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Correction to "Tensor Products of Modules and the Rigidity of Tor", Math. Annalen, 299 (1994), 449–476.

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Abstract This note makes a correction to the paper "Tensor products of modules and the ridigity of Tor", a correction which is needed due to an incorrect convention for the depth of the zero module.

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1 Correction

In the paper [4], a convention for the depth of the zero module was made which is incorrect and affects several statements and proofs in the paper. The depth of the zero module was set to be -1, instead of the correct convention, which is ∞. This has unexpected consequences in the statements of some of the results. Recall from [2] that a finitely generated module M over a Noetherian ring R is said to satisfy condition (S_n) if

$$depth(M_P) \geq \min\{n, \dim(R_P)\} \ for\ every\ P \in Spec(R).$$

Using the convention that the depth of the module 0 is -1 implies that a module M can satisfy (S_n) only if the support of M is all of $Spec(R)$, which is not what is needed for various statements in the text. In particular, if R

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is Gorenstein, then [2, 3.6] gives that a module satisfies S_2 if and only if it is reflexive, and this characterization requires the convention that the depth of the zero module be ∞.

Another example is in application of the depth lemma. If

$$0 \to M \to N \to K \to 0$$

is a short exact sequence of finitely generated modules over a Noetherian local ring R, and $depth(N) > depth(K)$, then $depth(M) = depth(K) + 1$. This is clearly false under the convention that the depth of 0 be -1, e.g., if $K = 0$. This necessitates additional arguments in some cases, e.g., when localizing a short exact sequence at a prime not in the support of one of the modules.

Changing the depth of the zero module to ∞ means that some statements need to be revised, and one proof in particular needs some more comment. The two results needing change are Corollary 2.6 and Theorem 2.7. Corollary 2.6 should be modified to the following statement:

Corollary 1 *Let R be a complete intersection, and let M and N be nonzero R-modules such that $Tor_i^R(M,N) = 0$ for all $i \geq 1$. If $M \otimes_R N$ is maximal Cohen-Macaulay, then so are M and N.*

The proof of this statement follows directly from Proposition 2.5, as in the text. Proposition 2.5 states:

Proposition 1 *Let R be a complete intersection. Let M and N be non-zero finitely generated R-modules such that $Tor_i^R(M,N) = 0$ for all $i \geq 1$. Then*

$$depth(M) + depth(N) = dim(R) + depth(M \otimes_R N).$$

One needs to note here that $M \otimes_R N$ cannot be zero since both M and N are nonzero, and R is local.

The only change in Corollary 2.6 is the deletion of the additional assertion that if $M \otimes_R N$ satisfies Serre's property (S_n), then so do M and N. This is true if the support of both modules is entire spectrum, but could fail otherwise.

Likewise, Theorem 2.7 should be contracted to the following statement:

Theorem 1 *(Second Rigidity Theorem) Let $R = S/(f)$ be a hypersurface, and let M and N be non-zero R-modules, at least one of which has constant rank. If $M \otimes_R N$ is reflexive, then $Tor_i^R(M,N) = 0$ for all $i \geq 1$.*

The additional conclusion that both M and N are reflexive is never used in the text (except in low dimension, in which case the fact that they are maximal Cohen-Macaulay follows from (2.6)). The problem with this additional conclusion is that if $N_P = 0$, then the hypothesis is satisfied, but one cannot necessarily conclude anything about M_P. One can only conclude results about *both* modules on the intersection of their supports.

The proof of (2.7) must be modified in a few spots to take into account localization at a prime not in the support of one of the modules, but this is easy to do. One main place this problem arises is in the argument that from the short exact sequence

$$0 \to M \otimes_R N \to N^{(n)} \to M_1 \otimes_R N \to 0$$

one can conclude that the associated primes of $M_1 \otimes_R N$ are all of height at most one. The proof given has a gap in case a prime P is not in the support of the tensor product $M \otimes_R N$. One must argue separately that if P is associated to $M_1 \otimes_R N$, and if $(M \otimes_R N)_P = 0$, then P is also associated to N, and hence has height zero as N is torsion-free.

Corollary 2.6 and Theorem 2.7 are used in the paper [5] and similar modifications must be made, e.g., in [5] pgs. 169–172.

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$$depth_{R_P}(M_P) \geq min\{n, dim(M_P)\} \; for \; every \; P \in Spec(R).$$

This is weaker than the version we use.

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