10y+y^2)}$$
$$= \sqrt{10y-y^2}$$

The cross section weighs

$$\pi r^2 \underset{\uparrow \text{density}}{d} = \pi(10y-y^2)62.4 \text{ pounds}$$

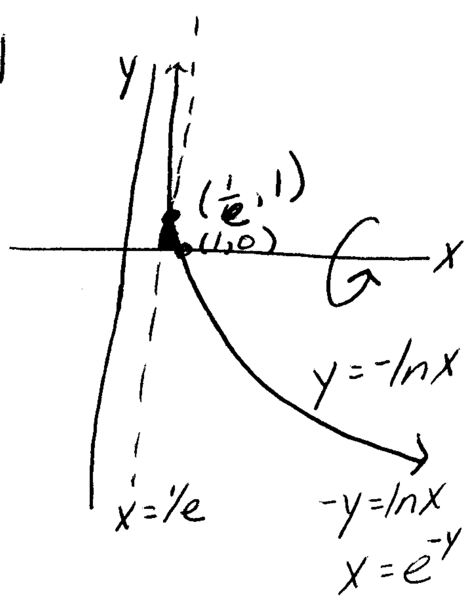
The cross section will travel the distance $(5-y)$ to the top of the tank

$$\text{Work} = \int_0^5 \pi(10y-y^2)62.4(5-y)\,dy$$

ⓑ Water in tank is 3 ft deep

$$\text{Work} = \int_0^3 \pi(10y-y^2)62.4(5-y)\,dy$$

5)

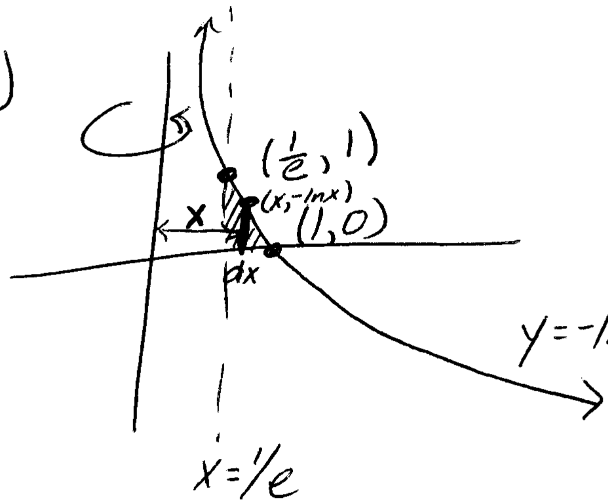


use disk or slices - cuts perpendicular to axis of rotation (x axis)
Thickness of slice is dx

$$\int_{1/e}^{1} \pi(-\ln x)^2\,dx$$

Note: It could also be done w/ shells

$$\int_0^1 2\pi y\left(e^{-y}-\tfrac{1}{e}\right)dy$$

5b) 

Shell method - shells are parallel to axis of rotation. (y-axis). Thickness of a shell is dx

$$\boxed{\int_{1/e}^{1} 2\pi \underset{\substack{\uparrow \\ \text{Radius}}}{x} \underset{\substack{\uparrow \\ \text{Height}}}{(-\ln x)}\, dx}$$

AGAIN: Could have been done by the disk (washer) method $\int_0^1 \pi\left((e^{-y})^2 - \left(\frac{1}{e}\right)^2\right) dy$

6) Use the Fundamental theorem of Calculus

$$\frac{d}{dx}\int_{-x}^{2x^3} e^t \cos\sqrt{t^2-4} + 1\, dt$$

chain Rule $\frac{d}{dx}(2x^3)$

$$= \left(e^{2x^3}\cos\sqrt{(2x^3)^2-4} + 1\right)(6x^2) - \left(e^{-x}\cos\sqrt{(-x)^2-4}+1\right)(-1)$$

$$= \boxed{\left(e^{2x^3}\cos\sqrt{4x^6-4} + 1\right)(6x^2) + \left(e^{-x}\cos\sqrt{x^2-4}+1\right)}$$

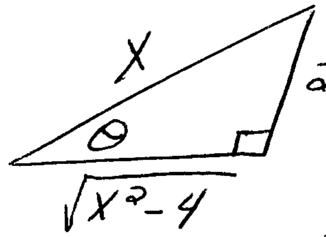
7a) $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x\,(1-\sin^2 x)\cos x\, dx$

$= \int (\sin^2 x - \sin^4 x)\cos x\, dx = \int u^2 - u^4\, du$

$u = \sin x$
$du = \cos x\, dx$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}$$

7b) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$



$\csc\theta = \frac{x}{2}$

$2\csc\theta = x$

$2\csc\theta\cot\theta\, d\theta = dx$

$\sin\theta = \frac{2}{x}$ $\qquad \tan\theta = \frac{2}{\sqrt{x^2-4}}$

$\sin^2\theta = \frac{4}{x^2}$ $\qquad \frac{1}{2}\tan\theta = \frac{1}{\sqrt{x^2-4}}$

$\frac{1}{4}\sin^2\theta = \frac{1}{x^2}$

$\sin^2\theta\csc\theta = \sin\theta$

$$\int\left(\frac{1}{x^2}\right)\left(\frac{1}{\sqrt{x^2-4}}\right)dx = \int \frac{1}{4}\sin^2\theta\left(\frac{1}{2}\tan\theta\right)2\csc\theta\cot\theta\, d\theta$$

$\tan\theta\cot\theta = 1$

$$= \frac{1}{4}\int \sin\theta\, d\theta = -\frac{1}{4}\cos\theta + C$$

$$= \boxed{-\frac{1}{4}\left(\frac{\sqrt{x^2-4}}{x}\right) + C}$$

4) $\displaystyle\int_1^\infty \frac{x-1}{x^2+x+1}\,dx$ for large values of x our function acts like $\frac{x}{x^2} = \frac{1}{x}$

$f(x) = \frac{x-1}{x^2+x+1}$ $\qquad g(x) = \frac{1}{x}$ $\qquad \displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{x-1}{x^2+x+1}}{\frac{1}{x}}$

$$= \lim_{x\to\infty}\frac{x(x-1)}{x^2+x+1} = \lim_{x\to\infty}\frac{x^2-x}{x^2+x+1} = 1$$

$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges $\boxed{\text{So } \displaystyle\int_1^\infty \frac{x-1}{x^2+x+1}\,dx \text{ also diverges.}}$

8b) $\int_0^\infty e^{-x^2}dx = \int_0^\infty \frac{1}{e^{x^2}}\,dx = \int_0^1 \frac{1}{e^{x^2}}dx + \int_1^\infty \frac{1}{e^{x^2}}dx$

e^{x^2} grows very fast so intuition tells us the integral will converge, but we need to show this

$f(x) = \frac{1}{e^{x^2}}$ For large enough x $e^{x^2} > x^2$

So $\frac{1}{e^{x^2}} < \frac{1}{x^2} = g(x)$

$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} \frac{-1}{\cancel{x}} \Big|_1^t = \lim_{t\to\infty} \frac{-1}{t} + 1$

$= 1.$ So $\int_1^\infty e^{-x^2}dx < \int_1^\infty \frac{1}{x^2}dx = 1$

converges

$e^{-x^2} = \frac{1}{e^{x^2}}$ is continuous on the interval $[0,1]$.

So $\int_0^1 e^{-x^2}dx$ is finite (i.e. the integral converges)

Together we get $\boxed{\int_0^\infty e^{-x^2}dx \text{ converges.}}$

I could not do $\int_0^\infty \frac{1}{x^2}dx$ to compare because as x gets close to 0, the graph $\frac{1}{x^2}$ blows up and then we still have $\frac{1}{x^2} > \frac{1}{e^{x^2}}$.

But $\int_0^\infty \frac{1}{x^2}dx$ diverges so it doesn't help us.

Note: Denominator does NOT factor

9a) $$\int \frac{16w}{w^2-2w+17}\,dw = \int \frac{16w-16+16}{w^2-2w+17}\,dw$$

$$= 8\int \frac{2w-2}{w^2-2w+17}\,dw + 16\int \frac{1}{w^2-2w+17}\,dw$$

$\uparrow u = w^2-2w+17$
$du = 2w-2\,dw$

$w^2-2w+17 = w^2-2w+1+16$
$= (w-1)^2+4^2$
$= 16\left(\left(\frac{w-1}{4}\right)^2+1\right)$

$$= 8\ln|w^2-2w+17| + \frac{16}{16}\int \frac{1}{\left(\frac{w-1}{4}\right)^2+1}\,dw$$

$\uparrow u = \left(\frac{w-1}{4}\right)$
$du = \frac{1}{4}\,dw$

$$= 8\ln|w^2-2w+17| + 4\int \frac{1}{u^2+1}\,du$$

$$= 8\ln|w^2-2w+17| + 4\tan^{-1}(u) + C$$

$$= \boxed{8\ln|w^2-2w+17| + 4\tan^{-1}\left(\frac{w-1}{4}\right) + C}$$

9b) The denominator factors $(x+5)(x-2)$

$$\int \frac{5x+4}{(x+5)(x-2)}\,dx = \int \frac{A}{x+5}\,dx + \int \frac{B}{x-2}\,dx$$

$$= A\ln|x+5| + B\ln|x-2| + C$$

$$\frac{A}{x+5} + \frac{B}{x-2} = \frac{A(x-2)+B(x+5)}{(x+5)(x-2)} = \frac{5x+4}{(x+5)(x-2)}$$

$$(A+B)x - 2A + 5B = 5x+4$$

$A+B=5$
$-2A+5B=4$
$7B=14$
$B=2$ $A=3$

So $\int \frac{5x+4}{(x+5)(x-2)}\,dx = \boxed{3\ln|x+5| + 2\ln|x-2| + C}$

10) $\int_{-1}^{2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\int_{-1}^{2} \sqrt{1+(4x^3)^2}\,dx}$

$y = x^4$

$\frac{dy}{dx} = 4x^3$

11) $\left|\text{MAX error}\right| = \frac{1}{2}\left| f(b) - f(a)\right| \frac{b-a}{n}$

where f is a monotone function on the interval $[a,b]$ and we are using Trapezoidal rule with n trapezoids.

$\frac{1}{4}10^{-8} \geq \frac{1}{2}\left|\frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{1}{\pi}\sin(0)\right|\left(\frac{\frac{\pi}{4}-0}{n}\right)$

$= \frac{1}{2}\left(\frac{1}{\pi}\right)\left(\frac{\pi}{4n}\right) = \frac{1}{8n}$

So $n \geq \frac{4\times10^8}{8} = \frac{10^8}{2} = \frac{100000000}{2}$

$= 50000000 = \boxed{5\times10^7}$