Countable Sets

Definition 1. A set \( S \) is countable if \( |S| = |\mathbb{Z}^+| \). A set \( S \) is at most countable if \( |S| \leq |\mathbb{Z}^+| \).

Or equivalently,

- A set \( S \) is countable if there is a bijection \( f : S \to \mathbb{Z}^+ \).

- A set \( S \) is at most countable if there is an injective function \( f : S \to \mathbb{Z}^+ \).

Theorem 2. A subset of at most countable set is at most countable. In particular, an infinite subset of a countable set is countable.

Corollary 3. A set is at most countable if and only if it is either finite or countable.

Theorem 4. The image of a countable set under any mapping is at most countable.

Theorem 5. The union of countably many countable sets is countable.

Theorem 6. The Cartesian product of a finite number of countable sets is countable.

Theorem 7. The set of all finite subsets of a countable set is countable.

By Theorem 7 it follows that the set of all finite subsets of the set of natural numbers is countable.

Theorem 8. The set of all integers \( \mathbb{Z} \) and the set of all rational numbers \( \mathbb{Q} \) are countable.