Theorem 1.3

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A fabulous theorem

Theorem

The ring $\operatorname{Int}(D)$ *satisfies ACCP if and only if* D *satisfies ACCP.*

Proving this fabulous theorem ($\Rightarrow$)

- Suppose $\mathrm{Int}(D)$ satisfies ACCP, and let a_nD be an increasing sequence of principal ideals of D.
- Note that $a_n\mathrm{Int}(D)$ is also increasing, so there is some n_0 such that for all $n \geq n_0$, $a_n\mathrm{Int}(D) = a_{n_0}\mathrm{Int}(D)$.

Proving this fabulous theorem ($\Rightarrow$)

- Suppose Int(D) satisfies ACCP, and let $a_n D$ be an increasing sequence of principal ideals of D.
- Note that $a_n\text{Int}(D)$ is also increasing, so there is some n_0 such that for all $n \geq n_0$, $a_n\text{Int}(D) = a_{n_0}\text{Int}(D)$.
- Then $a_n = \lambda_n a_{n_0}$, where λ_n is a unit of Int(D) and therefore a unit of D.
- So $a_n D = a_{n_0} D$ for all $n \geq n_0$ (from HWII 1b).
- Thus D satisfies ACCP.

Proving this fabulous theorem ($\Leftarrow$)

- Suppose D satisfies ACCP, and let $a_n\text{Int}(D)$ be an increasing sequence of nonzero principal ideals of $\text{Int}(D)$.
- Then $f_nK[x]$ is an increasing sequence of principal ideals in $K[x]$, so it stabilizes.

Proving this fabulous theorem ($\Leftarrow$)

- Suppose D satisfies ACCP, and let $a_n \text{Int}(D)$ be an increasing sequence of nonzero principal ideals of $\text{Int}(D)$.
- Then $f_n K[x]$ is an increasing sequence of principal ideals in $K[x]$, so it stabilizes.
- That is, there exists n_0 such that, for $n \geq n_0$, $f_n K[x] = f_{n_0} K[x]$, so $f_n = \lambda_n f_{n_0}$, where λ_n is a nonzero element of K, and therefore a unit.

Proving this fabulous theorem ($\Leftarrow$)

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Case 1: $D = K$

- Note that in this case, $\text{Int}(D) = K[x]$, so we are done.

Proving this fabulous theorem ($\Leftarrow$, cont'd)

Case 2: $D \neq K$

- In this case, D is infinite, so $\exists a \in D$ such that $f_{n_0}(a) \neq 0$.

Proving this fabulous theorem ($\Leftarrow$, cont'd)

Case 2: $D \neq K$

- In this case, D is infinite, so $\exists a \in D$ such that $f_{n_0}(a) \neq 0$.
- The sequence $f_n(a)D$ is increasing and stabilizes.

Proving this fabulous theorem ($\Leftarrow$, cont'd)

Case 2: $D \neq K$

- In this case, D is infinite, so $\exists a \in D$ such that $f_{n_0}(a) \neq 0$.
- The sequence $f_n(a)D$ is increasing and stabilizes.
- Thus $\exists m_0 \geq n_0$ such that, for $n \geq m_0$, $f_n(a) = \mu_n f_{m_0}(a)$, where μ_n is a unit of D.
- It follows that $f_n = \mu_n f_{m_0}$

Proving this fabulous theorem ($\Leftarrow$, cont'd)

Case 2: $D \neq K$

- In this case, D is infinite, so $\exists a \in D$ such that $f_{n_0}(a) \neq 0$.
- The sequence $f_n(a)D$ is increasing and stabilizes.
- Thus $\exists m_0 \geq n_0$ such that, for $n \geq m_0$, $f_n(a) = \mu_n f_{m_0}(a)$, where μ_n is a unit of D.
- It follows that $f_n = \mu_n f_{m_0}$

- Therefore $f_n\mathrm{Int}(D) = \mu_n f_{m_0}\mathrm{Int}(D)$ (again from HWII 1b).
- So Int$D$ satisfies ACCP.