# VANISHING THEOREMS FOR COMPLETE INTERSECTIONS

Craig Huneke, David A. Jorgensen and Roger Wiegand

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The starting point for this work is Auslander's seminal paper [A]. The main focus of his paper was to understand when the tensor product $M \otimes_R N$ of finitely generated modules $M$ and $N$ over a regular local ring $R$ is torsion-free. This condition forces the vanishing of a certain Tor module associated to $M$ and $N$, which in turn, by Auslander's famous rigidity theorem, implies that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. The vanishing of Tor carries a great deal of information; for example it implies the "depth formula" $\operatorname{depth}(M) + \operatorname{depth}(N) = \dim(R) + \operatorname{depth}(M \otimes_R N)$. From this formula one can deduce, for example, the highly non-trivial fact that if $M \otimes_R N$ is torsion-free and non-zero, then $M$ and $N$ must both be torsion-free.

In [HW] Huneke and Wiegand generalized some of Auslander's results to hypersurfaces, though with some unavoidable extra hypotheses. In particular, they proved a rigidity theorem [HW,(2.4)] and the following vanishing theorem:

**0.1. Theorem** [HW,(2.7)]. *Let $R$ be a hypersurface, and let $M$ and $N$ be non-zero finitely generated $R$-modules such that $M \otimes_R N$ is reflexive. Assume, in addition, that one of $M, N$ has constant rank. Then $\operatorname{Tor}_i^R(M,N) = 0$ for every $i \geq 1$.*

They also showed [HW, (2.5)] that vanishing of Tor implies the depth formula (for arbitrary complete intersections). A consequence is that both of the modules in the theorem above must be reflexive.

Given that regular rings and hypersurfaces are complete intersections of codimension 0 and 1, respectively, and that torsion-free and reflexive are equivalent to Serre's conditions $(S_1)$ and $(S_2)$, respectively, it seems reasonable to raise the question of whether over a complete intersection of codimension $c$, if $M \otimes_R N$ satisfies $(S_{c+1})$ then $\operatorname{Tor}_i^R(M,N) = 0$ for every $i \geq 1$. There are certain necessary conditions which must be satisfied. If $\operatorname{Tor}_i^R(M,N) = 0$ for every $i \geq 1$ then the depth formula holds for $M_p$ and $N_p$ for every prime $p$ in $R$,

$$\operatorname{depth}(M_p) + \operatorname{depth}(N_p) = \dim(R_p) + \operatorname{depth}((M \otimes_R N)_p),$$

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and the latter sum is at least

$$\min\{\dim(R_p) + c + 1,\ 2 \cdot \dim(R_p)\}$$

since $M \otimes_R N$ satisfies $(\mathrm{S}_{c+1})$. In particular, both $M$ and $N$ must satisfy $(S_{c+1})$. Two of our main results, Theorems 2.4 and 2.7, give an affirmative answer to this question for complete intersections of codimension 2 and 3, assuming certain local depth assumptions on the modules $M$ and $N$ (which are weaker than those forced by the conclusion).

Recent beautiful work of Avramov [Av] and Avramov and Buchweitz [AvB] assigns to each finitely generated module $M$ over a complete intersection $R$ of codimension $c$ an algebraic variety, $V(M)$, (reduced but not necessarily irreducible) in affine $c$-space. This variety is called the *support variety.* They prove that the vanishing of all high Tors of two finitely generated modules $M$ and

$N$ over a complete intersection of codimension $c$ is equivalent to the statement that $V(M) \cap V(N) = \{0\}$. This work yields a powerful new perspective on the vanishing of Tors, but we have been unable to apply the theory to

yield our main theorems.

## 1. Preparations

In this section we define terms and prove several preliminary results, some of which are of independent interest.

**Conventions.**

Throughout, all rings are commutative and Noetherian, and all modules are finitely generated. For a ring $R$ and a non-negative integer $j$, we set $\mathbf{X}^j(R) := \{p \in \mathrm{Spec}(R) \mid \mathrm{depth}(R_p) \le j\}$. Most of our applications are to Cohen-Macaulay rings, where $\mathrm{depth}(R_p) = \mathrm{height}(p)$ for all $p$. More generally, if $R$ satisfies Serre's condition $(\mathrm{S}_{j+1})$ then $\mathbf{X}^j(R)$ is exactly the set of primes $p$ such that $\mathrm{height}(p) \le j$. For a subset $X \subseteq \mathrm{Spec}(R)$, an $R$-module $M$ is said to be *free of constant rank on* $X$ (or *free of rank* $r$ *on* $X$ ) provided there is an integer $r$ such that $M_p \cong R_p^r$ for every $p \in X$. An $R$ module $M$ is torsion-free provided every non-zero-divisor of $R$ is a non-zero-divisor on $M$. The minimal number of generators for an $R$-module $M$ is denoted $\mu_R(M)$.

A *complete intersection* is local ring of the form $(R, \mathfrak{m}) = S/(\underline{f})$, where $(\underline{f}) = (f_1, \dots, f_c)$ is a regular sequence in the maximal ideal of the regular local ring $(S, \mathfrak{n})$. The *codimension,* $\mathrm{codim}(R)$, of the complete intersection $(R, \mathfrak{m})$ is $\mu_R(\mathfrak{m}) - \dim(R)$. If $(\underline{f}) \subseteq \mathfrak{n}^2$, then $\mathrm{codim}(R) = c$, the length of the regular sequence.

**The Depth Formula.**

One motivation for trying to prove that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \ge 1$ (over a complete intersection $R$) is that this condition gives a nice formula for the depth of $M \otimes_R N$. We quote the following result from [HW, Prop. 2.5]:

**1.1 Theorem.** *Let $M$ and $N$ be non-zero finitely generated modules over the complete intersection $R$. If $\operatorname{Tor}_i^R(M,N)=0$ for all $i\geq 1$, then $M$ and $N$ satisfy the "depth formula"* $\operatorname{depth}(M)+\operatorname{depth}(N)=\dim(R)+\operatorname{depth}(M\otimes_R N)$.

**Bourbaki Ideals.**

A *Bourbaki sequence* for an $R$-module $M$ is an exact sequence

$$0\to F\to M\to I\to 0,$$

in which $F$ is a free $R$-module, $I$ is an ideal, and $I_p$ is free of rank 1 on $\mathbf{X}^1(R)$. If such a sequence exists, we say that $I$ is a *Bourbaki ideal* for $M$. Note that every Bourbaki ideal contains a non-zero-divisor, and that the existence of a Bourbaki sequence for $M$ forces $M$ to be torsion-free (and free of constant rank on $\mathbf{X}^1(R)$). The following lemma will allow us to use Bourbaki ideals to test vanishing of Tor:

**1.2 Lemma.** *Let $M$ be an $R$ module with a Bourbaki ideal $I$, and let $N$ be a torsion-free $R$-module. Then $\operatorname{Tor}_i^R(M,N)\cong\operatorname{Tor}_i^R(I,N)$ for all $i\geq 1$.*

*Proof.* Choose a Bourbaki sequence $0\to F\to M\to I\to 0$, and apply $\_\otimes_R N$. The long exact sequence of Tor shows that $\operatorname{Tor}_i^R(M,N)\cong\operatorname{Tor}_i^R(I,N)$ for all $i\geq 2$. The end of the long exact sequence of Tor is

$$0\to\operatorname{Tor}_1^R(M,N)\to\operatorname{Tor}_1^R(I,N)\xrightarrow{\alpha}F\otimes_R N\to M\otimes_R N\to I\otimes_R N\to 0.$$

Since $I$ contains a non-zero-divisor, $I_p=R_p$ for every $p\in\operatorname{Ass}(R)$, and it follows that $\operatorname{Tor}_1^R(I,N)$ is torsion. Since $F\otimes_R N$ is torsion-free, $\alpha=0$, that is, $\operatorname{Tor}_1^R(M,N)\cong\operatorname{Tor}_1^R(I,N)$ as desired. □

The next lemma is well known, but for lack of a good reference we include the proof.

**1.3 Lemma.** *Let $I$ an ideal of $R$ containing a non-zero-divisor. Let $Q_1,\dots,Q_n$ be prime ideals, and assume that $I_{Q_i}$ is principal for each $i$. Then $I$ is isomorphic to an ideal $J\not\subseteq Q_1\cup\cdots\cup Q_n$.*

*Proof.* Let $S$ be the complement in $R$ of the union of the primes in $\{Q_1,\dots,Q_n\}\cup\operatorname{Ass}(R)$. Since $I_S$ is principal there is an element $x\in I$ such that $I_S=R_Sx$. For each $p\in\operatorname{Ass}(R)$ we have $R_p=I_p=R_px$, and it follows that $x$ is a non-zero-divisor of $R$. Choose $s\in S$ such that

$$sI\subseteq Rx. \tag{1.3.1}$$

Put $J=\frac{s}{x}I$. Then $J$ is an ideal of $R$ by (1.3.1), and it is isomorphic to $I$. Suppose $J\subseteq Q_i$ for some $i$. Put $Q=Q_i$. We have $\frac{s}{x}I\subseteq Q$, so $sI\subseteq xQ$. Localizing at $Q$, we have $I_Q\subseteq QR_Qx\subseteq QI_Q$. By Nakayama's lemma $I_Q=0$, contradiction. □

**1.4 Theorem.** *Let $R$ be a ring satisfying Serre's condition ($S_2$), and let $M$ be a non-zero torsion-free $R$-module. Let $S$ be a finite set of prime ideals of $R$, and assume that $M$ is free of constant rank on $\mathbf{X}^1(R) \cup S$. Then $M$ has a Bourbaki ideal $I \nsubseteq \bigcup S$.*

*Proof.* We begin by observing that $M$ can be embedded in $R^r$, where $r$ is the common rank of the free $R_p$-modules $M_p$, $p \in \mathbf{X}^1(R) \cup S$. It suffices to show that $M \hookrightarrow K^r$, where $K$ is the total quotient ring of $R$. But $M \hookrightarrow M \otimes_R K$ as $M$ is torsion-free, and $M \otimes_R K \cong K^r$ because $M$ is free of rank $r$ on $\operatorname{Ass}(R)$.

We now proceed by induction on $r$. If $r = 1$, then $M$ is isomorphic to an ideal $I$ of $R$. Since $I_p \cong R_p$ for every $p \in S \cup \operatorname{Ass}(R)$ we can apply Lemma 1.3 to replace $I$ by an ideal outside the union of the primes in $S \cup \operatorname{Ass}(R)$.

Assume $r \geq 2$. Recall that a *patch* is a subset $X$ of $\operatorname{Spec}(R)$ that is closed in the patch (constructible) topology. This means that $X = \bigcap_\alpha W_\alpha$, where each $W_\alpha$ is either open or closed in the Zariski topology. Let $Y = \mathbf{X}^1(R) \cup S$. Since $R$ satisfies ($\mathrm{S}_2$), $\mathbf{X}^1(R)$ is exactly the set of primes of height at most 1, and it is easy to see that this set is a patch (see, for example [W, (2.1)]). Since $\mu_{R_p}(M_p) \geq 2$ for each prime $p \in Y$ and since $Y$ is a finite union of patches of dimension at most one, (2.3) of [W] implies that there exists a $Y$-basic element $m \in M$. Put $N = M/Rm$. Then $N_p$ is free for each $p \in Y$. In order to use induction, we need to show that $N$ is torsion-free, that is, $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(R)$.

First note that $(0 : m) = 0$. For, if $0 \neq t \in (0 : m)$, there is a prime $Q \in \operatorname{Ass}(R)$ such that $t$ is non-zero in $R_Q$, contradicting the fact that $R_Q m \cong R_Q$. Suppose, now, that $p \in \operatorname{Ass}(N) - \operatorname{Ass}(R)$. Since $M_p \hookrightarrow R_p^r$, $M_p$ has depth at least 1. Applying the depth lemma to the exact sequence

$$0 \to R_p m \to M_p \to N_p \to 0,$$

we see that $\operatorname{depth}(R_p m) = 1$. Since $R_p m \cong R_p$, $p \in \mathbf{X}^1(R)$. But then $N_p$ is free, and $\operatorname{depth}(N_p) = \operatorname{depth}(R_p) = 1$, contradicting the assumption that $p \in \operatorname{Ass}(N)$.

By induction, $N$ has a free submodule $G$ such that $N/G \cong I$, where $I$ is an ideal not contained in the union of the primes in $S \cup \operatorname{Ass}(R)$ and $I_p$ is principal for each $p \in \mathbf{X}^1(R)$. Writing $G = F/Rm$, we see that $F \cong G \oplus Rm$, which is free. Since $M/F \cong I$ we're done. □

**Orientable Modules.**

Recall that an $R$-module $M$ is called *orientable* (see [B]) provided

(1) $M$ is free of constant rank, say $r$, on $\mathbf{X}^1(R)$, and
(2) $(\bigwedge^r M)^{**} \cong R$.

The following proposition summarizes the properties of orientable modules we will need in the sequel.

**1.5 Proposition.** *Let $R$ be a ring.*

(1) *Free modules are orientable.*
(2) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of $R$-modules, and assume each $M_i$ is free of constant rank on $\mathbf{X}^1(R)$. If two of the $M_i$ are orientable, so is the third.*
(3) *Let $I$ be an orientable ideal of $R$. Either $I$ is principal or $I$ is isomorphic to an ideal of grade at least 2.*

*Proof.* (1) is clear, and (2) is part (c) of [B, (2.8)]. To prove (3), let $I$ be a non-principal orientable ideal of $R$. Since $R$ is Noetherian we may assume that $I$ is maximal in its isomorphism class. Since $I \neq 0$, there exists $p \in \operatorname{Ass}(R)$ such that $I_p \neq 0$. Therefore the rank of $I$ (in the definition of orientability) is 1. It follows that $I_p = R_p$ for every $p \in \operatorname{Ass}(R)$, that is, $I$ contains a non-zero-divisor. Therefore we can identify $I^*$ with its ideal-theoretic inverse (in the total quotient ring of $R$).

If $I^{**} = R$, then $I^* = R^*$, which implies that $\operatorname{Ext}^1_R(R/I, R) = 0$. Since also $\operatorname{Hom}_R(R/I, R) = 0$ (as $I$ contains a non-zero-divisor), $\operatorname{grade}(I) \geq 2$, as desired. If $I^{**} \neq R$, write $I^{**} = Rx$, where $x$ is a non-unit and non-zero-divisor of $R$. Put $J = (I : x)$. Then $xJ = I$, so $I \cong J$, which contradicts maximality of $I$. □

**The Pushforward.**

Let $M$ be a torsion-free module over a Gorenstein ring $R$. We start with a short exact sequence $0 \to K \to (R^\mu)^* \to M^* \to 0$, where $\mu := \mu_R(M^*)$. Now dualize, getting $0 \to M^{**} \to R^\mu \to K^* \to \operatorname{Ext}^1_R(M^*, R) \to 0$. If we compose the natural map $0 \to M \to M^{**}$ with $0 \to M^{**} \to R^\mu$, we get a short exact sequence

$$0 \to M \to R^\mu \to M_1 \to 0. \tag{PF}$$

We will refer to any short exact sequence obtained in this way as a "pushforward" of $M$. The properties we will need are summarized in the following:

**1.6 Proposition.** *Given a torsion-free module $M$ over the Gorenstein ring $R$, consider the pushforward (PF). Let $p \in \operatorname{Spec}(R)$.*

(1) *$M_p$ is free if and only if $(M_1)_p$ is free.*
(2) *If $M_p$ is a maximal Cohen-Macaulay $R_p$-module so is $(M_1)_p$.*
(3) $\operatorname{depth}_{R_p} (M_1)_p \geq \operatorname{depth}_{R_p} M_p - 1$.
(4) *If $M$ satisfies $(S_k)$ then $M_1$ satisfies $(S_{k-1})$.*

*Proof.* Property (1) is clear from the construction, and (3) follows from the depth lemma applied to (PF). For (2), suppose $M_p$ is a maximal Cohen-Macaulay $R_p$-module. Then $M_p^*$ is a maximal Cohen-Macaulay $R_p$-module. Hence $K_p$ is a maximal Cohen-Macaulay $R_p$-module, and as $M_p$ is reflexive, we conclude that $(M_1)_p \cong K_p^*$ is a maximal Cohen-Macaulay $R_p$-module. Property (4) now follows from (2) and (3). □

**Quasi-Liftings.**

Suppose that $R := S/(f)$ where $S$ is Gorenstein and $f$ is a non-zero-divisor of $S$. Let $M$ be a torsion-free $R$-module with pushforward $0 \to M \to R^\mu \to M_1 \to 0$ ($\mu := \mu_R(M^*)$). We call the $S$-module $E$ in the short exact sequence $0 \to E \to S^\mu \to M_1 \to 0$ a *quasi-lifting* of $M$ (relative to the presentation $R = S/(f)$). In some sense (see Proposition 1.7 (2) below), a quasi-lifting of $M$ is actually a type of lifting of both $M$ and its pushforward $M_1$.

Our next proposition summarizes some of the elementary properties of a quasi-lifting.

**Proposition 1.7.** *Suppose that $R := S/(f)$ where $S$ is Gorenstein and $f$ is a non-zero-divisor of $S$. Let $M$ be a torsion-free $R$-module with pushforward $0 \to M \to R^\mu \to M_1 \to 0$ ($\mu := \mu_R(M^*)$), and let $E$ be the corresponding quasi-lifting of $M$.*

(1) *$E$ is a finitely generated torsion-free $S$-module.*
(2) *There exists a short exact sequence,*
$$0 \to M_1 \to E/fE \to M \to 0.$$
(3) *Let $p \in Spec(S)$. Then $E_p$ is free if $f \notin p$. If $f \in p$ and $M_p$ is $R_p$-free, then $E_p$ is $S_p$-free.*
(4) *For all primes $p \in V(f) \subseteq Spec(S)$, $depth(E_p) = depth((M_1)_p) + 1$.*
(5) *Assume that $M$ is torsion-free. Then $E/fE$ is torsion-free over $R$ iff $M$ is reflexive.*
(6) *For all $R$-modules $N$, $\operatorname{Tor}_i^S(E, N) \cong \operatorname{Tor}_i^R(E/fE, N)$.*

*Proof.* Recall the exact sequences,

$$0 \to M \to R^m \to M_1 \to 0, \tag{1.7.1}$$

and

$$0 \to E \to S^m \to M_1 \to 0. \tag{1.7.2}$$

Statement (1) follows at once from (1.7.2) as $E$ is a submodule of $S^m$.

Applying $\_ \otimes_S R$ to (1.7.2), we get the exact sequence $0 \to M_1 \to E/fE \to R^m \to M_1 \to 0$. Here we use that $\operatorname{Tor}_1^S(M_1, R) \cong M_1$. This exact sequence combined with (1.7.1) yields (2).

If $f \notin p$, then localizing (1.7.2) at $p$ immediately gives that $E_p \cong S_p^m$. Assume that $f \in p$ and $M_p$ is free. By Proposition 1.6 (1), $(M_1)_p$ is also free. Part (2) of this proposition then implies that $(E/fE)_p$ is free over $R_p$ and hence $E_p$ is free over $S_p$. This proves (3).

Item (4) follows at once from (1.7.2) and the fact that $S$ is Cohen-Macaulay of dimension $1+\dim(R)$.

For (5), suppose $M$ is reflexive as an $R$-module. Then $M_1$ is torsion-free by Proposition 1.6 (4). The exact sequence of (2) then proves that $E/fE$ is torsion-free. Conversely, suppose that $E/fE$ is torsion-free. The sequence of (2) shows that $M_1$ is torsion-free and then (1.7.1) proves that $M$ must be reflexive.

Finally, (6) is a standard result on change of rings. See, for example, [Ma, Lemma 2, p. 140]. □

In the proof of our main results on vanishing of Tor, we first do the pivotal case where the dimension and codimension are equal. We will use the standard *change-of-rings* long exact sequence expressing the relationship between the Tors over $S$ and the Tors over $R$ (see Murthy [Mu] or Lichtenbaum [L]). For the convenience of the reader we recall this sequence. Let $(S,\mathfrak{n})$ be a local ring and put $R=S/(f)$, where $f$ is a non-zero-divisor in $\mathfrak{n}$. Fix $R$-modules $M$ and $N$. For compactness of notation we let $\mathrm{T}_i^R=\mathrm{Tor}_i^R(M,N)$ and $\mathrm{T}_i^S=\mathrm{Tor}_i^S(M,N)$.

With the notation established above, we have the following exact sequence:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
T_j^R & \longrightarrow & T_{j+1}^S & \longrightarrow & T_{j+1}^R & \longrightarrow & \\
T_{j-1}^R & \longrightarrow & T_j^S & \longrightarrow & T_j^R & \longrightarrow & \\
\vdots & & \vdots & & \vdots & & \\
T_1^R & \longrightarrow & T_2^S & \longrightarrow & T_2^R & \longrightarrow & \\
T_0^R & \longrightarrow & T_1^S & \longrightarrow & T_1^R & \longrightarrow & 0.
\end{array}$$

Our extension to higher dimensions depends on the following technical result:

**1.8 Theorem.** *Let $R=S/(f)$, where $f$ is a non-zero-divisor of the complete intersection $S$. Let $M$ and $N$ be torsion-free $R$-modules with quasi-liftings $E$ and $F$, respectively. Assume that the following hold for some $c\geq 1$:*

(1) *$M\otimes_R N$ satisfies Serre's condition $(S_{c+1})$,*
(2) *$M$ and $N$ both satisfy $(S_c)$ and*
(3) *$\mathrm{Tor}_i^R(M,N)_p=0$ for each $i\geq 1$ and each $p\in\mathbf{X}^c(R)$.*

*Then $E\otimes_S F$ satisfies $(S_c)$.*

*Proof.* We have pushforward short exact sequences

$$0\to M\to R^m\to M_1\to 0, \tag{1.8.1}$$
$$0\to N\to R^n\to N_1\to 0, \tag{1.8.2}$$

and the short exact sequences

$$0 \to E \to S^m \to M_1 \to 0, \tag{1.8.3}$$

$$0 \to F \to S^n \to N_1 \to 0. \tag{1.8.4}$$

Suppose $p$ is a prime of $R$ with $\dim(R_p) \le c$. Then (3) and the long exact sequence of Tor (applied to (1.8.2)) imply that $\mathrm{Tor}_i^R(M, N_1)_p = 0$ for all $i \ge 2$. Since $M_p$ and $(N_1)_p$ are maximal Cohen-Macaulay $R_p$-modules, it follows from the equivalence of (1) and (2) of [J1, (2.1)] that

$$\mathrm{Tor}_i^R(M, N_1)_p = 0 \text{ for every } i \ge 1. \tag{1.8.5}$$

Similarly

$$\mathrm{Tor}_i^R(M_1, N)_p = \mathrm{Tor}_i^R(M_1, N_1)_p = 0 \text{ for all } i \ge 1. \tag{1.8.6}$$

Suppose now that $p$ is a prime of $S$ with $\dim(S_p) \le c + 1$. We claim that

$$\mathrm{Tor}_i^S(E, F)_p = \mathrm{Tor}_i^S(E, N_1)_p = \mathrm{Tor}_i^S(M_1, F)_p = 0 \text{ for every } i \ge 1. \tag{1.8.7}$$

If $f \notin p$, then $E_p$ and $F_p$ are free $S$-modules (Proposition 1.7 (3)). Thus, in proving (1.8.7) we may

assume that $f \in p$. Then $\dim(R_p) \le c$. The various Tors over $R_p$ and $S_p$ are nicely untangled by the change-of-rings long exact sequence given above. Applying this sequence to (1.8.6), we see that $\mathrm{Tor}_i^S(M_1, N_1) = 0$ for all $i \ge 2$. Now (1.8.7) follows by the usual shifting of Tor along the exact sequences (1.8.3) and (1.8.4).

Apply $\_ \otimes_R N$ to (1.8.1). By (1.8.6), $\mathrm{Tor}_1^R(M_1, N)$ is torsion. Since $M \otimes_R N$ is torsion-free, we see that $\mathrm{Tor}_1^R(M_1, N) = 0$ and

$$0 \to M \otimes_R N \to N^m \to M_1 \otimes_R N \to 0 \tag{1.8.8}$$

is exact.

Now we tensor the short exact sequence of Proposition 1.7 (2) with $N$, getting the exact sequence

(1.8.9)

$$\mathrm{Tor}_1^R(E/fE, N) \to \mathrm{Tor}_1^R(M, N) \xrightarrow{\beta} M_1 \otimes_R N \to E/fE \otimes_R N \to M \otimes_R N \to 0.$$

We claim that $M_1 \otimes_R N$ is torsion-free, that is, it satisfies ($S_1$). To see this let $p$ be a prime ideal of $R$. If $\mathrm{height}(p) \ge 2$ we have $\mathrm{depth}((M \otimes_R N)_p) \ge 2$ and $\mathrm{depth}(N_p) \ge 1$, and (1.8.8) implies that $\mathrm{depth}((M_1 \otimes_R N)_p) \ge 1$. If $\mathrm{height}(p) = 1$, then $M_p$ and $N_p$ are maximal Cohen-Macaulay, and so is $(M_1)_p$ by (2) of (1.6).

By (3) and the depth formula (1.1), $(M_1 \otimes_R N)_p$ is maximal Cohen-Macaulay, as desired.

Since $\mathrm{Tor}_1^R(M,N)$ is torsion (by hypothesis (3)) we see that $\beta = 0$ in (1.8.9). Thus we have an exact sequence

$$0 \to M_1 \otimes_R N \to E/fE \otimes_R N \to M \otimes_R N \to 0, \tag{1.8.10}$$

and it follows that $E/fE \otimes_R N$ is torsion-free.

If we now apply $E/fE \otimes_R \_$ to (1.8.2), we get

(1.8.11)

$$0 \to \mathrm{Tor}_1^R(E/fE, N_1) \to E/fE \otimes_R N \to E/fE \otimes_R R^n \to E/fE \otimes_R N_1 \to 0.$$

We claim that $\mathrm{Tor}_1^S(E, N_1) = 0$. We know at least that $\mathrm{Tor}_1^S(E, N_1)$ is torsion, by (1.8.7). But $\mathrm{Tor}_1^S(E, N_1) \cong \mathrm{Tor}_1^R(E/fE, N_1)$ by Proposition 1.7 (6). Since $E/fE \otimes_R N$ is torsion-free, our claim follows from the exact sequence (1.8.11), which we may now rewrite as

$$0 \to E/fE \otimes_R N \to E/fE \otimes_R R^n \to E/fE \otimes_R N_1 \to 0. \tag{1.8.12}$$

If we now tensor (1.8.4) with $E$, we get the short exact sequence

$$0 \to E \otimes_S F \to E^n \to E \otimes_S N_1 \to 0. \tag{1.8.13}$$

To see that $E \otimes_S F$ satisfies $(\mathrm{S}_c)$, let $p \in \mathrm{Spec}(S)$. If $f \notin p$, then (1.8.13) yields an isomorphism $(E \otimes_S F)_p \cong E_p^n$. But $E_p \cong S_p^m$ (by Proposition 1.7), and so $\mathrm{depth}(E \otimes_S F)_p = \dim S_p$. Hence we may assume that $f \in p$.

Suppose first that $\dim S_p \le c+1$. Since $\dim R_p \le c$, $M_p$ and $N_p$ are maximal Cohen-Macaulay, whence so are $(M_1)_p$ and $(N_1)_p$ by statement (2) of (1.7). It follows that $E_p$ and $F_p$ are maximal Cohen-Macaulay $S_p$-modules. Now (1.8.7) and the depth formula (1.1) imply that $(E \otimes_S F)_p$ is maximal Cohen-Macaulay.

Now suppose that $\dim S_p > c+1$. Then $\dim(R_p) \ge c+1$. By hypothesis (1) $\mathrm{depth}((M \otimes_R N)_p) \ge c+1$ and by (1.8.8) $\mathrm{depth}(N_p) \ge c$, so that $\mathrm{depth}(M_1 \otimes_R N)_p \ge c$. By (1.8.10), $\mathrm{depth}((E/fE \otimes_R N)_p) \ge c$. Now $(E/fE)_p$ has depth at least $c-1$ by part (3) of 1.6 and part (2) of (1.7). Therefore $\mathrm{depth}((E/fE \otimes_R N_1)_p) \ge c-1$ by (1.8.12). Note that $E/fE \otimes_R N_1 \cong E \otimes_S N_1$, and consider (1.8.13) localized at $p$. We have $\mathrm{depth}(E_p) \ge 1 + \mathrm{depth}((M_1)_p) \ge \mathrm{depth}(M_p) \ge c$. It follows that $\mathrm{depth}((E \otimes_S F)_p) \ge c$, as desired. □

**Rigidity.**

The famous rigidity theorem of Auslander, saying that if a single Tor involving a pair of finitely generated modules over an unramified regular local ring vanishes then all higher Tors vanish, can be extended to complete intersections by modifying the number of consecutive Tors which are assumed to be zero. This was done by Murthy [Mu, Theorem 1.6], whose result states that if $R = S/(x_1, ..., x_c)$ is a complete intersection (with $S$ a regular local ring and $x_1, ..., x_c$ a regular sequence), $M$ and $N$ are two finitely generated $R$-modules,

and either $\mathrm{Tor}^R_{i+j}(M, N) = 0$ for $0 \le j \le c + 1$ or $\mathrm{Tor}^R_{i+j}(M, N) = 0$ for $0 \le j \le c$ and $S$ is unramified, then $\mathrm{Tor}^R_k(M, N) = 0$ for all $k \ge i$. The extension of Auslander's rigidity result to ramified regular local rings by Lichtenbaum [L] allows one to remove the extra assumption of Murthy in case $S$ is ramified: if $\mathrm{Tor}^R_{i+j}(M, N) = 0$ for $0 \le j \le c$ then $\mathrm{Tor}^R_k(M, N) = 0$ for all $k \ge i$ in both the ramified and unramified cases. For our work we need a similar rigidity theorem — one where only $c$ consecutive Tors are assumed to be zero even when $S$ is ramified.

We will say a complete intersection $R$ is *unramified* provided $R = S/(\underline{f})$ where $S$ is an unramified regular local ring and $\underline{f}$ is a regular sequence in the square of the maximal ideal of $S$. The following is an extension of [HW, (2.4)]. A sharper form of the case $i > d$ can be found in [J1, (2.6)].

**1.9 Theorem.** *Let $R$ be a complete intersection of dimension $d$ and codimension $c \ge 1$, and let $M$ and $N$ be $R$-modules. Assume*

(1) *$M \otimes_R N$ has finite length,*
(2) $\dim(M) + \dim(N) < d + c$,
(3) $\mathrm{Tor}^R_i(M, N) = \cdots = \mathrm{Tor}^R_{i+c-1}(M, N) = 0$ *for some $i > 0$, and*
(4) *either $R$ is unramified or $i > d$.*

*Then $\mathrm{Tor}^R_j(M, N) = 0$ for all $j \ge i$.*

*Proof.* We proceed by induction on $c$, the case $c = 1$ being the First Rigidity Theorem (2.4) of [HW]. Assume $c > 1$, and write

$R = S/(f)$, where $S$ is a complete intersection of codimension $c - 1$ and is unramified if $i \le d$. Applying (3) to the change-of-rings long exact sequence of Tor, one sees immediately that $\mathrm{Tor}^S_{i+1}(M, N) = \cdots = \mathrm{Tor}^S_{i+c-1}(M, N) = 0$. By induction we have $\mathrm{Tor}^S_j(M, N) = 0$ for all $j \ge i + 1$, and another glance at the long exact sequence shows that now $\mathrm{Tor}^R_j(M, N) = \mathrm{Tor}^R_{j+2}(M, N) = 0$ for all $j \ge i$. Since $c \ge 2$ it follows from (3) that $\mathrm{Tor}^R_j(M, N) = 0$ for all $j \ge i$. □

## 2. Vanishing Theorems

In this section we state and prove our vanishing theorems, for complete intersections of codimensions 2 and 3. One should note that the hypotheses that $M$ and $N$ satisfy $(\mathrm{S}_c)$ are not so strong as they might seem, as the conclusion of the theorems imply the depth formula (1.1), which in turn implies $M$ and $N$ satisfy $(\mathrm{S}_{c+1})$. On the other hand, it is not clear to us that one really needs to assume $M$ and $N$ satisfy

$(S_c)$: we have no example to the contrary. The last section of this paper deals with trying to prove the modules must be reflexive in case the tensor product is reflexive.

The following easy inductive argument will be used in the proofs of both of the main theorems (2.4) and (2.7).

**2.1 Lemma.** *Let $R$ be a Gorenstein ring, let $c \geq 1$, and let $M$ and $N$ be $R$-modules such that*

(1) *$M$ satisfies Serre's condition $(S_c)$,*
(2) *$N$ satisfies $(S_{c-1})$,*
(3) *$M \otimes_R N$ satisfies $(S_c)$, and*
(4) *$M_p$ is free for each $p \in \mathbf{X}^{c-1}(R)$.*

*Put $M_0 := M$, and for $i = 1, \ldots, c$ let*

$$0 \to M_{i-1} \to F_i \to M_i \to 0 \tag{2.1.1}$$

*be the pushforward. Then $\operatorname{Tor}_i^R(M_c, N) = 0$ for $i = 1, \ldots, c$.*

*Proof.* For $i = 1, \ldots, c$, $(M_i)_p$ is free for each $p \in \mathbf{X}^{c-1}(R)$ by (1.7). Since $c \geq 1$, (4) implies that $\operatorname{Tor}_1^R(M_i, N)$ is torsion. Tensoring (2.1.1) with $N$ yields the exact sequence

$$0 \to \operatorname{Tor}_1^R(M_i, N) \to M_{i-1} \otimes_R N \to F_i \otimes_R N \to M_i \otimes_R N \to 0. \tag{2.1.2}$$

Set $i = 1$, and note that $M_0 \otimes_R N$ is torsion-free since it satisfies $(S_c)$ and $c \geq 1$. It follows that $\operatorname{Tor}_1^R(M_1, N) = 0$.

Assume inductively that for some $j$ with $1 \leq j < c$, $M_{j-1} \otimes_R N$ satisfies $(S_{c-j+1})$ and $\operatorname{Tor}_1^R(M_j, N) = 0$. We claim that $M_j \otimes_R N$ satisfies $(S_{c-j})$. To see this, let $p \in \operatorname{Spec}(R)$, set $i = j$, and localize the short exact sequence (2.1.2) at $p$. If $\dim(R_p) \leq c - j$ then $(M_j)_p$ is free and $N_p$ is maximal Cohen-Macaulay, so $(M_j \otimes_R N)_p$ is maximal Cohen-Macaulay. If, on the other hand, $\dim(R_p) \geq c-j+1$, then $\operatorname{depth}((M_{j-1} \otimes_R N)_p) \geq c - j + 1$ and $\operatorname{depth}(N_p) \geq c - j$. It follows from (2.1.2) that $\operatorname{depth}((M_j \otimes_R N)_p) \geq c - j$, proving the claim.

In particular, $M_j \otimes_R N$ is torsion-free since $j < c$. Letting $i = j + 1$ in (2.1.2), we see that $\operatorname{Tor}_1^R(M_{j+1}, N) = 0$. By induction we have $\operatorname{Tor}_1^R(M_i, N) = 0$ for $i = 1, \ldots, c$. The conclusion follows by shifting along the sequences (2.1.1). $\square$

In the proofs of our main results, $c$ will be the codimension, and we will use (1.9) in conjunction with (1.2) to force the vanishing of *all* $\operatorname{Tor}_i^R(M_r, N)$. Then, upon shifting back, we'll have $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. As we will see, there is a serious obstruction to pushing our results beyond codimension 3.

**Codimension 2.**

For complete intersections of codimension 2, the pivotal case of our problem occurs in dimension 2. Our theorem in that case (2.3) is a little stronger than the general case, so we will state it separately. It is convenient to isolate the following step of the proof:

**2.2 Theorem.** *Let $R$ be a complete intersection with* $\dim(R) = \operatorname{codim}(R) = 2$*, and let $M$ and $N$ be torsion-free $R$-modules. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^1(R)$, and*
(2) *$N$ is free of constant rank on* $\operatorname{Ass}(R)$.

*If* $\operatorname{Tor}_i^R(M,N) = \operatorname{Tor}_{i+1}^R(M,N) = 0$ *for some $i \geq 1$, then* $\operatorname{Tor}_j^R(M,N) = 0$ *for all $j \geq i$.*

*Proof.* By (2) we can embed $N$ in a free module $F$ in such a way that $F/N$ is torsion. (See the argument at the beginning of the proof of (1.4).) If $F/N$ has finite length, put $S = \emptyset$; otherwise, let $S$ be the set of minimal primes of $(N : F)$. Since $N_p = F_p$ for every $p \in \operatorname{Ass}(R)$, the primes in $S$ all have height 1. By Theorem 1.4, $M$ has a Bourbaki ideal $I \not\subseteq \bigcup S$. Then $(R/I)\otimes_R(F/N)$ has finite length, and both $R/I$ and $F/N$ have dimension at most 1. By (1.2) we have $\operatorname{Tor}_i^R(I,N) = \operatorname{Tor}_{i+1}^R(I,N) = 0$, and it will suffice to show that $\operatorname{Tor}_j^R(I,N) = 0$ for all $j \geq i$.

We have $\operatorname{Tor}_{i+2}^R(R/I,F/N) = \operatorname{Tor}_{i+3}^R(R/I,F/N) = 0$ by the long exact sequence of Tor, and now (1.9) implies that $\operatorname{Tor}_j^R(R/I,F/N) = 0$ for every $j \geq i+2$. Shifting back down, we get $\operatorname{Tor}_j^R(I,N) = 0$ for all $j \geq i$ as desired. □

**2.3 Theorem (dimension 2).** *Let $R$ be a complete intersection of codimension* 2 *and dimension $d \leq 2$, and let $M$ and $N$ be $R$-modules. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^1(R)$,*
(2) *$N$ is free of constant rank on* $\operatorname{Ass}(R)$,
(3) *$M$ is maximal Cohen-Macaulay.*

*If $M \otimes_R N$ is maximal Cohen-Macaulay, then* $\operatorname{Tor}_i^R(M,N) = 0$ *for all $i \geq 1$.*

*Proof.* We assume both $M$ and $N$ are non-zero. Note that as $R$ has dimension two, maximal Cohen-Macaulay modules are exactly the reflexive modules. If $d \leq 1$ there is nothing to prove since $M$ is free by assumption. Therefore we assume that $d = 2$.

We begin with a standard argument (see [A]) that reduces to the case where $N$ is torsion-free. Let $T$ be the torsion submodule of $N$, and let $\bar{N} = N/T$. Applying $M \otimes_R \_$ to the exact sequence $0 \to T \to N \to \bar{N} \to 0$ gives an exact sequence

$$\to \operatorname{Tor}_1^R(M,\bar{N}) \to M \otimes_R T \xrightarrow{\alpha} M \otimes_R N \to M \otimes_R \bar{N} \to 0.$$

Since $M \otimes_R T$ is torsion and $M \otimes_R N$ is torsion-free, $\alpha = 0$. Thus $M \otimes_R \bar{N}$ is maximal Cohen-Macaulay. Since $N_p = \bar{N}_p$ for each $p \in \operatorname{Ass}(R)$ we have, assuming the torsion-free case of the theorem, that $\mathrm{Tor}_1^R(M,\bar{N})=0$. This forces $M\otimes_R T=0$, so $T=0$, that is, $N=\bar{N}$. Therefore we assume from now on that $N$ is torsion-free.

Now $N$ satisfies ($S_1$), and we can apply Lemma 2.1 with $c=2$. We have $\mathrm{Tor}_1^R(M_2,N)=\mathrm{Tor}_2^R(M_2,N)=0$. By (2.2) $\mathrm{Tor}_j(M_2,N)=0$ for all $j\geq 1$. By shifting along the sequences (2.1.1), we get $\mathrm{Tor}_i^R(M,N)=0$ for all $i\geq 1$. □

**2.4 Theorem (arbitrary dimension).** *Let $R$ be a complete intersection of codimension 2 and dimension $d$, and let $M$ and $N$ be $R$-modules. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^1(R)$,*
(2) *$N$ is free of constant rank on $\mathrm{Ass}(R)$,*
(3) *$M$ and $N$ satisfy ($S_2$).*

*If $M\otimes_R N$ satisfies ($S_3$), then $\mathrm{Tor}_i^R(M,N)=0$ for all $i\geq 1$.*

*Proof.* We may assume $d\geq 3$ by (2.3). Write $R=S/(f)$, where

$S$ is a hypersurface of dimension $d+1$. Form the quasi-liftings $E$ and $F$ of $M$ and $N$, respectively, relative to this presentation of $R$. We will apply Theorem 1.8 in the case $c=2$. By Theorem 2.3, hypothesis (3) of Theorem 1.8 is satisfied. Therefore $E\otimes_S F$ satisfies ($S_2$) over the hypersurface $S$. Moreover, both $E$ and $F$ are free of constant rank on $\mathbf{X}^0(S)$ since $M_1$ and $N_1$ are torsion $S$-modules. By Theorem 0.1, $\mathrm{Tor}_i^S(E,F)=0$ for all $i\geq 1$. Looking at (1.8.4), we see that $\mathrm{Tor}_i^S(E,N_1)=0$ for all $i\geq 2$. Since $\mathrm{Tor}_i^S(E,N_1)\cong\mathrm{Tor}_i^R(E/fE,N_1)$ for all $i\geq 1$, we have $\mathrm{Tor}_i^R(E/fE,N_1)=0$ for all $i\geq 2$. Hence, by (1.8.2), we have

$$\mathrm{Tor}_i^R(E/fE,N)=0 \text{ for all } i\geq 1; \tag{2.4.1}$$

and now Proposition 1.7 (2) shows that

$$\mathrm{Tor}_{i+1}^R(M,N)\cong\mathrm{Tor}_i^R(M_1,N)\quad\text{for all } i\geq 1. \tag{2.4.2}$$

The short exact sequence (1.8.1) yields

$$\mathrm{Tor}_{i+1}^R(M_1,N)\cong\mathrm{Tor}_i^R(M,N)\quad\text{for all } i\geq 1. \tag{2.4.3}$$

Putting (2.4.2) and (2.4.3) together, we have

$$\mathrm{Tor}_{i+2}^R(M,N)\cong\mathrm{Tor}_i^R(M,N)\quad\text{for all } i\geq 1. \tag{2.4.4}$$

In light of (2.4.4), it suffices to show that $\mathrm{Tor}_1^R(M,N)=\mathrm{Tor}_2^R(M,N)=0$.

Since $\beta=0$ in (1.8.9), (2.4.1) shows that $\mathrm{Tor}_1^R(M,N)=0$. Finally, from (1.8.8) we have that $\mathrm{Tor}_1^R(M_1,N)=0$, and now (2.3.1) says $\mathrm{Tor}_2^R(M,N)=0$. This finishes the proof of the theorem. □

Although we have already mentioned the next corollary in the text, we separate this consequence of Theorem 2.4 to emphasize the main theme: that weaker depth conditions on the tensor product of two modules over a complete intersection often force much stronger depth conditions on the modules themselves. See also [CI] and [J2] for recent work on depth formulas for modules over local rings, related to the vanishing of Tors.

**Corollary 2.5.** *Let $R$ be a complete intersection of codimension 2 and dimension $d \geq 3$, and let $M$ and $N$ be $R$-modules. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^1(R)$,*
(2) *$N$ is free of constant rank on $\operatorname{Ass}(R)$,*
(3) *$M$ and $N$ satisfy $(S_2)$.*
(4) *$M \otimes_R N$ satisfies $(S_3)$.*

*Then depth$(M)$ + depth$(N) \geq d + 3$.*

*Proof.* By Theorem 2.4, all the positive Tors of $M$ and $N$ vanish. The result is then immediate from Theorem 1.1. □

**Codimension 3.**

For codimension 3, in order to use the rigidity theorem (1.9), we must assume that our complete intersection is unramified.

Once again it is helpful to isolate the critical result on rigidity.

**2.6. Theorem.** *Let $(R, \mathfrak{m})$ be an unramified complete intersection with $\dim(R) = \operatorname{codim}(R) = 3$. Suppose $M$ and $N$ are torsion-free $R$-modules. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^2(R)$,*
(2) *$N$ is orientable.*

*If for some $i \geq 1$ we have $\operatorname{Tor}_i^R(M, N) = \operatorname{Tor}_{i+1}^R(M, N) = \operatorname{Tor}_{i+2}^R(M, N) = 0$ , then $\operatorname{Tor}_j^R(M, N) = 0$ for all $j \geq i$.*

*Proof.* We may assume $N$ is not free. Therefore we can choose, by (1.4) and (1.6), a Bourbaki ideal $J$ for $N$ with height$(J) \geq 2$. If $J$ is $\mathfrak{m}$-primary, put $S = \emptyset$. Otherwise, let $S$ be the set of minimal prime ideals of $J$. Since height$(p) = 2$ for each $p \in S$ and $M$ is free of constant rank on $\mathbf{X}^2(R)$, (1.4) provides a Bourbaki ideal $I$ for $M$ with $I \not\subseteq \bigcup S$. The hypothesis on the vanishing Tors implies, by (1.2) and dimension shifting, that $\operatorname{Tor}_{i+2}^R(R/I, R/J) = \operatorname{Tor}_{i+3}^R(R/I, R/J) = \operatorname{Tor}_{i+4}^R(R/I, R/J) = 0$. Since $\sqrt{I + J} = \mathfrak{m}$ and $\dim R/I + \dim R/J < d + r$, (1.9) implies that $\operatorname{Tor}_j^R(R/I, R/J) = 0$ for all $j \geq i$. Shifting back, we have $\operatorname{Tor}_j^R(I, J) = 0$ for all $j \geq i$, and another application of (1.2) completes the proof. □

**2.7 Theorem (dimension 3).** *Suppose that $R$ is an unramified complete intersection of codimension 3 and dimension $d \leq 3$, and let $M$ and $N$ be $R$-modules. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^2(R)$,*
(2) *$N$ is orientable,*
(3) *$M$ satisfies ($S_3$).*
(4) *$N$ satisfies ($S_2$)*

*If $M \otimes_R N$ is maximal Cohen-Macaulay, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

*Proof.* If $d \leq 2$ there is nothing to prove. Therefore we assume that $d = 3$. We apply Lemma 2.1 with $c = 3$. We have $\operatorname{Tor}_1^R(M_3, N) = \operatorname{Tor}_2^R(M_3, N) = \operatorname{Tor}_3^R(M_3, N) = 0$. By (2.6) $\operatorname{Tor}_j(M_3, N) = 0$ for all $j \geq 1$. By shifting along the sequences (2.1.1), we get $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. □

**2.8 Theorem (arbitrary dimension).** *Suppose that $R$ is an unramified complete intersection of codimension 3 and dimension $d$. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^2(R)$,*
(2) *$N$ is orientable,*
(3) *$M$ and $N$ satisfy ($S_3$).*

*If $M \otimes_R N$ satisfies($S_4$), then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

*Proof.* We can assume $d \geq 4$. Form the quasi-liftings $E$ and $F$ of $M$ and $N$, respectively. By (2.7) we get $\operatorname{Tor}_i^R(M, N)_p = 0$ for all $i \geq 1$ and for all $p \in \mathbf{X}^3(R)$. Therefore, by (1.8) with $c = 3$, $E \otimes_S F$ satisfies (S$_3$). Note that $E$ is free of constant rank on $\mathbf{X}^1(S)$ and $F$ is free of constant rank on $\mathbf{X}^0(S)$. Therefore, by (2.4), $\operatorname{Tor}_i^S(E, F) = 0$ for all $i \geq 1$, which yields $\operatorname{Tor}_i^R(E/fE, N) = 0$ for all $i \geq 1$. As in the proof of Theorem 2.4, we have $\operatorname{Tor}_{i+2}^R(M, N) \cong \operatorname{Tor}_i^R(M, N)$ for all $i \geq 1$. Therefore it is enough to show that $\operatorname{Tor}_1^R(M, N) = \operatorname{Tor}_2^R(M, N) = 0$.

Since $\beta = 0$ in (1.8.9), we have $\operatorname{Tor}_1^R(M, N) = 0$. Lastly, since Proposition 1.7 (2) shows that $\operatorname{Tor}_{i+1}^R(M, N) \cong \operatorname{Tor}_i^R(M_1, N)$ for all $i \geq 1$, and since $\operatorname{Tor}_1^R(M_1, N) = 0$ (by (1.8.8)), we get $\operatorname{Tor}_2^R(M, N) = 0$. This completes the proof. □

As in Corollary 2.5, we also can use Theorem 2.8 to give strong depth conditions under the assumptions of (2.8).

**Corollary 2.9.** *Suppose that $R$ is an unramified complete intersection of codimension 3 and dimension $d \geq 4$. Assume*

(1) *$M$ is free of constant rank on $\mathbf{X}^2(R)$,*
(2) *$N$ is orientable,*
(3) *$M$ and $N$ satisfy ($S_3$).*
(4) *$M \otimes_R N$ satisfies($S_4$).*

*Then depth(M) + depth(N) $\geq d+4$.*

*Proof.* Theorem 2.7 gives that all positive Tors of $M$ and $N$ vanish. Since $R$ is a complete intersection, the depth formula Theorem 1.1 holds, which implies the conclusion. □

**Higher Codimension.**

The whole source of the difficulty in our proofs, and the obstruction to

extending them to higher codimension, lies in hypothesis (1) in (1.9), the rigidity theorem of [HW] — namely, that $M \otimes_R N$ have finite

length. It is conceivable that (1.9) is still true without this hypothesis, but an entirely different proof would be needed, since the proof in [HW] relied on Euler characteristics.

Our use of Bourbaki ideals and orientability were designed to replace the modules $M$ and $N$ by ideals $I$ and $J$ such that $I+J$ is $\mathfrak{m}$-primary, in order to be able to apply (1.9). One might ask why we can't simply *hypothesize* the existence of a Bourbaki ideal of large height for one of the modules. To see the difficulty in this approach, suppose we wanted to generalize our Theorem 2.6 to codimension 4 and dimension 4. If we had a Bourbaki ideal $J$ of height 3 for the module $N$, we would have $\operatorname{depth}(J) \leq 1+\dim(R/J) = 2$. This in turn would force $N$ to have depth at most 2, which would be too low to make our arguments go through. Of course, one might have two Bourbaki ideals $I$ and $J$, each of height two, such that $I+J$ is $\mathfrak{m}$-primary, provided the dimension of $R$ is not too much bigger than its codimension. However in this case

for every prime $p$ not equal to $\mathfrak{m}$, either $M_p$ or $N_p$ would have to be free. This is a strong restriction on any type of induction.

It would be nice to get rid of the hypotheses, in (2.4) and (2.7), that $M$ is free on $\mathbf{X}^1(R)$, respectively, $\mathbf{X}^2(R)$. In this context we might

point out that if neither $M$ nor $N$ has constant rank on $\mathbf{X}^0(R)$ there is not much hope of proving vanishing. For example, take $R = k[[x,y]]/(xy)$, and let $M = Rx$. Then $M \otimes_R M \cong M$, which is maximal Cohen-Macaulay, yet $\operatorname{Tor}_i^R(M,M)$ is nonzero for $i$ odd.

## 3. Integrality and $M \otimes_R N$

In this section we explore to what extent one can remove some of the depth assumptions on $M$ or $N$. We focus on the case in which the tensor product is reflexive. One would like to conclude that $M$ and $N$ must be reflexive also. The best we can do at this time is the following quite general theorem and corollary.

**3.1 Theorem.** *Suppose that $R$ is a domain, and let $M$ and $N$ be $R$-modules such that $M \otimes_R N$ is reflexive and $N$ is torsion-free. Then for all $\phi \in \operatorname{Hom}_R(M, R)$, we have $\phi(M)N = \phi(M)N^{**}$.*

*Proof.* From the natural map $N \hookrightarrow N^{**}$ we have a short exact sequence $0 \to N \to N^{**} \to L \to 0$ with $L$ having grade $\geq 2$. We tensor this with $M$. Since $M \otimes_R N$ is torsion-free and $\operatorname{Tor}_1^R(M, L)$ is torsion, we get the short exact sequence

$$(3.1.1) \qquad 0 \to M \otimes_R N \to M \otimes_R N^{**} \to M \otimes_R L \to 0.$$

As $M \otimes_R N$ satisfies $(\mathrm{S}_2)$, and $M \otimes_R L$ has grade at least two, $\operatorname{Ext}_R^1(M \otimes_R L, M \otimes_R N) = 0$. Hence (3.1.1) splits. Let $\alpha : M \otimes_R N^{**} \to M \otimes_R N$ be the left splitting. Let $\phi \in \operatorname{Hom}_R(M, R)$ and consider the commutative diagram

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\;j\;} & M \otimes_R N^{**} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \phi(M)N & \xrightarrow[\;i\;]{} & \phi(M)N^{**} \end{array}$$

where the vertical maps are the natural maps and the horizontal maps are the inclusions. Since $L \cong N^{**}/N$ is torsion, we just need to show that $\alpha$ induces a splitting of $i$ in order to prove the theorem.

Given $a \in \phi(M)N^{**}$, set $\beta(a) := \pi\alpha(b)$, where $b$ is a preimage of $a$ under $\rho$. We claim that $\beta$ is well-defined.

Suppose that $b$ and $b'$ in $M \otimes_R N^{**}$ are two preimages of $a$. Let $x \in R$ be any element satisfying $xN^{**} \subseteq N$. We have $\rho(b) = \rho(b')$, and so $\rho(xb) = \rho(xb')$. Since $xb$ and $xb'$ are in $M \otimes_R N$, this is the same as saying $\rho j(xb) = \rho j(xb')$. By commutativity, this gives $i\pi(xb) = i\pi(xb')$. Hence $\pi(xb) = \pi(xb')$, and this is the same as saying $\pi\alpha(xb) = \pi\alpha(xb')$. Since $\phi(M)N$ is torsion-free, we get $\pi\alpha(b) = \pi\alpha(b')$, which is what we wanted to show.

It's now an easy exercise to show that $\beta$ is a splitting. $\square$

**3.2 Corollary.** *Suppose that $R$ is a local domain and let $M$ and $N$ be $R$-modules with $M^* \neq 0$ and $N$ torsion-free. If $M \otimes_R N$ is reflexive, then $N^{**}$ is integral over $N$.*

*Proof.* This follows from the general fact that if $R$ is a domain, $N$ is a submodule of $M$, and $IM \subseteq IN$ for some non-zero ideal $I$ of $R$, then $M$ is integral over $N$ by the valuative criterion for a module to be integral over a submodule. (See, for example, [R, Definition, p.5].) $\square$

## References

[A] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631–647.

[Av] L. Avramov, *Modules of finite virtual projective dimension*, Invent. math. **96** (1989), 71-101.

[AvB] L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, preprint.

[B] W. Bruns, *The Buchsbaum-Eisenbud structure theorems and alternating syzygies*, Comm. Algebra **15** (1987), 873–925.

[CI] S. Choi and S. Iyengar, *On a depth formula for modules over local rings*, preprint.

[HW] C. Huneke and R. Wiegand, *Tensor products of modules and the rigidity of Tor*, Math. Ann. **299** (1994), 449-476.

[J1] D. A. Jorgensen, *Complexity and* Tor *on a complete intersection*, J. Algebra **211** (1999), 578–598.

[J2] D. A. Jorgensen, *A generalization of the Auslander-Buchsbaum formula*, J. Pure Applied Algebra (to appear).

[L] S. Lichtenbaum, *On the vanishing of* Tor *in regular local rings*, Illinois J. Math. **10** (1966), 220–226.

[Ma] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, 1990.

[Mu] M. P. Murthy, *Modules over regular local rings*, Illinois J. Math. **7** (1963), 558–565.

[R] D. Rees, *Reductions of modules*, Math. Proc. Camb. Phil. Soc. **101** (1987), 431–449.

[W] R. Wiegand, *Dimension functions on the prime spectrum*, Comm. Algebra **3** (1975), 459–480.