Propositions 1.4 and 1.5

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- $\rho(x) = \frac{L_D(x)}{\ell_D(x)}$, the elasticity of x
- $\rho(D) = \sup\{L_D(x)/\ell_D(x) | x \in D\}$, the elasticity of D

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- Recall Lemma 1.1:
  - units of $Int(D)$ are units of D
  - $d \in D$ is irreducible in $D \iff d$ is irreducible in $Int(D)$
- For $d \in D$, factorization into irreducibles is the same in D and $Int(D)$.

Proof of Proposition 1.4

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$$\leq \sup\{L(f)/\ell(f) | f \in Int(D)\}$$

□

Proposition (1.5)

Let D be a BFD. For each $\alpha \in D$ and each $f \in Int(D)$,

$$L_{Int(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq deg(f(x)) + L_D(f(\alpha))$$

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- D satisfies ACCP $\Rightarrow$ $Int(D)$ satisfies ACCP $\Rightarrow$ $Int(D)$ is atomic
- factorization into a finite number of irreducibles is possible

Proof of 1.5, cont.

Suppose

$$f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)$$

where $\lambda_i \in D$ is nonzero, nonunit, and $g_j(x) \in Int(D)$ is nonconstant.

Proof of 1.5, cont.

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$$s \leq L_D(\lambda_1 \cdots \lambda_s)$$

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$$L_D(\lambda_1 \cdots \lambda_s) \leq L_D(\lambda_1 \cdots \lambda_s g_1(\alpha) \cdots g_t(\alpha)) = L_D(f(\alpha))$$

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Recall:

- $L_D(u) = 0$

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Recall:

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Conclude $s \leq L_D(f(\alpha))$.

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where $\lambda_i \in D$ is nonzero, nonunit, and $g_j(x) \in Int(D)$ is nonconstant.

$$t \leq L_{K[x]}(\lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)) = L_{K[x]}(f(x))$$

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$$L_{K[x]}(f(x)) \leq deg(f(x))$$

Conclude that

$$t \leq L_{K[x]}(f(x)) \leq deg(f(x)).$$

Proof of 1.5, cont.

Combine these inequalities to get

$$t + s \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq deg(f(x)) + L_D(f(\alpha)).$$

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Since $L_{Int(D)}(f(x)) = \sup\{s + t | f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)\}$, we have that

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Combine these inequalities to get

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Since $L_{Int(D)}(f(x)) = \sup\{s + t | f(x) = \lambda_1 \cdots \lambda_s g_1(x) \cdots g_t(x)\}$, we have that

$$L_{Int(D)}(f(x)) \leq L_{K[x]}(f(x)) + L_D(f(\alpha)) \leq deg(f(x)) + L_D(f(\alpha)). \square$$