IMMERSE 2009 — Algebra

Homework I:

1. Let $R$ be a commutative ring with unity. Then, for all $a, b, c \in R$ prove the following properties:
   (a) Left (right) cancellation law with respect to addition: if $a + b = a + c \ (b + a = c + a)$, then $b = c$;
   (b) $0a = 0$;
   (c) $(-a)b = a(-b) = -(ab)$;
   (d) $(-a)(-b) = ab$;
   (e) $-a = (-1)a$.

2. Decide which of the following sets under the given operations are commutative rings with unity/integral domains/fields:
   (a) $\mathbb{Q}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$, under the operations of addition and multiplication as in $\mathbb{R}$;
   (b) $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$, under the operations of addition and multiplication as in $\mathbb{R}$;
   (c) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, under the operations of addition and multiplication as in $\mathbb{C}$;
   (d) $n\mathbb{Z}, \ n > 1$, with usual addition and multiplication.

3. Prove the cancellation property for integral domains: If $ab = ac$ in an integral domain $R$, then $b = c$.

4. ★ Prove that any finite integral domain is a field.

5. ★ Let $R$ be an integral domain and $p(x), q(x)$ be nonzero elements of $R[x]$. Then
   (a) $R[x]$ is an integral domain;
   (b) $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$;
   (c) the units of $R[x]$ are just the units of $R$.

6. ★ Let $R = \{f \in \mathbb{Q}[x] : f(x) \in \mathbb{Z} \forall x \in \mathbb{Z}\}$. Prove that $R$ is a subring of $\mathbb{Q}[x]$ that contains $\mathbb{Z}[x]$. Conclude that $R$ is an integral domain.

7. Describe the quotient ring of the Gaussian integers $\mathbb{Z}[i]/(2 - i)$.

8. Let $R$ be a ring and $I$ be an ideal of $R$. Prove:
   (a) $I = R$ if and only if $I$ contains a unit.
   (b) $R$ has at least one maximal ideal.
   (c) The ideal $(0)$ is maximal if and only if $R$ is a field.

9. (a) Show that the ideal $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.
   (b) Show that the ideal $(x^2 + 1)$ is not prime in $\mathbb{Z}_2[x]$.

10. ★ Let $R$ be a ring and $I$ an ideal of $R$.
    (a) Prove that $I$ is prime if and only if $R/I$ is an integral domain.
    (b) Prove that $I$ is maximal if and only if $R/I$ is a field.
    (c) Every maximal ideal of $R$ is a prime ideal.

11. (a) Prove that $\mathbb{Z}_n$ is a ring for all $n \in \mathbb{Z}^+$. Furthermore, $\mathbb{Z}_p$ is a field if and only if $p$ is prime.
    (b) Prove that $\mathbb{Z}[t]/p\mathbb{Z}[t] \cong \mathbb{Z}_p[t]$.

Supplemental Reading and Exercises

- Read Chapter 1 (pages 1–5) of M. F. Atiyah and I. G. MacDonald Introduction to Commutative Algebra.
- Work exercises # 1 and # 2.