Math Excel 106 **Worksheet 6** Fall 2007 — Thursday, 09/13/2007

1. For each part below, find the slope of the function's graph at $x = a$. Then find an equation for the line tangent to the graph there, and sketch the curve and tangent together.

   (a) $f(x) = 3x^2 - 6x + 1, \quad a = 2$

   (b) $g(x) = \dfrac{2}{x+1}, \quad a = 1$

   (c) $h(x) = \sqrt{x^2+1}, \quad a = 1$

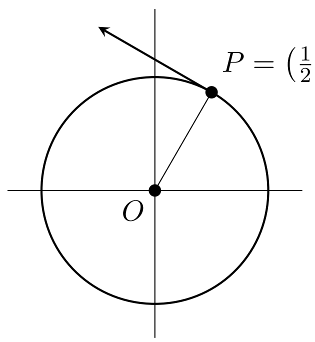
   (d) $y(x) = \dfrac{x+1}{x-1}, \quad a = 3$

   (e) $\varphi(x) = x^{-3}, \quad a = 1$

   (f) $\psi(x) = (x^2+1)^{-1}, \quad a = 0$

2. On a circle with center O, finding a tangent line at a point P can be accomplished by finding the equation of a line which (i) passes through the point P and (ii) is perpendicular to the line segment OP.

   A rock on the end of a string one meter in length is being swung around in a circle. (Imagine that this is being done on an ice rink, with the circle lying flat on the ice, so that gravity has no effect on the motion of the rock and the effects of friction can be ignored.) Let the center O of the circle be the point $(0, 0)$ in the plane, and suppose that the rock is traveling counterclockwise. When the rock reaches the point $P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, the string breaks.



   (a) Find the equation of the line describing the path of the rock after the string breaks.

   (b) Supposing that the rock was traveling around the circle at a constant rate of 1.2 meters per second, what are the coordinates of the rock 3 seconds after the string breaks?

   (c) When will the rock be a distance of 20 meters from the point O?

3. What is the rate of change of the volume of a ball $\left(V = (4/3)\pi r^3\right)$ with respect to the radius when the radius is $r = 2$?

4. What is the value of

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}?$$

   (Recall that 100!, read "100 factorial", is the number $100 \times 99 \times 98 \times \cdots \times 3 \times 2 \times 1$.)

5. The function $f(\theta) = (\sin\theta)/\theta$ is not defined at $\theta = 0$. (Why?) Is it possible to define $f(0)$ in such a way that f is continuous at $\theta = 0$? Explain your answer.

6. Draw a careful graph of the function
$$f(x) = \frac{x(x^2-1)}{|x^2-1|}.$$
Note that $f(x)$ is not defined at $x = 1$ or -1. Can $f(x)$ be extended so as to be continuous at $x = \pm 1$? Explain your answer.

7. Graph the function $f(x) = 2x^3 - 3x^2 - 12x + 3$. Use the graph to guess the values of x at which the line tangent to the graph has a slope of 0. Then prove that the slope of the tangent line at these values of x really is 0. In what way are these values of x important?

8. Let $f(x) = x^3 - 2x^2 + 7x - 1$. Find
$$\lim_{x\to 5} f(x).$$
How did you find this limit? What property of the function $f(x)$ allows you to find the limit in this way? Why? How do you know this function has this property?

9. Consider the function $g(x) = x + 3$. Show that
$$\lim_{x\to 0} g(x) = 3.$$
On the other hand, we can write $g(x) = (x^2 + 3x)(1/x)$, so
$$\lim_{x\to 0} g(x) = \lim_{x\to 0} (x^2+3x)\left(\frac{1}{x}\right).$$
Now $\lim_{x\to 0}(x^2 + 3x) = 0$, and we know that 0 times any number is 0, so we can say that
$$\lim_{x\to 0} g(x) = \lim_{x\to 0} (x^2+3x)\left(\frac{1}{x}\right) = 0 \times \text{something} = 0.$$
But we know that $\lim_{x\to 0} g(x) = 3$, so this is a contradiction! What is wrong here? (Remember this example, because this is a common mistake.)

10. Four people come across a bridge on a moonless night, which they need to cross. After examining the rickety structure, they decide that no more than two people should risk crossing at once, for fear of overloading the bridge. However, they have only one flashlight among them, which they need to have to cross the bridge in order to avoid stepping on rotten planks or falling through gaping holes into the river. Furthermore, the four people require different amounts of time to cross the bridge: the fastest can cross in one minute, the slowest needs ten minutes, and the others need two minutes and five minutes, respectively. When two people cross the bridge together, they must walk at the pace of the slower person.

    They realize that two people must cross the bridge first, and then one of these people needs to return across the bridge with the flashlight; the flashlight must be shuttled back and forth in this fashion until everyone has safely crossed. How can the four people cross the bridge in the minimum amount of time?