**Exam 1 review key**
**Math Excel, fall 2007**

1. Determine the following limits.

(a) $\lim_{x\to 1^+} \frac{1-3x}{x-1}$

As $x$ approaches 1, the numerator of this fraction approaches $-2$, and the denominator approaches 0. In other words, if $x$ is very close to 1, the numerator will be very close to $-2$, and the denominator will be very close to 0. When we consider the fraction as a whole, then, we see that we are dividing a number very close to $-2$ by a very small number, which means that the value of the fraction will be very large. As the denominator gets even smaller, the value of the fraction will become even larger, without bound. Therefore, the limit of the entire fraction is infinite. We now need to decide if this limit is $+\infty$ or $-\infty$.

We are looking for the limit as $x$ approaches 1 from the right. Look at the signs of the numerator and the denominator in this case. If $x$ is a number that is very close to 1, but is just a little bit bigger than 1, then the numerator will be negative ($1-3x$ will be very close to $-2$), but the denominator will be positive ($x-1$, while being very close to zero, will always be positive if $x$ is a little bigger than 1). So overall, if $x$ is a little bigger than 1, the fraction will be negative (negative divided by positive is negative). Hence we must have

$$\lim_{x\to 1^+} \frac{1-3x}{x-1} = -\infty.$$

Note that technically this limit does not exist, because there is not a *real number* that the fraction is approaching as $x$ approaches 1. However, it is meaningful to say that this limit "is" $-\infty$.

(b) $\lim_{x\to 1^-} \frac{1-3x}{x-1}$

Since this is exactly the same problem as in part (a), except that we are considering the limit as $x$ approaches 1 from the left, everything in the first paragraph above still applies. Thus this limit is also infinite.

To determine the sign of the answer, we again look at the signs of the numerator and the denominator. If $x$ is a number that is very close to 1, but is just a little smaller than 1 (because we are considering the limit as $x$ approaches 1 from the left), then the numerator will be negative ($1-3x$ will be very close to $-2$), and the denominator will also be negative ($x-1$ will always be negative if $x$ is a little smaller than 1). So the fraction itself must be positive, and we get

$$\lim_{x\to 1^-} \frac{1-3x}{x-1} = +\infty.$$

Again, technically speaking, this limit does not exist, although it makes sense to say that the limit "is" $+\infty$.

(c) $\lim_{x\to 1} \frac{1-3x}{x-1}$

Now we are considering the two-sided limit of this fraction as $x$ approaches 1. Since we already know both of the one-sided limits, we can use that to determine the two-sided limit. It is always true that the two-sided limit is equal to the one-sided limits, as long as the one-sided limits are the same; but if the one-sided limits are different, then the two-sided limit does not exist. In this case, the limit from the right is $-\infty$, while the limit from the left is $+\infty$. Since these are not the same, the two-sided limit does not exist; i.e.,

$$\lim_{x\to 1} \frac{1-3x}{x-1} \quad \text{does not exist.}$$

Here when we say the limit does not exist, we mean it really doesn't exist. It doesn't make sense to say that the two-sided limit is $-\infty$, because the function approaches $-\infty$ only when $x$ approaches 1 from the right, and it doesn't make sense to say that the two-sided limit is $+\infty$, because the function approaches $+\infty$ only when $x$ approaches 1 from the left.

2. Determine the intervals on which $f(x)$ is continuous.

   (a) $f(x) = \begin{cases} x-1, & \text{if } x < 4; \\ \sqrt{x+5}, & \text{if } x \geq 4. \end{cases}$

   To the left of 4, $f(x)$ is equivalent to $x-1$. Now $x-1$ is a polynomial, and polynomials are always continuous everywhere, so $f(x)$ is continuous everywhere to the left of 4.

   To the right of 4, $f(x)$ is equivalent to $\sqrt{x+5}$. This is not a polynomial, but it *is* the square root of a polynomial. Such a function is continuous everywhere, except at places where it's undefined (because there's a negative number under the square root). It turns out that we don't have to worry about this, because $f(x)$ is equal to $\sqrt{x+5}$ only if $x \geq 4$, so the value under the square root will always be at least $4+5=9$. Thus $f(x)$ is continuous everywhere to the right of 4.

   The only thing we haven't checked yet is what happens right at 4, where the two "pieces" of the function meet. Intuitively speaking, in order for $f(x)$ to be continuous at $x=4$, the two pieces have to come together at the same point. Mathematically, in order for $f(x)$ to be continuous at $x=4$, three things have to be true: the function $f(x)$ must be *defined* at $x=4$, the *limit must exist* as $x$ approaches 4, and this limit must be *equal* to the value of the function at $x=4$.

   We know $f(x)$ is defined at $x=4$, because $f(4)=\sqrt{4+5}=3$. Let's examine the limit of $f(x)$ as $x$ approaches 4. As $x$ approaches 4 from the left, $f(x)$ will always be $x-1$ (since $x$ will be a little less than 4), so

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x-1) = 4-1 = 3.$$

   As $x$ approaches 4 from the right, $f(x)$ will always be $\sqrt{x+5}$ (since $x$ will be a little more than 4), so

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x+5} = \sqrt{4+5} = 3.$$

   Since the two one-sided limits exist and are equal, we can conclude that

$$\lim_{x\to 4} f(x) = 3.$$

   Since $f(x)$ is defined at $x=4$, the limit of $f(x)$ as $x$ approaches 4 exists, and this limit is equal to the value of the function at $x=4$, we know that $f(x)$ is continuous at $x=4$.

   Therefore $f(x)$ is continuous for all real numbers. The interval on which $f(x)$ is continuous is $(-\infty, +\infty)$.

   (b) $f(x) = \dfrac{e^x}{x+2}$

   The numerator of this fraction is $e^x$, which is continuous everywhere. The denominator of this fraction is $x+2$, which is a polynomial and is thus also continuous everywhere. Therefore, the only places that $f(x)$ can be discontinuous are places where the denominator equals zero.

If $x+2=0$, then $x=-2$. So at $x=-2$, the denominator of the fraction is zero, and $f(x)$ is undefined (and hence discontinuous). But for all other values of $x$, the denominator of the fraction is nonzero, so $f(x)$ is defined and continuous. Thus the intervals on which $f(x)$ is continuous are $(-\infty,-2)$ and $(-2,+\infty)$.

3. Consider the function $f(x)=\sqrt{x+1}$. Use the definition of derivative to compute $f'(3)$. Find the equation of the tangent line to $f(x)$ at $x=3$.

By the definition of the derivative,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

So we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{[by definition]}\\
&= \lim_{h\to 0}\frac{\sqrt{(x+h)+1}-\sqrt{x+1}}{h} && \text{[substituting } f(x)=\sqrt{x+1}\text{]}\\
&= \lim_{h\to 0}\left(\frac{\sqrt{(x+h)+1}-\sqrt{x+1}}{h}\right)\left(\frac{\sqrt{(x+h)+1}+\sqrt{x+1}}{\sqrt{(x+h)+1}+\sqrt{x+1}}\right)\\
&\qquad \text{[multiplying top and bottom by the conjugate of the numerator]}\\
&= \lim_{h\to 0}\frac{(x+h)+1-(x+1)}{h\big(\sqrt{(x+h)+1}+\sqrt{x+1}\big)}\\
&= \lim_{h\to 0}\frac{h}{h\big(\sqrt{(x+h)+1}+\sqrt{x+1}\big)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{(x+h)+1}+\sqrt{x+1}} && \text{[canceling } h\text{]}\\
&= \frac{1}{\sqrt{(x+0)+1}+\sqrt{x+1}} && \text{[substituting } h=0\text{]}\\
&= \frac{1}{\sqrt{x+1}+\sqrt{x+1}}\\
&= \frac{1}{2\sqrt{x+1}}.
\end{aligned}$$

Therefore

$$f'(x)=\frac{1}{2\sqrt{x+1}}.$$

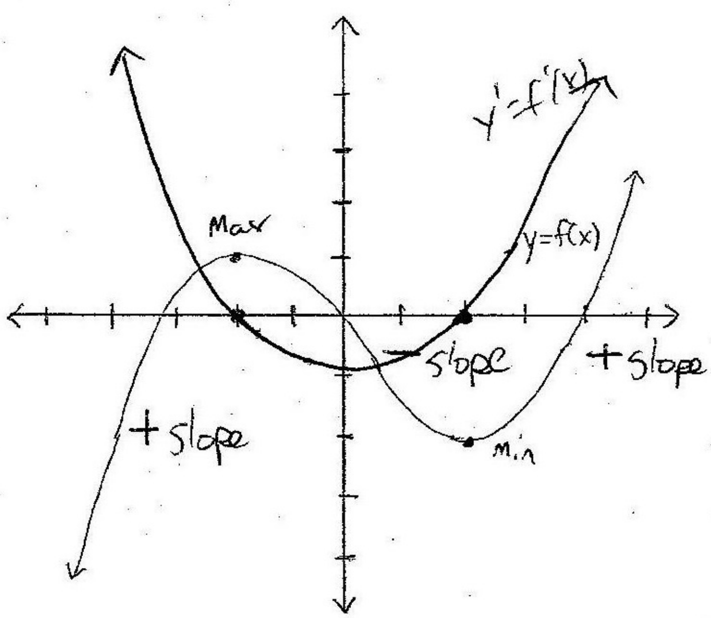
To find $f'(3)$, we simply plug in 3:

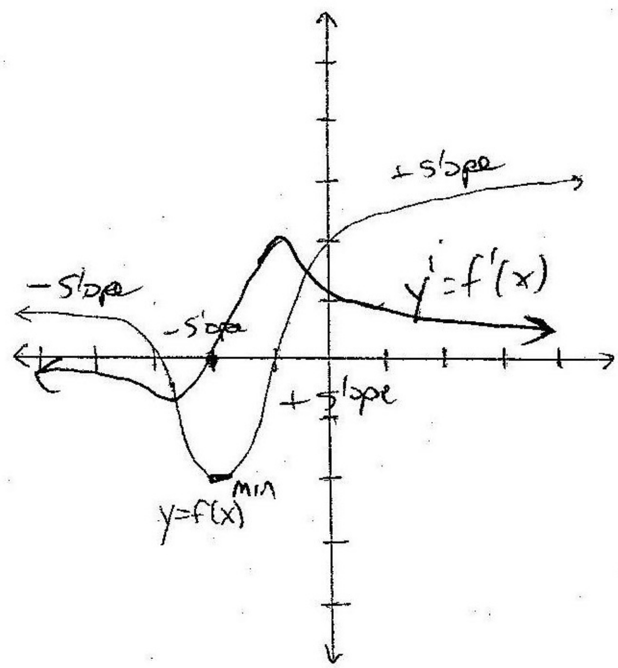
$$f'(3)=\frac{1}{2\sqrt{3+1}}=\frac{1}{2\sqrt{4}}=\frac{1}{2\cdot 2}=\frac{1}{4}.$$

What this tells us is that the derivative at $x=3$ is $1/4$. Since the derivative at a point is the slope of the tangent line at that point, we know that the slope of the tangent line at $x=3$ is $1/4$. To find the equation of the tangent line, we can use the point-slope formula, if we can find a point that is on the tangent line. We know that the tangent line touches the function at $x = 3$, and the value of the function there is $f(3) = \sqrt{3+1} = 2$, so the tangent line passes through the point $(3, 2)$. Therefore, the equation for the tangent line to $f(x)$ at $x = 3$ is

$$\begin{aligned}(y - y_1) &= m(x - x_1),\\ (y - 2) &= \frac{1}{4}(x - 3),\\ y &= \frac{x}{4} + \frac{5}{4}.\end{aligned}$$

4. Given the graph of the function $y = f(x)$ below, sketch a graph of the derivative of $y = f'(x)$.




It is worth pointing out a few key features of these graphs.

In the left graph, we first noticed that the function $f(x)$ reaches a local maximum at $x = -2$ and a local minimum at $x = 2$; at these points, the slope of the tangent line is 0, so we know the derivative at these points is 0. In other words, $f'(-2) = 0$ and $f'(2) = 0$. So we plotted those two points first; these are the big dots on the graph of $y = f'(x)$. Now, between $x = -2$ and $x = 2$ the function $f(x)$ has a negative slope, so between $x = -2$ and $x = 2$ the value of the derivative will be negative. When $x < -2$, the function $f(x)$ has a positive slope, so the value of the derivative will be positive; the same is true when $x > 2$. Hence the derivative should have positive values for $x < -2$, a value of zero when $x = -2$, negative values for $-2 < x < 2$, a value of zero when $x = 2$, and positive values for $x > 2$. This suggests a parabola-like shape, which is what we sketched for the graph of the derivative.

The graph of the derivative $f'(x)$ reaches a minimum value at about $x = 0$, or maybe a little to the right of $x = 0$; in other words, at this value of $x$, the derivative is the most negative that it will ever be.

What does this mean in terms of the original function $f(x)$? It means that at this value of $x$, the *slope* is the most negative that it will ever be. This can be confirmed by looking at the graph of the original function: at about $x = 0$, or maybe a little to the right of $x = 0$, the function $f(x)$ has the steepest negative slope that it has anywhere.

As $x \to -\infty$, the derivative $f'(x)$ increases, apparently without bound. In other words, as $x$ becomes more and more negative, the value of the derivative becomes greater and greater (always being positive). So, as $x$ becomes more and more negative, the slope of the original function $f(x)$ should become greater and greater. This is what we see in the original graph. As $x$ becomes more and more negative, the slope of $f(x)$ becomes steeper and steeper, and is always positive (once we're beyond $x = -2$, of course). The same thing happens as $x \to +\infty$, i.e., as $x$ becomes greater and greater in the positive direction.

In the right graph, we began as before by identifying places where the slope of the tangent line is zero. This happens at $x = -2$, where the function $f(x)$ reaches a minimum value. So the value of the derivative at $x = -2$ must be zero; we plotted this point as the big dot on the graph of $y = f'(x)$. To draw the rest of the graph of the derivative, we saw that the slope of the function $f(x)$ is negative everywhere to the left of $x = -2$ and positive everywhere to the right of $x = -2$. So the values of the derivative $f'(x)$ should be negative for $x < -2$ and positive for $x > -2$.

Now, there appears to be a horizontal asymptote, $y = 1$, as $x \to -\infty$. Because of this, the function $f(x)$ is steepest at about $x = -5/2$, and then becomes flatter as $x$ becomes more negative. In other words, the slope of $f(x)$ is most negative at about $x = -5/2$, and becomes closer and closer to zero as $x$ becomes more negative (because the function $f(x)$ is flattening out). So we were able to see that the graph of the derivative $f'(x)$ must reach its most negative value at about $x = -5/2$, and as $x$ becomes more negative than that, the graph of the derivative must approach zero (while always remaining negative). This explains the shape of the graph of the derivative to the left of $x = -2$.

A similar thing happens to the right of $x = -2$. There seems to be a horizontal asymptote, $y = 3$, as $x \to +\infty$. The function $f(x)$ is steepest at about $x = -1$, and then becomes flatter as $x$ increases beyond $-1$ (the function has to flatten out, because it keeps getting closer and closer to this horizontal asymptote, but can never reach it). Thus the derivative $f'(x)$, which is the slope of the function $f(x)$, should reach its maximum value at $x = -1$, and then decrease toward zero beyond that. This gives the shape of the graph of $f'(x)$ to the right of $x = -2$.

5. Use the derivative rules to compute the derivative of $y = \dfrac{x^2 e^x}{2x - 1}$.

Let $g(x) = x^2 e^x$ and $h(x) = 2x - 1$, to make it easier to talk about the numerator and the denominator of this fraction. Then

$$y = \frac{x^2 e^x}{2x - 1} = \frac{g(x)}{h(x)}.$$

This is a quotient, so we will use the quotient rule:

$$y' = \frac{h(x)g'(x) - g(x)h'(x)}{\big(h(x)\big)^2}. \tag{1}$$

Now we need to figure out what $g'(x)$ and $h'(x)$ are, so that we can plug them in.

We see that $g(x) = x^2 e^x$ is a product. Let $k(x) = x^2$ and $\ell(x) = e^x$, just to give names to the parts of $g(x)$. So

$$g(x) = x^2 e^x = k(x)\ell(x).$$

The product rule tells us that

$$g'(x) = k(x)\ell'(x) + k'(x)\ell(x). \tag{2}$$

Since $k(x) = x^2$ is simply a power of $x$, the power rule tells us that $k'(x) = 2x^1 = 2x$. Since $\ell(x) = e^x$, and we know that the derivative of $e^x$ is $e^x$, we have $\ell'(x) = e^x$. So we can plug these back into formula (2) to get

$$g'(x) = k(x)\ell'(x) + k'(x)\ell(x) = (x^2)(e^x) + (2x)(e^x).$$

Let's now try to find $h'(x)$. We have $h(x) = 2x-1$, which is a polynomial. To compute the derivative of a polynomial, we simply compute the derivative of each term and add all these together (this is the sum rule); we can use the power rule and the constant-multiple rule to compute the derivatives of the individual terms. Using these rules, we find that

$$h'(x) = 2x^0 - 0 = 2.$$

Now that we know that $g'(x) = x^2e^x + 2xe^x$ and $h'(x) = 2$, we can substitute these back into formula (1):

$$\begin{aligned} y' &= \frac{h(x)g'(x) - g(x)h'(x)}{\big(h(x)\big)^2} \\ &= \frac{(2x-1)(x^2e^x + 2xe^x) - (x^2e^x)(2)}{(2x-1)^2} \\ &= \frac{(2x^3e^x + 4x^2e^x - x^2e^x - 2xe^x) - 2x^2e^x}{(2x-1)^2} \\ &= \frac{2x^3e^x + x^2e^x - 2xe^x}{(2x-1)^2}. \end{aligned}$$

6. Calculate the following limits. Show all your work.

   (a) $\lim_{x\to 0}(\cos^2 x + 6)$

   We know that $\cos x$ is continuous everywhere, and so is 6, so $\cos^2 x + 6$ is continuous everywhere. Since it is continuous, the limit at any point is equal to the value of the function at that point. Therefore, to find the limit at $x = 0$, we can simply find the *value* of the function at $x = 0$ by plugging in 0, and because the function is continuous this value will be equal to the limit. So

   $$\lim_{x\to 0}(\cos^2 x + 6) = \cos^2(0) + 6 = 1^2 + 6 = 7.$$

   Almost all of the functions you will come across in this course are continuous everywhere, except where there's an obvious problem (such as zero in a denominator or a negative number under a square root). This includes all polynomials; all roots; all rational functions; the exponential function $e^x$; the trig functions $\sin x$, $\cos x$, and $\tan x$ (although be sure to note that $\tan x$ is not defined when $\cos x = 0$); and compositions of these functions. So you can usually use this trick of computing the limit of the function by computing the value of the function, unless the function is not defined there. Then you have to find the limit in some other way.

(b) $\lim_{x\to 0} \frac{xe^{-2x+1}}{x^2+x}$

Here we cannot simply plug in $x = 0$, because that would make the denominator zero. So we'll use some algebraic manipulations to change the fraction into something that we *can* plug $x = 0$ into.

$$\begin{aligned}
\lim_{x\to 0} \frac{xe^{-2x+1}}{x^2+x} &= \lim_{x\to 0} \frac{x(e^{-2x+1})}{x(x+1)} && [\text{factor out } x] \\
&= \lim_{x\to 0} \frac{e^{-2x+1}}{x+1} && [\text{cancel } x] \\
&= \frac{e^{-2\cdot 0+1}}{0+1} && [\text{plug in } x = 0] \\
&= e. && [\text{evaluate}]
\end{aligned}$$

(c) $\lim_{x\to \pi/2} \frac{\cos x}{\cot x}$

Again, we cannot simply plug in $x = \pi/2$, because $\cot(\pi/2) = 0$. We need to remember some trigonometry for this one.

$$\begin{aligned}
\lim_{x\to \pi/2} \frac{\cos x}{\cot x} &= \lim_{x\to \pi/2} (\cos x)\left(\frac{1}{\cot x}\right) \\
&= \lim_{x\to \pi/2} (\cos x)(\tan x) && [\cot x = 1/\tan x] \\
&= \lim_{x\to \pi/2} (\cos x)\left(\frac{\sin x}{\cos x}\right) && [\tan x = (\sin x)/(\cos x)] \\
&= \lim_{x\to \pi/2} \sin x && [\text{cancel } \cos x] \\
&= \sin\left(\frac{\pi}{2}\right) && [\text{plug in } x = \pi/2] \\
&= 1.
\end{aligned}$$

(d) $\lim_{x\to 0} \frac{x^2 - \sin x}{2x}$

Once again, if we try to plug in $x = 0$, the denominator becomes zero, so that's no good. Instead, we'll try splitting the fraction:

$$\lim_{x\to 0} \frac{x^2 - \sin x}{2x} = \lim_{x\to 0} \left(\frac{x^2}{2x} - \frac{\sin x}{2x}\right) = \lim_{x\to 0} \left(\frac{x}{2} - \frac{1}{2} \cdot \frac{\sin x}{x}\right).$$

Now, we can use limit laws to conclude that

$$\lim_{x\to 0} \left(\frac{x}{2} - \frac{1}{2} \cdot \frac{\sin x}{x}\right) = \left(\lim_{x\to 0} \frac{x}{2}\right) - \left(\frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x}\right) = \left(\lim_{x\to 0} \frac{x}{2}\right) - \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin x}{x}\right). \tag{3}$$

*Important:* This is true only if $\lim_{x\to 0}(x/2)$ and $\lim_{x\to 0}(\sin x)/x$ both exist! If either one of these limits turns out not to exist (this includes limits which are $-\infty$ or $+\infty$), then this step isn't valid.

By plugging in $x = 0$, we can easily find that

$$\lim_{x \to 0} \frac{x}{2} = \frac{0}{2} = 0,$$

and we know that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

So the two limits on the right side of equation (3) exist after all, and therefore, by plugging these limits in, we have

$$\left(\lim_{x \to 0} \frac{x}{2}\right) - \frac{1}{2}\left(\lim_{x \to 0} \frac{\sin x}{x}\right) = 0 - \frac{1}{2}(1) = -\frac{1}{2}.$$

Hence

$$\lim_{x \to 0} \frac{x^2 - \sin x}{2x} = -\frac{1}{2}.$$

(e) $\lim\limits_{x \to +\infty} \dfrac{3x^3 - 2x + 1}{4x^3 + x^2 - 5x}$

We have a rational function (i.e., a fraction whose numerator and denominator are both polynomials), and we're trying to find the limit as $x \to +\infty$. The trick to use in this case is to divide both the top and the bottom of the fraction by the highest power of $x$ that appears in the denominator. In this case, this is $x^3$. So we get

$$\lim_{x \to +\infty} \frac{3x^3 - 2x + 1}{4x^3 + x^2 - 5x} = \lim_{x \to +\infty} \frac{\frac{3x^3}{x^3} - \frac{2x}{x^3} + \frac{1}{x^3}}{\frac{4x^3}{x^3} + \frac{x^2}{x^3} - \frac{5x}{x^3}} = \lim_{x \to +\infty} \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{4 + \frac{1}{x} - \frac{5}{x^2}}.$$

Now, as $x \to +\infty$, that is, as $x$ gets bigger and bigger and bigger, the terms $2/x^2$, $1/x^3$, $1/x$, and $5/x^2$ are going to get smaller and smaller and smaller. The limit of these terms as $x \to +\infty$ is 0. So if $x$ is very big, these terms are insignificant when compared to the rest of the numerator or the rest of the denominator, and we can ignore them. Hence

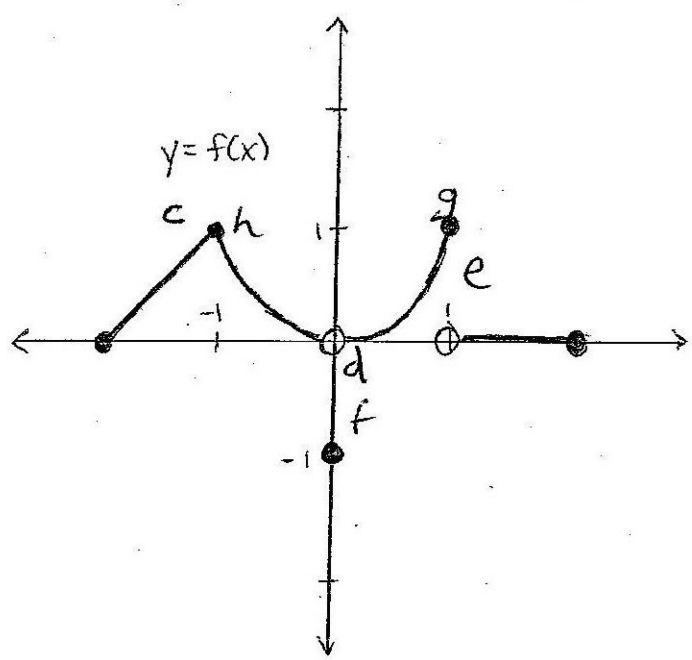
$$\lim_{x \to +\infty} \frac{3 - \frac{2}{x^2} + \frac{1}{x^3}}{4 + \frac{1}{x} - \frac{5}{x^2}} = \lim_{x \to +\infty} \frac{3}{4} = \frac{3}{4}.$$

So

$$\lim_{x \to +\infty} \frac{3x^3 - 2x + 1}{4x^3 + x^2 - 5x} = \frac{3}{4}.$$

This trick of dividing both the top and the bottom by the highest power of $x$ in the denominator works just as well if we are trying to find the limit of a rational function as $x \to -\infty$.

7. Consider the function below.




(a) Find the domain of $f(x)$.

The domain of $f(x)$ is the set of all values of $x$ for which the function is defined. We can see by examining the graph that there is a value for $f(x)$ at every value of $x$ between $x = -2$ and $x = 2$ (including these endpoints); in other words, there is no value of $x$ in $[-2, 2]$ for which $f(x)$ is undefined. Outside of this range, the function $f(x)$ is not defined. So the domain of $f(x)$ is $[-2, 2]$ (or, equivalently, the domain of $f(x)$ is the set of all $x$ such that $-2 \leq x \leq 2$).

(b) Find the range of $f(x)$.

The range of $f(x)$ is the set of all values of $y$ for which $y = f(x)$ for some value of $x$. In other words, it is the set of all of the values of $y$ that you can get out of the function, if you put in an appropriate value of $x$. We can see from the graph that any value of $y$ between 0 and 1, including 0 and 1, is the output of the function for some value of $x$; additionally, we can get $f(x) = -1$ if we plug in $x = 0$. So the range of $f(x)$ is the set of all $y$ such that either $0 \leq y \leq 1$ or $y = -1$; equivalently, using set notation, we can say that the range of $f(x)$ is $[0, 1] \cup \{-1\}$.

(c) Find $\lim_{x \to -1} f(x)$.

As $x$ approaches $-1$ from either the left or the right, the value of the function $f(x)$ gets closer and closer to 1. Therefore,

$$\lim_{x \to -1} f(x) = 1.$$

(d) Find $\lim_{x \to 0} f(x)$.

It is true that the *value* of the function at $x = 0$ is $-1$; that is, $f(0) = -1$. But when we talk about the *limit* of the function as $x$ approaches 0, we don't care what the function does right at $x = 0$; we care only what the function does as $x$ gets *close* to 0.

In this case, as $x$ gets closer and closer to 0, but without quite reaching 0, the value of the function $f(x)$ gets closer and closer to 0. (It is only when $x$ actually reaches 0 that the function suddenly jumps to $-1$.) So

$$\lim_{x\to 0} f(x) = 0.$$

(e) Find $\lim_{x\to 1} f(x)$.

As $x$ approaches 1 from the left, the function $f(x)$ approaches 1. So

$$\lim_{x\to 1^-} f(x) = 1.$$

On the other hand, as $x$ approaches 1 from the right, the function $f(x)$ approaches 0, so

$$\lim_{x\to 1^+} f(x) = 0.$$

Since the two one-sided limits are not equal, the two-sided limit does not exist. In other words, there is no single value that the function $f(x)$ approaches as $x$ approaches 1. Hence

$$\lim_{x\to 1} f(x) \quad \text{does not exist.}$$

(f) For which values of $x$ does $f(x)$ have a removable discontinuity?

The first criterion for having a removable discontinuity at a point is to be discontinuous there. If a function is continuous somewhere, it certainly doesn't have a removable discontinuity there. By examining the graph of $f(x)$, we can see that the only places where $f(x)$ is not continuous are at $x = 0$ and $x = 1$. At these values of $x$, the *limit* of $f(x)$ does not equal the *value* of $f(x)$.

So if $f(x)$ is to have a removable discontinuity somewhere, it must be at $x = 0$ or at $x = 1$, or both. What does it mean for a discontinuity to be removable? It means that if we redefine the value of the function at a single point, we can make it continuous. Looking at the graph, we see that if we changed the value of $f(0)$ from $-1$ to 0, we would make the function continuous at $x = 0$. Intuitively, the reason that $f(x)$ is not continuous at $x = 0$ is because there is a "hole" in the graph there, and if we could change the value of $f(0)$ then we could "plug" that hole and fix the problem. Mathematically, we know that $f(x)$ has a removable discontinuity at $x = 0$ because $f(x)$ is not continuous there, but the limit of $f(x)$ does exist there. So $f(x)$ has a removable discontinuity at $x = 0$.

On the other hand, the discontinuity at $x = 1$ is different. Intuitively, the reason that the function is discontinuous at $x = 1$ is because there is a sudden jump in the graph. There is no way to change the value of the function *at a single point* to repair this problem, so this discontinuity is not removable. Mathematically, we know that the discontinuity at $x = 1$ is not removable because the limit of $f(x)$ as $x$ approaches 1 does not exist.

(g) For which values of $x$ does $f(x)$ have a non-removable discontinuity?

As we determined in the answer to part (f), the discontinuity at $x = 1$ is not removable. So $f(x)$ has a non-removable discontinuity at $x = 1$.

(h) For which values of $x$ does $f(x)$ fail to be differentiable?

A function must be continuous at a point in order to be differentiable there; hence $f(x)$ is not differentiable at $x = 0$ or $x = 1$, because it is not continuous at these values of $x$.

However, it is not enough that a function be continuous in order for the function to be differentiable. If the function has a “corner” somewhere, then it is not differentiable there. This is because there is more than one way to draw a tangent line at that point, so it doesn't make sense to talk about the slope of “the” tangent line. In this case, $f(x)$ has a “corner” at $x = -1$, so $f(x)$ is not differentiable at $x = -1$.

For all values of $x$ other than $x = -1$, $x = 0$, and $x = 1$, the function $f(x)$ is continuous and smooth, so it is differentiable. Thus, the only values of $x$ for which $f(x)$ fails to be differentiable are $x = -1$, $x = 0$, and $x = 1$.