Lemma 1.1 and Prop 1.2

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Description of Int(D)

D is an integral domain with quotient field K.

$$Int(D) = \{f \in K[X] | f(D) \subseteq D\}.$$

NOTE: The coefficients of polynomials come from K.

Important Claims

Claim 1: $D \subseteq Int(D)$.

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Claim 2: Constant polynomials in Int(D) are elements of D.

Lemma 1.1 part i

Lemma (1.1i)

The units of Int(D) are the units of D.

Proof.

Suppose $u(x)$ is a unit in Int(D). Show $u(x)$ is a unit in D.

Since $u(x)$ is a unit in Int(D), there exists $(u(x))^{-1} \in Int(D)$ such that $u(x)u(x)^{-1} = 1$.

It follows that $deg(u(x)) + deg((u(x))^{-1} = 0$. Therefore, $deg(u(x)) = deg(u(x))^{-1} = 0$.

Hence, $u(x)$ and $u(x)^{-1}$ are constant polynomials, so $u(x) = c_1 \in D$ and $(u(x))^{-1} = c_1^{-1} \in D$.

□

Lemma 1.1 part i (cont'd)

Proof.

Suppose u is a unit in D. Show u is a unit in Int(D).
Since u is a unit in D, there exists $u^{-1} \in D$ such that $uu^{-1} = 1$.
But elements of D are in Int(D). Therefore, $u^{-1} \in Int(D)$. □

Lemma 1.1 part ii

Lemma (1.1ii)

Given $d \in D$, d is irreducible in Int(D) if and only if it is irreducible in D.

Proof.

Suppose $d \in D$ is irreducible in Int(D), show it is irreducible in D.

Let $d = aq$, $a \in D$, $q \in D$. Show either a or q is a unit in D.

Since d is irreducible in Int(D) and $d = aq$, then either a or q is a unit in Int(D). (NOTE: This makes sense because since a and q are elements of D they are also elements of Int(D)).

From Lemma 1.1 part(i) it follows that a or q is a unit in D. So, d is irreducible in D. □

Lemma 1.1 part ii (cont'd)

Proof.

Suppose d is irreducible in D, show it is irreducible in Int (D).
Let $d = a(x)q(x)$, $a(x), q(x) \in Int(D)$, show either $a(x)$ or $q(x)$ is a unit in Int(D).

Observe $deg(d) = 0$ so $deg(a(x)) = 0$ and $deg(q(x)) = 0$.
Therefore, $a(x), q(x)$ are constant polynomials a and q, $a \in D$ and $q \in D$.

Since d is irreducible in D and $d = aq$, $a, q \in D$, then either a or q is a unit.
By Lemma 1.1 part(i) either a or q has to be a unit in Int(D).

□

Proposition 1.2

Proposition

If Int(D) is atomic, then D is atomic.

Proof.

Let $d \in D \subseteq Int(D)$. Since Int(D) is atomic, $d = f_1(x)...f_n(x)$ where $f_1, ..., f_n \in Int(D)$ and $f_1, ..., f_n$ are irreducible.

Since $deg(d) = 0$, then $deg(f_1(x)) + ... + deg(f_n(x)) = 0$. Hence, $deg(f_1(x)) = ... = deg(f_n(x)) = 0$.

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Since $deg(d) = 0$, then $deg(f_1(x)) + ... + deg(f_n(x)) = 0$. Hence, $deg(f_1(x)) = ... = deg(f_n(x)) = 0$.

So, $f_1, ..., f_n$ are constant polynomials. Therefore, $f_1...f_n \in D$. Further, by Lemma 1.1 part (ii), since $f_1, ..., f_n$ are irreducible in Int(D) then they are irreducible in D.
So, $d = f_1...f_n$, is a product of irreducible elements in D. Hence, D is atomic. □