VII. Krull Dimension & Fractional Ideals

**Definition 1.** A ring \( R \) has **Krull dimension** \( n \) (or just the **dimension** \( n \)) if \( n \) is the maximum positive integer such that \( P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \) is a chain of \( n+1 \) distinct prime ideals in \( R \). The dimension of \( R \) is infinite if \( R \) has arbitrary long chains of distinct prime ideals.

**Example 2.**
1. Every field has dimension 0.
2. Every PID that is not a field has dimension 1.
3. If \( F \) is a field, then \( F[x_1, x_2, \ldots, x_n] \) has dimension at least \( n \).
4. If \( F \) is a field, then \( F[x_1, x_2, \ldots, x_n, \ldots] \) has infinite dimension.

**Lemma 3.** Let \( I_1, I_2, \ldots, I_n \) be ideals in a ring \( R \).

1. The product

\[
I_1I_2 \cdots I_n = \left\{ \text{all finite sums } \sum_i a_{1i}a_{2i}\cdots a_{ni} \text{ where } a_{ki} \in I_k, \text{ for } k = 1, 2, \ldots, n \right\}
\]

is an ideal of \( R \) contained in each \( I_j, j = 1, 2, \ldots, n \).

2. If \( P \) is a prime ideal of \( R \) with \( I_1I_2 \cdots I_n \subseteq P \), then \( I_j \subseteq P \) for some \( j \).

**Theorem 4.** Let \( I \) be a nonzero ideal of a Noetherian integral domain \( D \). Then \( I \) contains a product of nonzero prime ideals.

**Corollary 5.** Let \( I \) be an ideal of a Noetherian ring \( R \). Then \( I \) contains a product of prime ideals.

**Definition 6.** Let \( D \) be an integral domain with quotient field \( K \). A \( D \)-submodule \( I \) of \( K \) is a **fractional ideal** of \( D \) if \( dI \subseteq D \) for some nonzero \( d \in D \). \( d \) is called a **denominator** of \( I \).

**Example 7.**
1. Any ideal of \( D \) is a fractional ideal. Conversely, every fractional ideal of \( D \) that is contained in \( D \) is an ideal of \( D \).
2. The set \((1/2)\mathbb{Z}\) of rational numbers with denominators 1 or 2 is a fractional ideal of \( \mathbb{Z} \).
3. Let \( F \) be a field. The set \( F[x] \subset F(x) \) is a fractional ideal of \( F[x^2, x^3] \).
Lemma 8. Let $D$ be an integral domain with quotient field $K$.

1. If $I$ is a finitely generated $D$-submodule of $K$, then $I$ is a fractional ideal of $D$.

2. If $D$ is Noetherian then $I$ is a fractional ideal of $D$ if and only if $I$ is a finitely generated $D$-module of $K$.

3. If $I$ and $J$ are fractional ideals of $D$ with denominators $d$ and $r$ respectively, then $I \cap J$, $I + J$ and $IJ$ are fractional ideals with denominators $d$ (or $r$), $dr$ and $dr$, respectively.
Homework VII:

1. Prove that $D$ is an integral domain of Krull dimension 0 if and only if $D$ is a field.

2. A ring $R$ satisfies the **descending chain condition (DCC) on (prime) ideals** if whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a decreasing chain of (prime) ideals of $R$, then there is a positive integer $n$ such that $I_k = I_n$ for all $k \geq n$.

   Prove that an integral domain that satisfies DCC on all ideals has dimension 0.

   **Note:** A ring that satisfies DCC on all ideals is called Artinian.

3. Let $I_1, I_2, \ldots, I_n$ be ideals in a ring $R$. Prove that the product $I_1I_2\cdots I_n$ is an ideal of $R$ contained in each $I_j$, $j = 1, 2, \ldots, n$.

4. Let $R$ be a ring. If $I$ and $J$ are relatively prime (i.e. $I + J = R$), prove that $IJ = I \cap J$.

5. Let $D$ be an integral domain with quotient field $K$. If $K$ is a fractional ideal of $D$, prove that $D = K$. 