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MATH 107 Quiz 3 (solutions)
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Evaluate the following integrals.

1. $\displaystyle\int_0^2 \frac{dx}{8+2x^2}$

   Solution. With $x = a\tan\theta$, we have

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

   We will set $a = 2$, so $x = 2\tan\theta$ and $4 + x^2 = 4\sec^2\theta$. This gives $dx = 2\sec^2\theta\,d\theta$, and $\theta = \tan^{-1}(x/2)$. So

$$\begin{aligned}
\int_0^2 \frac{dx}{8+2x^2} &= \frac{1}{2}\int_0^2 \frac{dx}{4+x^2} \\
&= \frac{1}{2}\int_{x=0}^{x=2} \frac{2\sec^2\theta\,d\theta}{4\sec^2\theta} \\
&= \frac{1}{4}\int_{x=0}^{x=2} d\theta \\
&= \frac{1}{4}\theta\Big|_{x=0}^{x=2} \\
&= \frac{1}{4}\tan^{-1}(x/2)\Big|_0^2 \\
&= \frac{1}{4}\big(\tan^{-1}(1) - \tan^{-1}(0)\big) \\
&= \frac{1}{4}\left(\frac{\pi}{4} - 0\right) \\
&= \frac{\pi}{16}.
\end{aligned}$$

2. $\displaystyle\int \frac{x^2+15x-28}{(x+4)(x-2)^2}\,dx$

   Solution. We will use the method of partial fractions to decompose this fraction. Since each of the factors of the denominator is a linear polynomial, our equation is

$$\frac{x^2+15x-28}{(x+4)(x-2)^2} = \frac{A}{x+4} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

   Multiplying every term by $(x+4)(x-2)^2$ to clear the denominators, we get

$$\begin{aligned}
x^2+15x-28 &= A(x-2)^2 + B(x+4)(x-2) + C(x+4) \\
&= Ax^2 - 4Ax + 4A + Bx^2 + 2Bx - 8B + Cx + 4C \\
&= (A+B)x^2 + (-4A+2B+C)x + (4A-8B+4C).
\end{aligned}$$

   By matching coefficients of powers of x we obtain the following system of equations.

$$\begin{cases}
A + B = 1 & \text{(i)} \\
-4A + 2B + C = 15 & \text{(ii)} \\
4A - 8B + 4C = -28 & \text{(iii)}
\end{cases}$$

Equation (i) gives us

$$B = 1 - A.$$

Substituting this into Equation (ii), we get

$$\begin{aligned} -4A + 2(1 - A) + C &= 15, \\ -4A + 2 - 2A + C &= 15, \\ C &= 6A + 13. \end{aligned}$$

Now substituting both of these into Equation (iii), we find

$$\begin{aligned} 4A - 8(1 - A) + 4(6A + 13) &= -28, \\ 4A - 8 + 8A + 24A + 52 &= -28, \\ 36A &= -72, \end{aligned}$$

so

$$A = -2.$$

Therefore

$$B = 1 - A = 1 - (-2) = 3 \qquad \text{and} \qquad C = 6A + 13 = 6(-2) + 13 = 1.$$

So we have found that

$$\frac{x^2 + 15x - 28}{(x+4)(x-2)^2} = \frac{-2}{x+4} + \frac{3}{x-2} + \frac{1}{(x-2)^2}.$$

Thus our original integral becomes

$$\begin{aligned} \int \frac{x^2 + 15x - 28}{(x+4)(x-2)^2}\,dx &= \int \left(\frac{-2}{x+4} + \frac{3}{x-2} + \frac{1}{(x-2)^2} \right) dx \\ &= \int \frac{-2}{x+4}\,dx + \int \frac{3}{x-2}\,dx + \int \frac{1}{(x-2)^2}\,dx \\ &= -2 \int \frac{1}{x+4}\,dx + 3 \int \frac{1}{x-2}\,dx + \int (x-2)^{-2}\,dx \\ &= -2 \ln|x+4| + 3 \ln|x-2| - (x-2)^{-1} + K, \end{aligned}$$

where K is the arbitrary constant of integration (since we used the letter C for a different purpose earlier).