Theorem 1.6: $\rho(\mathrm{Int}(\mathbb{Z})) = \infty$

Elasticity for integral-valued polynomials

Paul-Jean Cahen, Jean-Luc Chabert
Journal for Pure and Applied Algebra 103 (1995) 303-311

Jenae Beauchamp

IMMERSE Program, University of Nebraska, Lincoln

July, 2, 2009

Proof.

Recall that the binomials

$$\binom{X}{n} = \frac{1}{n!}\prod_{i=0}^{n-1}(x-i)$$

are integral-valued (and form a basis of $\text{Int}(\mathbb{Z})$ as a $\mathbb{Z}$-module). Write

$$n!\binom{X}{n} = \prod_{i=0}^{n-1}(x-i);$$

it is clear that $(X-i)$ is irreducible (and will follow from more general results below, Example 2.3), hence there are n irreducible factors on the right-hand side whereas, on the left-hand side, $n!$ admits as many irreducible factors in $\text{Int}(\mathbb{Z})$ as it does in $\mathbb{Z}$ (Lemma 1.1). We conclude with a proof that this number of factors may eventually be greater than nm, whatever m. Let $n = p!$, where p is prime. Among the factors $1, 2, \ldots, n$ of $n!$, there are $n/2$ multiples of 2, $n/3$ multiples of 3, $n/5$ multiples of 5, and so on up to p, hence $n!$ admits (at least) $n(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p})$ irreducible factors. We are done, since the series of inverses of primes is divergent. □

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it is clear that $(X - i)$ is irreducible.

<u>Show:</u> $(x - i)$ is irreducible

[Def:] Let $x = ab$. If x is <u>irreducible</u> then either a or b is a unit.

Let $(x - i) \;=\; ab$

$$deg(x - i) \;=$$

Show: $(x - i)$ is irreducible

[Def:] Let $x = ab$. If x is irreducible then either a or b is a unit.

$$\text{Let } (x - i) = ab$$
$$deg(x - i) = 1$$

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[Def:] Let $x = ab$. If x is irreducible then either a or b is a unit.

$$\begin{aligned} \text{Let } (x-i) &= ab \\ deg(x-i) &= 1 \\ deg(ab) &= deg(a) + deg(b) = 1 \end{aligned}$$

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$$\begin{aligned} (x-i) &= a(cx+d) \\ &= acx + ad \end{aligned}$$

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$ac = 1$

$\therefore$ a is a unit.

$\therefore$ $(x - i)$ is irreducible.

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Hence there are n irreducible factors on the right-hand side whereas, on the left-hand side, $n!$ admits as many irreducible factors in $\mathrm{Int}(\mathbb{Z})$ as it does in $\mathbb{Z}$ [(Lemma 1.1) The units of $\mathrm{Int}(D)$ are the units of D].

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$$\frac{n}{2} = \frac{120}{2} = 60 \text{ multiples of } 2$$

$$\frac{n}{3} = \frac{120}{3} = 40 \text{ multiples of } 3$$

$$\frac{n}{5} = \frac{120}{5} = 24 \text{ multiples of } 5$$

…hence $n!$ admits <u>(at least) $n(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p})$ irreducible factors.</u>

...hence $n!$ admits (at least) $n(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p})$ irreducible factors.

$$120(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) = 124$$

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$$120(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}) \;=\; 124$$

To "admit (at least) 124 irreducible factors":
the value 120 can be broken down into (at least) 124 irreducible factors which live in $\mathrm{Int}(\mathbb{Z})$, and hence also live in $\mathbb{Z}$. (Lemma 1.1)

"We conclude with a proof that this number of factors may eventually be greater than nm, whatever m."

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$$L(n!) \;\geq\; n(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p})$$

Let

$$f(X) = n!\binom{X}{n} = \prod_{i=0}^{n-1}(x - i)$$

By def: $\mathrm{L}(f(X))$ is the upper bound of all possible lengths of $n!$ as a product of irreducible factors. Therefore, $\mathrm{L}(f(X))$ admits <u>at least</u> $n(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p})$ irreducible factors:

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Let $m = (\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p})$. Then

$$L(f(X)) \;\geq\; nm, \text{ whatever } m, \quad \text{and} \quad \ell(f(X)) \leq \; n$$

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$$\text{Elasticity} = \rho(f(X)) \ = \ \frac{L(f(X))}{\ell(f(X))} \geq \frac{nm}{n} = m$$

Elasticity of D: $\rho(\mathrm{D}) = \sup\ \{\ \rho(x) \mid \mathrm{x}$ nonzero nonunit $\}$
(The supremum (least upper bound) over the ratios of all possible lengths of $n!$ as a product of irreducible values).

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$$\rho(\mathrm{Int}(\mathbb{Z})) \;\geq\; \sup \left\{ \rho\left(n! \binom{X}{n} \right) \right\}$$

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$$\rho(\mathrm{Int}(\mathbb{Z})) \ \geq \ \sup\left\{\rho\left(n!\binom{X}{n}\right)\right\} \ = \ \infty$$