IMMERSE 2009 — Algebra

Historical Introduction

0 C.E. Probably no one knew what an integral-valued polynomial was, and certainly no one cared.

1668 Isaac Newton used finite differences to tabulate square and cube roots. James Gregory later gave a corresponding formula. The general idea, now called the *Gregory-Newton interpolation formula*, is the following: Given a function $y = f(x)$ containing a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ we wish to find a degree $n-1$ polynomial which matches the given function at these n points. We do so recursively as follows:

P_0: Set $P_0(x) = y_1$, the constant polynomial which matches our given function at (x_1, y_1).

P_1: Given two points (x_1, y_1) and (x_2, y_2), we set P_1 to be the linear function

$$P_1(x) = P_0(x) + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

through these two points.

P_k: Suppose we now have $k+1$ points. For $1 \leq k \leq n$, we define P_k in terms of P_{k-1} as follows: We have that $P_{k-1}(x_i) = y_i$ for $1 \leq i \leq k$. Now set

$$P_k(x) = P_{k-1}(x) + \left(\frac{y_{k+1} - P_{k-1}(x_{k+1})}{(x_{k+1} - x_1)(x_{k+1} - x_2)\cdots(x_{k+1} - x_k)}\right)(x - x_1)(x - x_2)\cdots(x - x_k).$$

Example: Consider the following set of 5 points $(0, -5), (1, -3), (-1, -15), (2, 39), (-2, -9)$. Then

$$\begin{aligned}
P_0(x) &= -5, \\
P_1(x) &= -5 + 2x, \\
P_2(x) &= -5 + 2x - 4x(x-1), \\
P_3(x) &= -5 + 2x - 4x(x-1) + 8x(x-1)(x+1), \text{ and} \\
P_4(x) &= -5 + 2x - 4x(x-1) + 8x(x-1)(x+1) + 3x(x-1)(x+1)(x-2).
\end{aligned}$$

The general formula for the Gregory-Newton interpolating polynomial for a function f at points $0, h, 2h, 3h, \ldots, nh$ is

$$f(hx) \approx \sum_{k=0}^{n} \Delta^k f(0) \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

where $\Delta^k f(0)$ is the kth forward difference of f defined recursively by $\Delta^k f(0) = \Delta(\Delta^{k-1} f(0))$ with $\Delta f(0) = f(h) - f(0)$. For example,

$$\begin{aligned}
\Delta^2 f(0) = \Delta(\Delta f(0)) &= \Delta(f(h) - f(0)) = \Delta f(h) - \Delta f(0) \\
&= [f(2h) - f(h)] - [f(h) - f(0)] = f(2h) - 2f(h) + f(0).
\end{aligned}$$

Note that this is essentially a finite Taylor-MacLaurin expansion for f.

1600s Thomas Harriod developed the polynomials for $n = 5$ and Henry Briggs used these interpolating polynomials to tabulate logarithms.

Algebra If $h = 1$, then

$$f(x) \approx \sum_{k=0}^{n} \Delta^k f(0) \binom{x}{k};$$

$$\Delta f(0) = f(1) - f(0) \text{ and } \Delta^k f(0) = \Delta(\Delta^{k-1} f(0)) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(i).$$

If $f(x) \in \mathbb{Z}$ whenever $x \in \mathbb{Z}$, then we also must have that each $\Delta^k f(0)$ is an integer. The converse is also true since each $\binom{x}{k} \in \mathbb{Z}$.

1919 Following a suggestion of Hurwitz, both Georg Pólya and Alexander Ostrowski published a paper (in the same edition of the same journal, pp 97–116, 117–124 of J. Reine Angew. Math.) with the title *Über ganzwertige Polynome in algebraischen Zahlkörpen* (On integral-valued polynomials in algebraic number fields). These were the first papers to generally study integral-valued polynomials.

Given an algebraic number field K, their goal was to study polynomials $f(x)$ with coefficients in K which take "integral" values on the "integers" of K.

For example, $f(x) = \dfrac{x(x-1)}{i+1}$ is a polynomial with coefficients in $\mathbb{Q}[i]$ that satisfies $f(\alpha) \in \mathbb{Z}[i]$ whenever $\alpha \in \mathbb{Z}[i]$.

Pólya asked the following question: Does there exists a finite set $\{f_k\}$ of integral-valued polynomials such that f_k has degree k and such that each integral-valued polynomial of degree n can be written as $a_0 f_0 + a_1 f_1 + \cdots + a_n f_n$ for some appropriate subset $\{a_i\}$ of the ring of integers?

Together, Pólya and Ostrowski showed that the answer is yes if certain ring-theoretic conditions are satisfied. However, most of their results focused on rational integers; $\mathbb{Z} \subseteq \mathbb{Q}$.

1936 Thoralf Skolem was the first to consider the set of integral-valued polynomials as a ring.

1951 Ernst Strauss considered the set of integral-valued polynomials whose derivatives are also integral-valued.

1955 Nicolaas G. deBruijn considered the set of integral-valued polynomials whose finite divided differences are also integral-valued.

1959 Leonard Carlitz extended the results of deBruijn.

1960 – 2009 There are now well over 200 research articles and several texts dealing with integer valued polynomials. The 1995 paper *Elasticity for integral-valued polynomials* by Paul-Jean Cahen and Jean-Luc Chabert investigates how bad non-unique factorization can be in rings of integral-valued polynomials.