MATH 107 Homework 1 (Solutions)
Section 7.2, problems 21, 29, and 37
Assigned January 24, 2007
Due January 26, 2007

21. $\displaystyle\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$

Solution. We use the double-angle formula $\cos 2\theta = 1 - 2\sin^2\theta$ to obtain $1 - \cos 2\theta = 2\sin^2\theta$, which we can substitute into our integral.

$$\begin{aligned}
\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta &= \int_0^{\pi/2} \theta\sqrt{2\sin^2\theta}\,d\theta \\
&= \int_0^{\pi/2} \theta\sqrt{2}\sqrt{\sin^2\theta}\,d\theta \\
&= \sqrt{2}\int_0^{\pi/2} \theta\sqrt{\sin^2\theta}\,d\theta \\
&= \sqrt{2}\int_0^{\pi/2} \theta|\sin\theta|\,d\theta \qquad (1) \\
&= \sqrt{2}\int_0^{\pi/2} \theta\sin\theta\,d\theta \qquad (2)
\end{aligned}$$

$$\begin{bmatrix} u = \theta & dv = \sin\theta\,d\theta \\ du = d\theta & v = -\cos\theta \end{bmatrix}$$

$$\begin{aligned}
&= \sqrt{2}\left[-\theta\cos\theta + \int \cos\theta\,d\theta\right]_0^{\pi/2} \\
&= \sqrt{2}\Big[-\theta\cos\theta + \sin\theta\Big]_0^{\pi/2} \\
&= \sqrt{2}\left[\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-0\cos 0 + \sin 0)\right] \\
&= \sqrt{2}\,[(-0+1) - (0+0)] \\
&= \sqrt{2}\,(1) \\
&= \sqrt{2}.
\end{aligned}$$

Note that in line (1) we used $\sqrt{\sin^2\theta} = |\sin\theta|$, since a square root is always positive. However, since we are integrating from 0 to $\pi/2$, where $\sin\theta$ is positive, we can replace $|\sin\theta|$ by $\sin\theta$ in line (2).

29. $\displaystyle\int_0^{\pi/4} 4\tan^3 x\,dx$

Solution. We use the identity $\tan^2 x = \sec^2 x - 1$ and u-substitution to evaluate this integral.

$$\begin{aligned}
\int_0^{\pi/4} 4\tan^3 x\,dx &= 4\int_0^{\pi/4} \tan^3 x\,dx \\
&= 4\int_0^{\pi/4} \tan^2 x \tan x\,dx \\
&= 4\int_0^{\pi/4} (\sec^2 x - 1)\tan x\,dx \\
&= 4\int_0^{\pi/4} \tan x \sec^2 x\,dx - 4\int_0^{\pi/4} \tan x\,dx
\end{aligned}$$

$$\begin{bmatrix} u = \tan x \\ du = \sec^2 x\,dx \end{bmatrix}$$

$$\begin{aligned}
&= 4\int_{x=0}^{x=\pi/4} u\,du - 4\Big[\ln|\sec x|\Big]_0^{\pi/4} \qquad (1)\\
&= 4\left[\frac{u^2}{2}\right]_{x=0}^{x=\pi/4} - 4\left(\ln\left|\sec\frac{\pi}{4}\right| - \ln|\sec 0|\right) \\
&= 4\left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} - 4\left(\ln\sqrt{2} - \ln 1\right) \\
&= 2\left(\tan^2\frac{\pi}{4} - \tan^2 0\right) - 4\left(\frac{1}{2}\ln 2 - 0\right) \\
&= 2(1-0) - 2\ln 2 \\
&= 2 - 2\ln 2.
\end{aligned}$$

In line (1), we used $\int \tan x\,dx = \ln|\sec x| + C$. If we didn't know this, we could have derived it, as shown below.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$$\begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix}$$

$$\begin{aligned}
&= -\int \frac{1}{u}\,du \\
&= -\ln|u| + C \\
&= -\ln|\cos x| + C \\
&= \ln\left(|\cos x|^{-1}\right) + C \\
&= \ln|\sec x| + C.
\end{aligned}$$

Of course, the last two steps of this aren't really necessary; we could have just as well used $\int \tan x\,dx = -\ln|\cos x| + C$. However, $\int \tan x\,dx = \ln|\sec x| + C$ is the form usually shown in integral tables.

37. $\displaystyle\int_0^{\pi} \cos 3x \cos 4x \, dx$

Solution. We use the identity

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

to get

$$\begin{aligned}
\int_0^{\pi} \cos 3x \cos 4x \, dx &= \frac{1}{2}\int_0^{\pi} [\cos(3x-4x)+\cos(3x+4x)] \, dx \\
&= \frac{1}{2}\int_0^{\pi} [\cos(-x)+\cos 7x] \, dx \\
&= \frac{1}{2}\left[\sin x + \frac{1}{7}\sin 7x\right]_0^{\pi} \\
&= \frac{1}{2}\left[\left(\sin\pi + \frac{1}{7}\sin 7\pi\right) - \left(\sin 0 + \frac{1}{7}\sin(7\cdot 0)\right)\right] \\
&= \frac{1}{2}\left[\left(0 + \frac{1}{7}\cdot 0\right) - \left(0 + \frac{1}{7}\cdot 0\right)\right] \\
&= 0.
\end{aligned}$$