1. Prove that $D$ is an integral domain of Krull dimension 0 if and only if $D$ is a field.

Proof. ($\Rightarrow$) Let $D$ be an integral domain of Krull dimension 0. Need to show $D$ is a field. We know the only prime ideal of $D$ is $\langle 0 \rangle$. Further, we know every maximal ideal is prime. Then, $\langle 0 \rangle$ is a maximal ideal of $D$. Then, $D/\langle 0 \rangle \cong D$ is a field.

($\Leftarrow$) Let $D$ be a field. Since $D$ only has one prime ideal, $\langle 0 \rangle$, we know it is an integral domain of Krull dimension 0. □

2. A ring $R$ satisfies the **descending chain condition (DCC)** on (prime) ideals if whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a decreasing chain of (prime) ideals of $R$, then there is a positive integer $n$ such that $I_k = I_n$ for all $k \geq n$.

Prove that an integral domain that satisfies DCC on all ideals has dimension 0.

**Note:** A ring that satisfies DCC on all ideals is called Artinian.

Proof. First we note that in an integral domain zero is prime because if $ab = 0$ then either $a = 0$ or $b = 0$. Thus $\langle 0 \rangle$ is a prime ideal. Now we will show $\langle 0 \rangle$ is maximal, in fact in a ring satisfying DCC (an Artinian ring) every prime ideal is maximal, this is what we will show.

Consider $B = D/P$ where $P$ is a prime ideal and thus $B$ is an (Artinian) integral domain. Now let $x \in B$ be nonzero, then by DCC we know that for large enough $n$, $(x^{n+1}) = (x^n)$. This means we may write $x^{n+1}y = x^n$ for some nonzero $y \in B$. Because we are in an integral domain we may cancel $x^n$ on both sides leaving $xy = 1$ which means $x$ is a unit. Since this was any arbitrary nonzero $x \in B$, we conclude $B$ is a field, and therefore $P$ is maximal.

So since we are in an integral domain satisfying DCC we can conclude the zero ideal is both prime and maximal. Thus $\langle 0 \rangle$ is the only element in a chain of prime ideals, which implies by definition the Krull dimension of $D$ is zero. □

3. Let $I_1, I_2, \ldots, I_n$ be ideals in a ring $R$. Prove that the product $I_1I_2\cdots I_n$ is an ideal of $R$ contained in each $I_j$, $j = 1, 2, \ldots, n$.

Proof. Let $a = \sum_{i=1}^{m} a_1a_{2i} \cdots a_{ni}$ and $b = \sum_{i=1}^{l} b_1b_{2i} \cdots b_{ni}$ be elements of $I_1I_2\cdots I_n$. Then $a - b = \sum_{i=1}^{m+l} c_1c_{2i} \cdots c_{ni}$, where the $c_{ji}$’s are the $a_{ji}$’s for $1 \leq i \leq m$ and the $-b_{ji}$’s for $m + 1 \leq i \leq m + n$. Since the $I_j$’s are ideals, $-b_{ji} \in I_j$ for all $i, j$, so $a - b \in I_1I_2\cdots I_n$. Now let $x \in R$. Then $xa = \sum_{i=1}^{m} a_1a_{2i} \cdots a_{ni} = \sum_{i=1}^{m} (xa_{1i})a_{2i} \cdots a_{ni}$.
but \( xa_i \in I_i \) for each \( i \) since \( I_i \) is an ideal. So by definition \( xa \in I_1 I_2 \cdots I_n \). So \( I_1 I_2 \cdots I_n \) is an ideal by the ideal test.

To show it is contained in each \( I_j \), note that for all \( i, j, a_1 a_2 \cdots a_m \in I_j \) since it is a product of some element of \( I_j \) and other elements of the ring. So, since each \( I_j \) is closed under addition, \( a \in I_j \) for each \( j \). Therefore \( I_1 I_2 \cdots I_n \subseteq I_j \) for each \( I_j \). \( \square \)

4. Let \( R \) be a ring. If \( I \) and \( J \) are relatively prime (i.e. \( I + J = R \)), prove that \( IJ = I \cap J \).

**Proof.** We first show \( IJ \subseteq I \cap J \). If \( a \in IJ \), we can write \( a = \sum_{k=1}^{n} i_k j_k \) for some elements \( i_k \in I \) and \( j_k \in J \). Since both \( I \) and \( J \) are ideals, we have that \( i_k j_k \) is an element of both \( I \) and \( J \). Furthermore, the sum \( \sum_{k=1}^{n} i_k j_k \) is also in both \( I \) and \( J \). Thus, \( a \in I \cap J \).

For the other containment, note that the hypothesis \( I + J = R \) implies that 1 can be written as \( 1 = i + j \), where \( i \in I \) and \( j \in J \). Let \( a \in I \cap J \) be given. Then \( a = 1 \cdot a = (i + j)a = ia + ja \). Since \( a \in J \), we have \( ia \in IJ \). Similarly, \( ja \in IJ \), so \( a = ia + ja \in IJ \). \( \square \)

5. Let \( D \) be an integral domain with quotient field \( K \). If \( K \) is a fractional ideal of \( D \), prove that \( D = K \).

**Proof.** Let \( D \) be an integral domain with quotient field \( K \), where \( K \) is a fractional ideal of \( D \).

\[ \subseteq \] We know that \( D \subseteq K \) by the definition of a quotient field.

\[ \supseteq \] Since \( K \) is a fractional ideal of \( D \), we know there exists \( d \in D \) nonzero such that \( dK \subseteq D \). But notice that since \( K \) is the quotient field, we have \( dK = K \). Thus \( D \supseteq K \).

Therefore \( D = K \). \( \square \)