Math Excel 106 **Worksheet 5** Fall 2007 — Tuesday, 09/11/2007

1. Work this problem without using a calculator.

   (a) Find the limit of each expression below as $x \to \infty$ and as $x \to -\infty$.

   $$\text{(i)}\quad \frac{x+7}{5x-2} \qquad\qquad \text{(ii)}\quad \frac{x+7}{5x^2-2} \qquad\qquad \text{(iii)}\quad \frac{x^2+7}{5x-2}$$

   (b) For parts (i), (ii), and (iii) above, find those functions with vertical asymptotes, and identify each asymptote. Do likewise with horizontal asymptotes.

   (c) For each function above, identify the coordinates of any points at which it crosses the x-axis and any points at which it crosses the y-axis (that is, find the x- and y-intercepts of the function).

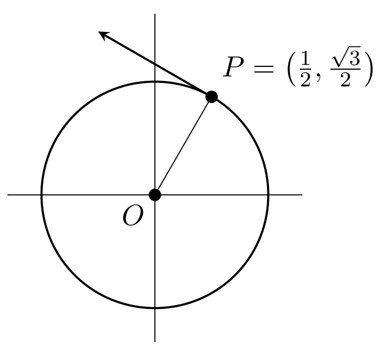
   (d) Graph each function above (without using a calculator!).

2. We know that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided that $M \neq 0$.

   If $M = 0$ and $L \neq 0$, what could $\lim_{x\to a} \frac{f(x)}{g(x)}$ be? What if both $M = 0$ and $L = 0$? Can you give any examples supporting your conclusions?

3. On a circle with center O, finding a tangent line at a point P can be accomplished by finding the equation of a line which (i) passes through the point P and (ii) is perpendicular to the line segment OP.

   A rock on the end of a string one meter in length is being swung around in a circle. (Imagine that this is being done on an ice rink, with the circle lying flat on the ice, so that gravity has no effect on the motion of the rock and the effects of friction can be ignored.) Let the center O of the circle be the point $(0, 0)$ in the plane, and suppose that the rock is traveling counterclockwise. When the rock reaches the point $P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, the string breaks.



   (a) Find the equation of the line describing the path of the rock after the string breaks.

   (b) Supposing that the rock was traveling around the circle at a constant rate of 1.2 meters per second, what are the coordinates of the rock 3 seconds after the string breaks?

   (c) When will the rock be a distance of 20 meters from the point O?

4. You put 100 rabbits into an enclosure at time $t = 0$. The number of rabbits after t years have elapsed is modeled by the function

$$p(t) = \frac{1000}{10 - t}.$$

Find the rabbit population after 7 years, 8 years, and 9 years. What catastrophic event does the population model $p(t)$ predict will happen in the not-so-distant future? How can you express this event mathematically?

5. What is the value of

$$\frac{1}{\log_2 100!} + \frac{1}{\log_3 100!} + \frac{1}{\log_4 100!} + \cdots + \frac{1}{\log_{100} 100!}?$$

(Recall that 100!, read "100 factorial", is the number $100 \times 99 \times 98 \times \cdots \times 3 \times 2 \times 1$.)

6. Are the following true or false? If false, state a correct version.

   (a) The function $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal.

   (b) The function $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$.

   (c) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist, then f has a removable discontinuity at $x = a$.

   (d) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.

   (e) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)/g(x)$ is continuous at $x = a$.

7. Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict the theorem below (Theorem 10, page 108)? Give reasons for your answer.

   THEOREM 10—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

8. (a) Find a function $f(x)$ that has a horizontal asymptote at $y = 2$ and a vertical asymptote at $x = -3$.

   (b) Find a function $g(x)$ such that $\lim_{x \to 1^+} g(x) = -\infty$ and $\lim_{x \to 1^-} g(x) = 1$.

   (c) Find a function $h(x)$ that is continuous at every point on the real line except $x = 2$ and $x = -5$.

   (d) Are the discontinuities in your function $h(x)$ removable or not? (How do you know?) Can you come up with another function that meets the requirements of part (c) for which you can answer this question in the opposite way?

   (e) Find a function $\varphi(x)$ that is defined for every value of x but which is not continuous at any value of x. (Hint: Between any two rational numbers there is an irrational number, and between any two irrational numbers there is a rational number.)

9. The function $f(\theta) = (\sin\theta)/\theta$ is not defined at $\theta = 0$. (Why?) Is it possible to define $f(0)$ in such a way that f is continuous at $\theta = 0$? Explain your answer.

10. Let $f(x) = x^3 - 2x^2 + 7x - 1$. Find
$$\lim_{x \to 5} f(x).$$
How did you find this limit? What property of the function $f(x)$ allows you to find the limit in this way? Why? How do you know this function has this property?

11. Consider the function $g(x) = x + 3$. Show that
$$\lim_{x \to 0} g(x) = 3.$$
On the other hand, we can write $g(x) = (x^2 + 3x)(1/x)$, so
$$\lim_{x \to 0} g(x) = \lim_{x \to 0} (x^2 + 3x) \left(\frac{1}{x} \right).$$
Now $\lim_{x \to 0} (x^2 + 3x) = 0$, and we know that 0 times any number is 0, so we can say that
$$\lim_{x \to 0} g(x) = \lim_{x \to 0} (x^2 + 3x) \left(\frac{1}{x} \right) = 0 \times \text{something} = 0.$$
But we know that $\lim_{x \to 0} g(x) = 3$, so this is a contradiction! What is wrong here? (Remember this example, because this is a common mistake.)

12. Find the following limits. Justify your answers.
    (a) $\lim_{x \to 2} (x^2 - 4x + 4) \left(\cos \dfrac{2x+3}{x^2 + x - 2} \right)$
    (b) $\lim_{z \to \infty} z \sin \left(\dfrac{1}{z} \right)$
    (c) $\lim_{t \to 1^+} \dfrac{2t - 7}{t - 1}$
    (d) $\lim_{t \to 1^-} \dfrac{2t - 7}{t - 1}$
    (e) $\lim_{\theta \to 3\pi/2} \sin \left(\theta + \dfrac{\pi}{2} \right) \tan(\theta)$

13. Three men check into a hotel and ask what the nightly rate is. The clerk says a room costs 30 dollars a night, so each of the men gives the clerk ten dollars, and they head up to the room.

    A while later, the clerk realizes he overcharged the men; the room they are staying in is only 25 dollars a night. So he takes five one-dollar bills from the cash box and hands them to the bellhop with instructions to return the money.

    On the way up to the room, the bellhop realizes that five dollars cannot be split evenly among three men, so he pockets two dollars and returns three dollars to the men.

    Now, each of the men initially paid ten dollars for the room, but later received a dollar back, so effectively each man paid nine dollars. In total, then, the room cost the men 27 dollars. With the two dollars the bellhop kept, this comes to 29 dollars. What happened to the missing dollar?