

Math 103: College Algebra and Trigonometry  
**Worksheet 1** (optional; 4 points)  
Assigned Monday, 13 October 2008  
Due Tuesday, 21 October 2008

**Rational and irrational numbers**

A *rational* number is a number that can be written as a fraction, with both the numerator and the denominator of the fraction being integers. (Negative integers are okay, and 0 is okay in the numerator.) For example,  $\frac{2}{3}$ ,  $\frac{355}{113}$ , and  $-\frac{7}{4}$  are all rational numbers. So are 0.6391 and  $-14.92$ , because these can be written as the fractions  $\frac{6391}{10\,000}$  and  $\frac{-1492}{100}$ . All integers are rational numbers, because they can be written as fractions with denominator 1.

It is perhaps surprising to learn that there are some numbers that *cannot* be written as fractions with integer numerators and denominators. Such numbers are called *irrational*. Examples of irrational numbers include the numbers  $\pi$ ,  $e$ , and  $\sqrt{2}$ . Every real number is either rational or irrational (but not both).

The ancient Greeks, and in particular the students of the Greek mathematician Pythagoras (for whom the Pythagorean theorem is named), believed that the whole universe could be understood through integers. They believed that every number was either itself an integer or could be written as a fraction of two integers—in other words, they believed that every number was rational. However, some time around the fifth century BC one of the Pythagoreans, a man named Hippasus, proved the startling fact that  $\sqrt{2}$  cannot be written as a fraction of two integers. According to legend, when he revealed his discovery to the other Pythagoreans, it disturbed their philosophy so much that Hippasus was drowned for heresy!

**Properties of rational and irrational numbers**

Rational numbers have decimal expansions that either terminate or repeat some sequence of digits over and over again forever. For example, the decimal expansion of the rational number  $\frac{5}{16}$  is 0.3125: it terminates after four decimal places. The decimal expansion of  $\frac{15}{7}$  is 2.142857142857142857...: the sequence “142857” repeats over and over again, endlessly. Sometimes there are some non-repeating digits before the “block” of repeating digits begins. For example, the decimal expansion of  $\frac{19}{12}$  is 1.58333333...: the 3’s repeat forever, but the 5 and the 8 only appear once. (In fact you can think of terminating decimals as simply a special case of repeating decimals, if you imagine a bunch of 0’s on the end:  $\frac{5}{16} = 0.3125000000\dots$ )

Irrational numbers, on the other hand, have decimal expansions that never terminate and never settle into a pattern of some repeating sequence of digits. For example, the first 100 digits of  $\pi$  are

$$\pi \approx 3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679\dots$$

The decimal expansion of  $\pi$  continues forever, and it never begins repeating the same sequence of digits over and over. This is because  $\pi$  is irrational. For two other examples, here are the first 100 digits of  $e$  and  $\sqrt{2}$ :

$$e \approx 2.7182818284590452353602874713526624977572470936999595749669676277240766303535475945713821785251664274\dots;$$
$$\sqrt{2} \approx 1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415727\dots$$

We can use the fact that the decimal expansions of irrational numbers do not repeat in order to construct irrational numbers. For example, the number

$$5.050050005000050000050000005\dots$$

is irrational, because there is no fixed sequence of digits which repeats over and over. There is a *pattern* to the digits of this number, but because there is one more 0 inserted each time a 5 occurs, it is not a repeating decimal, so it is irrational.

There are no “gaps” in the set of rational numbers: between any two rational numbers there is another rational number. For example,  $\frac{2}{3}$  is a rational number between  $\frac{1}{2}$  and  $\frac{3}{4}$ . Likewise, there are no “gaps” in the set of irrational numbers: between any two irrational numbers there is another irrational number. In fact, between any two real numbers whatsoever, there is some rational number and some irrational number (actually infinitely many of them). Mathematicians say that the set of rational numbers is *dense* in the set of real numbers, and so is the set of irrational numbers.

Square roots of positive integers are often irrational numbers. The very first irrational number discovered was  $\sqrt{2}$ , a square root of a positive integer. In fact, whenever a square root of a positive integer is not itself an integer, it is irrational. For example,  $\sqrt{9}$ ,  $\sqrt{16}$ , and  $\sqrt{3249}$  are integers ( $\sqrt{9} = 3$ ,  $\sqrt{16} = 4$ , and  $\sqrt{3249} = 57$ ), so these square roots are rational numbers; but  $\sqrt{5}$ ,  $\sqrt{11}$ , and  $\sqrt{2008}$  are not integers, so they are irrational numbers. (This is a special property of square roots—it’s not true in general. For instance,  $\frac{22}{7}$  is not an integer, but it *is* a rational number.)

Irrational numbers may seem to be rather strange creatures. Most of the numbers we use on a daily basis are rational numbers, with a few notable exceptions like  $\pi$ ,  $e$ , and  $\sqrt{2}$ . It may seem that there are “more” rational numbers than irrational numbers—that the irrational numbers are quirky “exceptions” hidden among the rational numbers. But the previous paragraph shows that there are infinitely many of both of them. In fact, there is a meaningful sense in which it can be said that there are *infinitely many times* as many irrational numbers as rational numbers! So actually it is the rational numbers which are unusual—when you consider the set of all real numbers, “almost all” of them are irrational.

### The Rational Zeros Theorem

Rational numbers are usually easier to work with than irrational numbers. When we are trying to solve a polynomial equation, it is nice if we can find some solutions that are rational numbers. Finding solutions that are irrational numbers is often more of a challenge. But once we find one solution to a polynomial equation, we can “divide out” that solution to get a simpler equation to solve. The textbook calls this simpler equation a *depressed equation*; see Section 5.5 for details.

The Rational Zeros Theorem, covered in Section 5.5 of the textbook, is a way to determine *all* of the *possible* rational zeros of a polynomial. It gives us an explicit list of the *only* possible zeros of the polynomial which are rational numbers. Any zeros of the polynomial that are rational numbers *must* be in this list; any zeros of the polynomial that are not in this list must be *irrational* numbers.

Not all of the rational numbers in the list produced by the Rational Zeros Theorem will be zeros of the polynomial; in fact, it’s possible that none of them are. Conversely, not all of the zeros of the polynomial have to be one of the numbers in the list produced by the Rational Zeros Theorem; if the polynomial has some zeros which are *irrational*, then those zeros will not appear in the list, because the list contains only *rational* numbers.

### Open questions

There are still many things which are not known about rational and irrational numbers.

For example, Johann Heinrich Lambert proved in 1761 that the numbers  $\pi$  and  $e$  are irrational. In 1794 Adrien-Marie Legendre proved that  $\pi^2$  is irrational. Alexander Gelfond and Theodor Schneider independently showed in 1934 that the number  $e^\pi$  (that is,  $e$  raised to the power of  $\pi$ ) is irrational. However, it is still unknown whether any of the numbers  $\pi + e$ ,  $\pi/e$ , or  $\ln \pi$  are rational. It seems unlikely that any of these can be written as a fraction of two integers, but so far no one has been able to prove that it’s impossible. If any of these numbers is rational, then that indicates a surprising (and so far unexplained) relationship between  $e$  and  $\pi$ .

There is a number called the Euler–Mascheroni constant, often written  $\gamma$ , which is important in higher mathematics. It was first described in 1735 by the Swiss mathematician Leonhard Euler. Its value is approximately

$$\gamma \approx 0.57721566490153286060651209008240243104215933593992 \dots$$

The number  $\gamma$  is useful because of the following approximation, which becomes a better and better approximation as the value of  $n$  gets larger and larger:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \approx \ln(n) + \gamma.$$

No one knows whether  $\gamma$  is rational or irrational. However, it has been shown that if it is rational, its denominator (in lowest terms) must be greater than  $10^{242080}$ . (This is an unbelievably huge number!)

Another interesting question is how “random” the digits of  $\pi$  are (or the digits of any other irrational number, for that matter). Since the digits of  $\pi$  never repeat and appear to have no particular pattern, it would be natural to suppose that every digit from 0 through 9 appears in the decimal expansion of  $\pi$  just as often as every other digit:  $\frac{1}{10}$  of the digits of  $\pi$  should be 0’s, another  $\frac{1}{10}$  should be 1’s, and so on. Similarly, it would be natural to suppose that every two-digit sequence 00, 01, 02, 03,  $\dots$ , 99 occurs equally often, and every three-digit sequence 000, 001, 002, 003,  $\dots$ , 999, and so on. It would be surprising if there were some bias to the digits of  $\pi$  (if, for example, the digit 6 occurs slightly more often than the other digits). Mathematicians expect that there is no such bias in the digits of  $\pi$ , but again no one has yet been able to prove that this is actually the case. (A number in which the digits show no bias in this sense is called a *normal number*. Very few numbers have been proven to be normal.)

Open questions like these are part of what makes mathematics interesting. These are questions to which no one in the world knows the answers.

### Problems

1. Show that  $-74.816$  is a rational number by writing it as a fraction of two integers.
2. Find a rational number between  $\frac{4}{7}$  and  $\frac{3}{5}$ . Find an irrational number between 6.4 and 6.5.
3. Let  $f(x) = 11x^3 - 41x^2 - 34x - 6$ .
  - (a) Use the Rational Zeros Theorem to list all possible rational zeros of  $f$ .
  - (b) Use the Intermediate Value Theorem to prove that  $f$  must have a zero somewhere between 4 and 5.
  - (c) Explain why the zero described in part (b) must be an irrational number.
  - (d) Find one rational zero of  $f$ .
  - (e) “Divide out” the rational zero you found in part (d). Find the zeros of the resulting polynomial; these are the remaining zeros of  $f$ .
  - (f) Check that the zeros of  $f$  you have found really are zeros of  $f$ . Explain how you know that you have found *all* of the zeros of  $f$ .
  - (g) Which of the zeros that you found is between 4 and 5? In part (c) you explained that this zero must be irrational. Now that you have an exact expression for it, why does it make sense that it should be irrational? In other words, by looking at what the zero actually is, give a different justification for the claim that it’s irrational.