Theorem 3.2

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Theorem (Theorem 3.2)

Let D be a domain with quotient field K such that

1. *$Int(D)$ is atomic,*
2. *there is a (nontrivial) discrete valuation ν on K,*
3. *there is a pseudo-principal ideal μ of D such that D/μ is finite.*

Then the elasticity of $Int(D)$ is infinite.

Definition

An ideal μ is pseudo-principal if there is an integer k and a nonunit $t \in D$ such that $\mu^k \subset Dt$.

Proof

Let n be an integer.
Since D has a pseudo-principal ideal μ by definition there exists a nonunit t in D and an integer k such that $\mu^k \subset Dt$.
Let $a = a_1 \cdots a_n \in (\mu^k)^n$, where $a_i \in \mu^k \subset Dt$.
Write each $a_i = d_i t$ for some $d_i \in D$. Take $d = d_1 \cdots d_n$.
Therefore a is of the form $a = dt^n$, hence $a \in Dt^n$. The ideal μ^{kn} is the set of finite sums of products of the form $a_1 \cdots a_n$ where $a_i \in \mu^k$. Therefore $\mu^{kn} \subset Dt^n$.

Proof

By hypothesis there is a (nontrivial) discrete valuation ν of K and a pseudo-principal ideal μ such that D/μ is finite.
Then by Lemma 3.1 there is a product $\prod_{i=0}^{q-1} f_i$ of irreducible factors of Int(D) with values in μ^{kn}, where q is the cardinal of D/μ.

Consider $h = 1/t^n \prod_{i=0}^{q-1} f_i$.
For every $\alpha \in D$ there exists an i, $1 \leq i \leq q-1$, such that $f_i(\alpha) \in \mu^{kn} \subset Dt^n \subseteq D$.
Hence $h = 1/t^n \prod_{i=0}^{q-1} f_i$ is integral valued.

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Hence $h = 1/t^n \prod_{i=0}^{q-1} f_i$ is integral valued.

Proof

Rewriting $t^n h = \prod_{i=0}^{q-1} f_i$.
There are (at least) $n+1$ factors on the left-hand side and exactly q irreducible factors on the right.
The elasticity of $\mathrm{Int}(D)$ is defined as $\rho(\mathrm{Int}(D)) = \sup\{m/k : \; x_1 \cdots x_m = y_1 \cdots y_k \text{ for } x_i, y_i \text{ irreducible elements of } \mathrm{Int}(D)\}$

$$\rho(\mathrm{Int}(D)) \geq \frac{n+1}{q}$$

Therefore the elasticity of $\mathrm{Int}(D)$ is infinite.

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Bibliography

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