Propositions 1.4 and 1.5

Becky Egg

IMMERSE 2009
University of Nebraska-Lincoln

July 2, 2009
Recall:

Suppose $x \in D$ such that $x = x_1 \cdots x_k$ with $x_i$ irreducible in $D$. Then
Recall:

Suppose $x \in D$ such that $x = x_1 \cdots x_k$ with $x_i$ irreducible in $D$. Then

- $k$ is the length of this factorization
Recall:

Suppose $x \in D$ such that $x = x_1 \cdots x_k$ with $x_i$ irreducible in $D$. Then

- $k$ is the length of this factorization
- $\ell_D(x)$ is the shortest length
- $\rho(D) = \sup \left\{ \frac{\ell_D(x)}{\ell_D(x)} \mid x \in D \right\}$, the elasticity of $D$
Recall:

Suppose $x \in D$ such that $x = x_1 \cdots x_k$ with $x_i$ irreducible in $D$. Then

- $k$ is the length of this factorization
- $\ell_D(x)$ is the shortest length
- $L_D(x)$ is the upper bound of the set of lengths
- $\rho(D) = \sup\left\{ \frac{L_D(x)}{\ell_D(x)} \mid x \in D \right\}$, the elasticity of $D$
Recall:

Suppose $x \in D$ such that $x = x_1 \cdots x_k$ with $x_i$ irreducible in $D$. Then

- $k$ is the length of this factorization
- $\ell_D(x)$ is the shortest length
- $L_D(x)$ is the upper bound of the set of lengths
- $\rho(x) = \frac{L_D(x)}{\ell_D(x)}$, the elasticity of $x$
Recall:

Suppose $x \in D$ such that $x = x_1 \cdots x_k$ with $x_i$ irreducible in $D$. Then

- $k$ is the length of this factorization
- $\ell_D(x)$ is the shortest length
- $L_D(x)$ is the upper bound of the set of lengths
- $\rho(x) = \frac{L_D(x)}{\ell_D(x)}$, the elasticity of $x$
- $\rho(D) = \sup\{L_D(x)/\ell_D(x)|x \in D\}$, the elasticity of $D$
Proposition (1.4)

The elasticity of $\text{Int}(D)$ is greater than or equal to the elasticity of $D$. 
Proposition (1.4)

The elasticity of $\text{Int}(D)$ is greater than or equal to the elasticity of $D$.

▶ Recall Lemma 1.1:
Proposition (1.4)

The elasticity of $\text{Int}(D)$ is greater than or equal to the elasticity of $D$.

- Recall Lemma 1.1:
  - units of $\text{Int}(D)$ are units of $D$
Proposition (1.4)

The elasticity of \( \text{Int}(D) \) is greater than or equal to the elasticity of \( D \).

- Recall Lemma 1.1:
  - units of \( \text{Int}(D) \) are units of \( D \)
  - \( d \in D \) is irreducible in \( D \) \iff \( d \) is irreducible in \( \text{Int}(D) \)
Proposition (1.4)

*The elasticity of \( \text{Int}(D) \) is greater than or equal to the elasticity of \( D \).*

- Recall Lemma 1.1:
  - units of \( \text{Int}(D) \) are units of \( D \)
  - \( d \in D \) is irreducible in \( D \) \iff \( d \) is irreducible in \( \text{Int}(D) \)
- For \( d \in D \), factorization into irreducibles is the same in \( D \) and \( \text{Int}(D) \).
Proof of Proposition 1.4

Proof.

\[ \rho(D) = \sup \{ \frac{L(c)}{\ell(c)} | c \in D \subset \text{Int}(D) \} \]
Proof of Proposition 1.4

Proof.

\[ \rho(D) = \sup \{ \frac{L(c)}{\ell(c)} | c \in D \subset \text{Int}(D) \} \leq \sup \{ \frac{L(f)}{\ell(f)} | f \in \text{Int}(D) \} \]
Proposition (1.5)\\
Let $D$ be a BFD. For each $\alpha \in D$ and each $f \in \text{Int}(D)$,
\[
L_{\text{Int}(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \deg(f(x)) + L_D(f(\alpha))
\]
Proposition (1.5)

Let $D$ be a BFD. For each $\alpha \in D$ and each $f \in \text{Int}(D)$,

$$L_{\text{Int}(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \deg(f(x)) + L_D(f(\alpha))$$

with the following conventions:
Proposition (1.5)

Let $D$ be a BFD. For each $\alpha \in D$ and each $f \in \text{Int}(D)$,

$$L_{\text{Int}(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \deg(f(x)) + L_D(f(\alpha))$$

with the following conventions:

- $L_D(u) = 0$ if $u$ is a unit in $D$
Proposition (1.5)

Let $D$ be a BFD. For each $\alpha \in D$ and each $f \in \text{Int}(D)$,

$$L_{\text{Int}(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \deg(f(x)) + L_D(f(\alpha))$$

with the following conventions:

- $L_D(u) = 0$ if $u$ is a unit in $D$
- $L_D(0) = \infty$
Proof of Proposition 1.5

- $D$ a BFD $\Rightarrow$ ACCP $\Rightarrow$ atomic
Proof of Proposition 1.5

- $D$ a BFD $\Rightarrow$ ACCP $\Rightarrow$ atomic
- $D$ satisfies ACCP $\Rightarrow$ $Int(D)$ satisfies ACCP $\Rightarrow$ $Int(D)$ is atomic
Proof of Proposition 1.5

- $D$ a BFD $\Rightarrow$ ACCP $\Rightarrow$ atomic
- $D$ satisfies ACCP $\Rightarrow$ $Int(D)$ satisfies ACCP $\Rightarrow$ $Int(D)$ is atomic
- factorization into a finite number of irreducibles is possible
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.

\[ s \leq L_D(\lambda_1 \cdots \lambda_s) \]
Proof of 1.5, cont.

Suppose
\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]
where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.

\[ s \leq L_D(\lambda_1 \cdots \lambda_s) \]

\[ L_D(\lambda_1 \cdots \lambda_s) \leq L_D(\lambda_1 \cdots \lambda_s g_1(\alpha) \cdots g_t(\alpha)) = L_D(f(\alpha)) \]
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.

\[ s \leq L_D(\lambda_1 \cdots \lambda_s) \]

\[ L_D(\lambda_1 \cdots \lambda_s) \leq L_D(\lambda_1 \cdots \lambda_s g_1(\alpha) \cdots g_t(\alpha)) = L_D(f(\alpha)) \]

Recall:

\[ L_D(u) = 0 \]
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in Int(D) \) is nonconstant.

\[ s \leq L_D(\lambda_1 \cdots \lambda_s) \]

\[ L_D(\lambda_1 \cdots \lambda_s) \leq L_D(\lambda_1 \cdots \lambda_s g_1(\alpha) \cdots g_t(\alpha)) = L_D(f(\alpha)) \]

Recall:
- \( L_D(u) = 0 \)
- \( L_D(0) = \infty \)
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.

\[ s \leq L_D(\lambda_1 \cdots \lambda_s) \]

\[ L_D(\lambda_1 \cdots \lambda_s) \leq L_D(\lambda_1 \cdots \lambda_s g_1(\alpha) \cdots g_t(\alpha)) = L_D(f(\alpha)) \]

Recall:

\[ L_D(u) = 0 \]

\[ L_D(0) = \infty \]

Conclude \( s \leq L_D(f(\alpha)) \).
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in Int(D) \) is nonconstant.

\[ t \leq L_{K[x]}(\lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)) = L_{K[x]}(f(x)) \]
Suppose
\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]
where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.

\[ t \leq L_K[x](\lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)) = L_K[x](f(x)) \]

\[ L_K[x](f(x)) \leq \deg(f(x)) \]
Proof of 1.5, cont.

Suppose

\[ f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x) \]

where \( \lambda_i \in D \) is nonzero, nonunit, and \( g_j(x) \in \text{Int}(D) \) is nonconstant.

\[ t \leq L_{K[x]}(\lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)) = L_{K[x]}(f(x)) \]

\[ L_{K[x]}(f(x)) \leq \deg(f(x)) \]

Conclude that

\[ t \leq L_{K[x]}(f(x)) \leq \deg(f(x)). \]
Proof of 1.5, cont.

Combine these inequalities to get

\[ t + s \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \deg(f(x)) + L_D(f(\alpha)). \]
Combine these inequalities to get

\[ t + s \leq L_{K[f]}(f(x)) + L_D(f(\alpha)) \leq \deg(f(x)) + L_D(f(\alpha)). \]

Since \( L_{\text{Int}(D)}(f(x)) = \sup\{s + t | f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)\} \), we have that
Proof of 1.5, cont.

Combine these inequalities to get

$$t + s \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \text{deg}(f(x)) + L_D(f(\alpha)) .$$

Since $L_{\text{Int}(D)}(f(x)) = \sup\{s + t| f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)\}$, we have that

$L_{\text{Int}(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq \text{deg}(f(x)) + L_D(f(\alpha))$. □