Homework IV:

1. (a) (Subfield test) Let $E$ be a field and let $F$ be a subset of $E$ with at least two elements. Prove that $F$ is a subfield of $E$ if, for any $a, b \in F$ ($b \neq 0$), $a - b$ and $ab^{-1}$ belong to $F$.

(b) $\star$ Let $E$ be a field and $\{F_\alpha\}$ be a family of subfields of $E$. Prove that $\bigcap \alpha F_\alpha$ is a subfield of $E$.

2. $\star$ Let $F$ be a field and let $p(x) \in F[x]$. Prove the following: If $f(x), g(x) \in F[x]$ such that $\deg f(x) < \deg p(x)$, $\deg g(x) < \deg p(x)$, and $f(x) + (p(x)) = g(x) + (p(x))$ in $F[x]/(p(x))$, then $f(x) = g(x)$.

3. $\star$ (Eisenstein’s Criterion for $\mathbb{Z}[x]$) Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x], \ n \geq 1.$$ 

If there is a prime $p$ such that

(a) $p$ doesn’t divide $a_n$,

(b) $p$ divides $a_i$ for $i \in \{0, 1, \ldots, n-1\}$, and

(c) $p^2$ doesn’t divide $a_0$,

then $f(x)$ is irreducible over $\mathbb{Q}$.

4. (a) Let $F$ be a field and let $p(x) \in F[x]$. Prove that $(p(x))$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible in $F[x]$.

(b) Let $F$ be a field, $I$ a nonzero ideal in $F[x]$, and $g(x)$ an element of $F[x]$. Prove that $I = (g(x))$ if and only if $g(x)$ is a nonzero polynomial of minimal degree in $I$.

5. Let $R$ be an integral domain that contains a field $F$ as a subring. If $R$ is finite-dimensional when viewed as a vector space over $R$, prove that $R$ is a field.

6. $\star$ Let $F \subseteq L \subseteq E$ be fields. Prove that

$$[E : F] = [E : L][L : F].$$

If one side of the above equation is infinite, then the other side is also infinite.

7. $\star$ Let $E$ be an extension of a field $F$. Prove that $F(a, b) = [F(a)](b) = [F(b)](a)$.

8. (a) Prove that the polynomial $f(x) = x^4 + 2x^3 + 6x^2 + x + 9 \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$. (Hint: Use Mod $p$ Irreducibility Test (Homework II #8).)

(b) Let $a$ be a root of $f(x)$ in some extension of $\mathbb{Q}$. Prove that $\sqrt{2} \not\in \mathbb{Q}(a)$. 

9. ★ Let \( F \subset L \subset E \) be fields. Prove that if \( E/L \) is an algebraic extension, and \( L/F \) is an algebraic extension, then \( E/F \) is an algebraic extension.

10. Prove the following: The field \( E \) is an algebraic closure of itself if and only if \( E \) is algebraically closed.

11. Suppose that \( F \) is a field and every irreducible polynomial in \( F[x] \) is linear. Prove that \( F \) is algebraically closed.