III. Noetherian Rings, Atomicity & Elasticity

Definition 1. A ring $R$ is Noetherian provided every ideal is finitely generated.

Example 2. All PID’s are Noetherian rings.

Theorem 3. Let $R$ be a ring. The following statements are equivalent.

1. Every ideal in $R$ is finitely generated.

2. Every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ stabilizes; i.e., there exists some $n$ for which $I_k = I_{k+1}$ for all $k \geq n$.

3. Every nonempty set of ideals in $R$ has a maximal element with respect to inclusion.

Lemma 4. Let $R$ be a ring and let $I$ be an ideal of the polynomial ring $R[x]$. The set $J$ consisting of all leading coefficients of polynomials contained in $I$ forms an ideal of $R$.

Theorem 5. (Hilbert Basis Theorem) If $R$ is a Noetherian ring, then so is the polynomial ring $R[x]$.

Example 6.

1. If $R$ is a Noetherian ring, $R[x_1,x_2,\ldots,x_n]$ is also Noetherian.

2. The ring $\mathbb{Z}[x_1,x_2,\ldots]$ is not Noetherian.

Proposition 7. If $R$ is a Noetherian ring and $I$ is a proper ideal of $R$, then $R/I$ is a Noetherian ring.

Example 8. Any nontrivial quotient of a polynomial ring in finitely many variables with coefficients in a Noetherian ring is Noetherian. For example, the following rings are Noetherian.

1. $\mathbb{Q}[x,y]/(x^2 + y^2 - 1)$

2. $\mathbb{Z}[x,y,z]/(2x,3y,5z)$
Definition 9. An integral domain $D$ is an **atomic domain** provided each nonzero nonunit element of $D$ can be expressed as a finite product of irreducible elements or atoms.

Example 10.

1. All integral domains that satisfy ACCP are atomic domains.
2. The ring $\mathbb{C}[\{x^r \mid r \in \mathbb{Q}^+\}]$ is not atomic.
3. The ring $\mathbb{C}[x^{1/3}, x^{1/2^5}, x^{1/2^27}, \ldots, x^{1/2^k} p_{k+1}, \ldots]$, where $p_k$ denotes the $k$th odd prime, illustrates that a ring can be atomic but not satisfy ACCP.

Definition 11. Let $D$ be an atomic domain and let $x \in D$ be a nonzero nonunit. If $x$ can be factored as $x = a_1a_2\cdots a_n$ with each $a_i$ irreducible, then $n$ is called the **length** of this factorization of $x$.

The symbol $l_D(x)$ denotes the length of the shortest factorization of $x$, and the symbol $L_D(x)$ denotes the upper bound of the lengths of all factorizations of $x$.

The **elasticity of an element** $x \in D$, denoted $\rho(x)$ is the (possibly infinite) ratio $\rho(x) = L_D(x)/l_D(x)$.

The **elasticity of the domain** $D$ is then

$$\rho(D) = \sup \left\{ \frac{m}{n} \mid x_1 \cdots x_m = y_1 \cdots y_n \text{ for } x_i \text{ and } y_j \text{ irreducible elements of } D \right\}.$$

Example 12.

1. $1 \leq \rho(D) \leq \infty$ for every atomic domain $D$.
2. If $D$ is a UFD, then $\rho(D) = 1$. The converse is not true.
3. $\rho(\mathbb{Z}) = 1$
4. Let $m$ and $n$ be positive integers with $m < n$ and $m \nmid n$. Then $\rho(\mathbb{Q}[x^m, x^n]) \geq n/m$.

Definition 13. An atomic domain $D$ is a **bounded factorization domain** or **BFD** if, for each nonzero nonunit $x \in D$, $L_D(x) < \infty$.

Example 14.

1. Every UFD is a BFD.
2. All Noetherian rings are BFDs.
3. If $R$ is a BFD then $R$ satisfies ACCP.
4. The ring $\mathbb{Q}[x^{1/2}, x^{1/3}, x^{1/5}, x^{1/7}, \ldots]$ is not a BFD but does satisfy ACCP.
Homework III:

1. (Special case of Proposition 6.1 and Proposition 6.2 (Chapter 6) of M. F. Atiyah and I. G. MacDonald “Introduction to Commutative Algebra”)

Let $R$ be a ring. Prove that the following statements are equivalent:

(a) Every ideal in $R$ is finitely generated.

(b) Every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ stabilizes; i.e., there exists some $n$ for which $I_k = I_{k+1}$ for all $k \geq n$.

(c) Every nonempty set of ideals in $R$ has a maximal element with respect to inclusion.

2. Let $R$ be a Noetherian ring and let $I$ be an ideal of $R$. Prove that $R/I$ is a Noetherian ring.

3. ∗ Prove that if $D$ is a BFD, then $D$ satisfies the ascending chain condition on principal ideals.

4. Prove that an atomic domain $D$ is a BFD if and only if $\rho(x) < \infty$ for every nonzero nonunit $x \in D$.

5. ∗ Let $D$ be an integral domain with quotient field $K$. The ring of integral-valued polynomials over $D$ is $\text{Int}(D) = \{ f \in K[x] \mid f(x) \in D \forall x \in D \}$.

(a) Prove that if $D = K$, then $\text{Int}(D) = K[x]$.

(b) Prove that $\text{Int}(D)$ is an integral domain.

(c) Prove for any positive integer $n$, \( x^n \) is an element of $\text{Int}(\mathbb{Z})$.

6. ∗ Let $\mathcal{P} = \{ p_k \}$ be an enumeration of the prime integers.

(a) Prove that $\mathcal{P}$ is an infinite set.

(b) Prove that the sum $\sum_{p \in \mathcal{P}} \frac{1}{p}$ is divergent.

7. ∗ Before Thursday’s presentations, carefully read Section 0 of the paper “Elasticity for integral-valued polynomials” by Paul-Jean Cahen and Jean-Luc Chabert.

Supplemental Reading and Exercises

- Read Propositions 6.1 and 6.2 (pages 74–75) and Chapter 7 (pages 80–82) of M. F. Atiyah and I. G. MacDonald “Introduction to Commutative Algebra.”

- Work exercises # 1 and # 4 from Chapter 7.