For notational purposes, let

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

represent the natural numbers,

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

represent the natural numbers adjoin 0 and

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

represent the integers.

Definition: If

$$n_1 < n_2 < \cdots < n_t$$

are natural numbers, then set

$$\langle n_1, n_2, \ldots, n_t \rangle = \{x_1 n_1 + x_2 n_2 + \cdots + x_t n_t \mid \text{ each } x_i \in \mathbb{N}_0\}.$$

$\langle n_1, n_2, \ldots, n_t \rangle$ is an additive submonoid of $\mathbb{N}_0$. Such a submonoid is known as a *numerical monoid.* The integers $n_1, \ldots, n_t$ are called the *generators* of $\langle n_1, n_2, \ldots, n_t \rangle$ and if they are relatively prime, then $\langle n_1, n_2, \ldots, n_t \rangle$ is called a *primitive numerical monoid.*

Examples:

$$\langle 1 \rangle = \{0, 1, 2, 3, 4, \ldots\} = \mathbb{N}$$

$$\langle 2, 3 \rangle = \{0, 2, 3, 4, 5, 6, \ldots\}$$

$$\langle 4, 9, 15 \rangle = \{0, 4, 8, 9, 12, 13, 15, 16, 17, 18, \ldots\}$$

$$\langle n, n+1, \ldots 2n-1 \rangle = \{0, n, n+1, n+2, \ldots\}$$

Facts:

- Every numerical monoid has a unique *minimal generating set*. This can be found by "throwing out" the unnecessary generators.

$$\langle 2, 3, 4\rangle = \langle 2, 3\rangle$$

- Every numerical monoid is "*isomorphic*" to a primitive numerical monoid (i.e., Given a numerical monoid S, there is a primitive numerical monoid S' and a bijection

$$f : S \to S'$$

such that for every x and y in S,

$$f(x + y) = f(x) + f(y).$$

Hence, to study numerical monoids, it suffices to consider only primitive ones (and we assume this through the rest of the talk).

An aside:

The Chicken McNugget Problem: If chicken McNuggets come in packages of 6, 9 and 20, is there any way to order 43 total mcnuggets without breaking up packages?

Translation of Chicken McNugget Problem: Is

$$43 \in \langle 6, 9, 20 \rangle?$$

How do we solve this? Consider

$$x_1 \cdot 6 + x_2 \cdot 9 + x_3 \cdot 20$$

for integers $0 \leq x_1 \leq 7$, $0 \leq x_2 \leq 4$ and $0 \leq x_3 \leq 2$. Thus I must check $8 \times 5 \times 3 = 120$ sums.

The Answer: NO!.

BUT, the McNugget problem does illustrate an important property of a primitive numerical monoid.

#	combination
44	$1 \cdot 6 + 2 \cdot 9 + 1 \cdot 20$
45	$5 \cdot 9$
46	$1 \cdot 6 + 2 \cdot 20$
47	$3 \cdot 9 + 1 \cdot 20$
48	$8 \cdot 6$
49	$1 \cdot 9 + 2 \cdot 20$

Observation: If $S = \langle n_1, \ldots, n_t \rangle$ and S contains the integers $m, m+1, m+2, \ldots, m+(n_1 - 1)$, then every integer $M > m$ is in S.

Theorem 1. *Let $S = \langle n_1, \ldots, n_t \rangle$ be a primitive numerical monoid. There exists a positive integer m such that if $M > m$, then $M \in S$.*

Sketch of Proof: From Number Theory we know there are *integers* $y_1, \ldots, y_t$ so that

$$1 = y_1 n_1 + \cdots + y_t n_t. \tag{\dagger}$$

Some of the coefficients y_i must be negative. Pick $b \in \mathbb{N}$ so that

$$b > \max\{|\ y_i\ |\ |\ 1 \leq i \leq t\}$$

and set $d = bn_1$. Let

$$m = dn_1 + dn_2 + \cdots + dn_t \in S.$$

I can now add 1 to m a total of $n_1 - 1$ times and obtain that

$$m, m+1, m+2, \ldots, m + (n_1 - 1)$$

are in S. The result now follows from the observation.

Definition. Let $S = \langle n_1, \ldots, n_t \rangle$ be a primitive numerical monoid. The largest $n \in \mathbb{N}$ such that $n \notin S$ is called the *Frobenius Number of* S and denoted $\mathcal{F}(S)$.

Example. $\mathcal{F}(\langle 6, 9, 20 \rangle) = 43$ (a.k.a. "the chicken McNugget number").

Theorem 2 (Sylvester 1884)**.** *Let* $S = \langle n_1, n_2 \rangle$ *be a primitive numerical monoid. Then*

$$\mathcal{F}(S) = (n_1 - 1)(n_2 - 1) - 1 = n_1 n_2 - n_1 - n_2.$$

AMAZING FACT: If $S = \langle n_1, \ldots, n_t \rangle$ and $t > 2$, then there is no known formula for the Frobenius number. For the case $t = 3$ there exists a "method" to compute $\mathcal{F}(S)$ due to Ralf Fröberg (1994).

Another aside:

MONOID MADNESS!

$$\begin{aligned}
\langle 6,9,20\rangle &= \text{chicken mcnugget monoid}\\
\langle 2,3,6,7,8\rangle &= \text{football monoid}\\
\langle 1,2,3,6,7,8\rangle &= \text{Canadian football monoid}\\
\langle 1,2,3\rangle &= \text{basketball monoid}\\
\langle 1,2\rangle &= \text{basketball monoid pre 3-point shot}\\
\langle 1\rangle &= \text{baseball monoid} = \text{hockey monoid}\\
\langle 1,6,12,18,24\rangle &= \text{beer monoid}\\
\langle 1/4,1/2,1\rangle &= \text{beer monoid by kegs}\\
\langle 0\rangle &= \text{soccer monoid}
\end{aligned}$$

Factorizations of Elements of a Numerical Monoid

Let's go back to the chicken McNugget Monoid.

$$44 = 4 \cdot 6 + 1 \cdot 20 = 1 \cdot 6 + 2 \cdot 9 + 1 \cdot 20.$$

We have the following contrast

$\mathbb{Z}$	$S = \langle n_1, \ldots, n_t \rangle$
prime numbers	minimal generators
factorizations are unique (up to order)	factorizations are not unique

This opens up a whole new area of investigation for numerical monoids.

If $S = \langle n_1, n_2, \ldots, n_t \rangle$ represents a minimal set of generators, then we will call the elements

$$n_1, n_2, \ldots, n_t$$

the *irreducible* elements of S.

Example: In $\langle 3, 4, 5\rangle$, 30 has many different factorizations. Setting

$$3x + 4y + 5z = 30$$

yields

x	y	z	length
0	0	6	6
1	3	3	7
2	1	4	7
0	5	2	7
2	6	0	8
3	4	1	8
4	2	2	8
5	0	3	8
6	3	0	9
7	1	1	9
10	0	0	10

Hence

$$\mathcal{L}(30) = \{6, 7, 8, 9, 10\}.$$

Example: In $\langle 3, 4, 5\rangle$, $\rho(30) = \frac{10}{6} = \frac{5}{3}$.

Question: What is $\rho(M) = \sup\{\rho(s) \mid s \in M\}$?

Theorem 3 (Matt Holden - Terri Moore, 2002 REU)**.** *Let* $S = \langle n_1, n_2, \ldots, n_t \rangle$ *be a primitive numerical monoid whose given generating set is minimal with* $t \geq 2$*.*

1. $\rho(S) = \frac{n_t}{n_1}$.

2. *There exists a rational number* q *with* $1 < q < \frac{n_t}{n_1}$ *such that* $\rho(s) \neq q$ *for all* $s \in S$*.*

Example: In $\langle 7, 10, 12\rangle$, 63 factors three different ways. Setting

$$7x + 10y + 12z = 63$$

they are

x	y	z	length
1	2	3	6
3	3	1	7
9	0	0	9

Hence

$$\mathcal{L}(63) = \{6, 7, 9\}.$$

Definition. If $\mathcal{L}(x) = \{m_1, \ldots, m_t\}$ with the m_i's listed in increasing order, then set

$$\Delta(x) = \{m_i - m_{i-1} \mid 2 \leq i \leq t\}$$

and

$$\Delta(S) = \bigcup_{0 \neq x \in S} \Delta(x).$$

If $\Delta(S) \neq \emptyset$, then $\min \Delta(S) = \gcd \Delta(S)$ and if $d = \gcd \Delta(S)$ and $\Delta(S)$ is finite we have that

$$\Delta(S) \subseteq \{d, 2d, \ldots, kd\}$$

for some positive integer k.

I asked the 2004 Trinity REU Algebra group (Kaplan, Craig Bowles - Cornell and Daniel Reiser - New Mexico State) to examine the structure of Delta sets of numerical monoids.

Theorem 4. *Let* $S = \langle n_1, n_2, \ldots, n_t \rangle$ *be a primitive numerical monoid. There exists a finite time algorithm to compute* $\Delta(S)$*.*

Is this Interesting? Here is an example that I believe clearly indicates the answer to this question is yes. Using the algorithm described above, one can easily show the following:

$$\Delta(\langle 7, 10, 12 \rangle) = \{1, 2\}$$

$$\Delta(\langle 7, 9, 12 \rangle) = \{1\}.$$

Theorem 5. **(1)** *If* $S = \langle n, n+k, n+2k, \ldots, n+tk\rangle$, *then*

$$\Delta(S) = \{k\}.$$

(2) *For any positive integers* k *and* t, *there exists a three generated numerical monoid* S *such that*

$$\Delta(S) = \{k, 2k, \ldots, tk\}.$$

(3) *For* $n \geq 3$,

$$\Delta(\langle n, n+1, n^2 - n - 1\rangle) = \{1, 2, .., n-2\} \cup \{2n-5\}.$$